The Chern-Simons Character of a Lattice Gauge Field

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Abstract

Let $\mathbf{u}$ be a generic $SU(2)$-valued lattice gauge field on a triangulation Λ of a manifold X of dimension ≥ 3. An algorithm is given for constructing from $\mathbf{u}$ a principal $SU(2)$-bundle ξ over X, and in ξ a piecewise smooth connection ω with the following property. The Chern-Simons character of the gauge field (ξ, ω) can be computed as the sum of local contributions $\hat{\chi}(\sigma)$, one for each 3-simplex σ of Λ, where $\hat{\chi}(\sigma)$ is the sum of a one-dimensional integral and five other terms which are volumes of elementary regions in the 3-sphere. For any 3-cycle $\sum_i a_i \sigma_i$, the sum $\sum_i a_i \hat{\chi}(\sigma_i)$, reduced mod $\mathbf{Z}$, is gauge-invariant. This character is related in the appropriate way to real and integral 4-cycles representing the second Chern class of ξ. If $\mathbf{u}$ has sufficiently small plaquette products, the isomorphism class of ξ is shown to depend only on $\mathbf{u}$, and not on details of the construction.

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1 Introduction

The Chern-Simons forms on the total space of a principal bundle with connection link the Chern-Weil-theoretic and the obstruction-theoretic approaches to characteristic classes. (We shall take up this theme in Theorem C.) When pulled down to the base, they give rise to the Chern-Simons characters [3, 2], with many applications to differential geometry. Recently interest in the Chern-Simons theory has grown remarkably, on the part of physicists as well as mathematicians. The reason is Witten's work [20], where the integral of the Chern-Simons form of an $SU(2)$-valued gauge field on a 3-manifold serves as the action in the path integrals of a topological quantum field theory. This turns out to have extensive applications to knot theory and to the theory of 3-manifolds.

Thus two well-explored parts of mathematics are joined by a chain with several fuzzy links, since the path-integral formalism, while powerful and intuitively very appealing, is not yet mathematically complete, and does not admit direct computation in general. Now lattice gauge fields (see Section 3.4) were invented by Wilson [19] as an attack on this very problem. It is in this spirit that we propose a lattice-theoretic construction of the Chern-Simons character, and an algorithm for its computation.

Our method involves a universal construction: following Dupont [4] we define a canonical connection in the total space of Milnor's universal G–bundle $\xi_\Delta G = (\pi_\Delta : E_\Delta G \to B_\Delta G)$, and a corresponding canonical Chern-Simons form. Then we construct from the lattice gauge field a coherent set of maps $\Gamma_\sigma : C_\sigma \to E_\Delta G$, one for each simplex σ in the lattice Λ (C_σ is a cube of the same dimension as σ), and use these maps to obtain a classifying map from Λ to $B_\Delta G$. For these maps to pull back the canonical Chern-Simons form in a computable way, the Γ_σ's must be defined compatibly with the natural cellular structure of $E_\Delta G$. This structure was established by Eilenberg and MacLane [5]and is called the bar construction. Finally the contribution to the Chern-Simons character from each 3–simplex σ is computed by integrating the pulled-back Chern-Simons form over a pseudo-section ψ (constructed from the lattice gauge field) above σ. For recent, related work on this topic, see [6, 7, 9, 21]. This paper amplifies, with some modifications, the preliminary account given in [16].

In our previous work [13, 14, 15] we have explained various methods for constructing a bundle (and, when appropriate, a connection) from lattice data. The method used here is a technical (but important) refinement of the method of [15]. The difference between the Γ_σ's and the analogous H_σ's [15]is that H_σ uses the product structure of C_σ, whereas Γ_σ is defined using an appropriate decomposition of C_σ into bar-products of lower–dimensional cubes. The H_σ's form a pseudo-section (called the "canonical pseudo-section" in [15]) but neither they nor the Γ_σ's have the property, of lifting a cycle to a cycle, necessary for the definition of the Chern-Simons character; hence the need for the more elaborate construction of ψ, which in fact contains the Γ_σ's (see Figure 5).

The rest of this section contains basic definitions and states the main results of this paper.

1.1 The Chern-Simons form as a differential character

Here we summarize results of Chern and Simons [3] and of Cheeger and Simons [2]. Let (ξ, ω) be an $SU(2)$-valued gauge field on a manifold X of dimension ≥ 3. That is, $\xi = (\pi : E \to X)$ is a principal $SU(2)$-bundle, and ω is an $\mathbf{su}(2)$-valued 1-form on E satisfying the usual conditions [8] (see also Prop. 4.2).

The real-valued 4-form $\tilde{p}$ on E given by

$$\tilde{p} = \frac{1}{8\pi^2}\mathrm{Tr}(\Omega \wedge \Omega),$$

where Ω is the curvature of ω, is the lift of a unique 4-form p on X:

$$\pi^*(p) = \tilde{p}.$$

The form p (the "topological charge density" of the gauge field when $\dim X = 4$) represents the second Chern class of ξ.

The form $\tilde{p}$ is always exact, no matter what the bundle ξ may be. In fact, a simple calculation shows that it is the coboundary of the 3-form $\tilde{\chi}$ defined on E by [3]

$$\tilde{\chi} = \frac{1}{8\pi^2}\mathrm{Tr}(\omega \wedge \Omega - \frac{1}{3}\omega \wedge \omega \wedge \omega).$$

This is the Chern-Simons form. Now it is impossible to find a 3-form α on X such that $\pi^*(\alpha) = \tilde{\chi}$; for on any $SU(2)$ fibre $\pi^*(\alpha)$ would vanish, while $\tilde{\chi}$ restricts to the volume form. However, $\tilde{\chi}$ can be represented as a "differential character" on X, as follows.

Let $\mathcal{Z}$ be the additive group of 3–cycles on X. For $Z \in \mathcal{Z}$, the restriction $\xi|_Z$ is trivial (for topological reasons) and so admits a section $\psi : Z \to \pi^{-1}(Z)$. Now set

$$\chi(Z) = \int_{\psi(Z)} \tilde{\chi}, \quad \text{reduced} \mod \mathbf{Z}.$$

It can be shown [2] that if $\psi' : Z \to \pi^{-1}(Z)$ is another section of $\xi|_Z$, then $\int_{\psi(Z)} \tilde{\chi} - \int_{\psi'(Z)} \tilde{\chi}$ is an integer; hence $\chi(Z)$ is independent of ψ. We will call this element of $\mathbf{R}/\mathbf{Z}$ the *Chern–Simons character of the gauge field* $(\xi, \tilde{\omega})$.

Remark 1.1 Suppose M is a 4-chain on X with $\partial M = Z$; if $Z \neq \emptyset$ then $\xi|_M$ is trivial, so let $\varphi : M \to \pi^{-1}(M)$ be a section; then

$$\int_M p = \int_{\varphi(M)} \tilde{p} = \int_{\varphi(Z)} \tilde{\chi} \equiv \chi(Z) \mod \mathbf{Z}.$$

Thus $\chi : \mathcal{Z} \to \mathbf{R}/\mathbf{Z}$ is indeed a differential character as defined in [2].

1.2 The Chern-Simons character of a lattice gauge field

By a (*simplicial*) *lattice* on X we mean a simplicial triangulation Λ of X, as in [14]. In the sections following this one we shall need an ordering (which can be arbitrary) of the vertices of Λ, say as $<0>, <1>, <2>, \ldots$. We shall call a simplicial lattice together with such an ordering an *ordered* (*simplicial*) *lattice*. Every simplex σ of an ordered lattice has an orientation induced by the ordering of its vertices, and when we write a simplex as $\sigma = < i_0, \ldots, i_k >$, we list the vertices according to their order.

Henceforth we restrict attention to 3-cycles Z which are chains on Λ; that is, $Z = \sum a_i \sigma_i$, where $\sigma_i \in \Lambda, \dim \sigma_i = 3$, σ_i has the orientation just described, $a_i \in \mathbf{Z}$, and $\sum a_i \partial \sigma_i = 0$.

Let $\Lambda^{(3)} = \bigcup\{\sigma : \dim \sigma = 3\}$, i.e. the the 3-skeleton of Λ. Then $\xi|_{\Lambda^{(3)}}$ is trivial, again for dimensional reasons. Let $\psi : \Lambda^{(3)} \to \pi^{-1}(\Lambda^{(3)})$ be a section or a pseudosection (see Section 5).

For every 3-simplex σ, set

$$\hat{\chi}(\sigma) = \int_{\psi(\sigma)} \tilde{\chi}.$$

This is a real-valued 3-cochain on Λ. The values $\hat{\chi}(\sigma)$ depend on ψ (as well as ω); but for any 3-cycle $Z = \sum a_i \sigma_i$, the element $\chi(Z) \in \mathbf{R}/\mathbf{Z}$, defined as

$$\chi(Z) = \sum a_i \hat{\chi}(\sigma_i) \quad \text{reduced} \mod \mathbf{Z},$$

is independent of ψ.

Now let $\mathbf{u} = \{u_{ij} :< ij >\in \Lambda\}$ be a generic (see below) $SU(2)$-valued lattice gauge field (see Section 3.4) on the ordered simplicial lattice Λ.

Our procedure will be to construct from $\mathbf{u}$ a principal $SU(2)$-bundle ξ, a connection in ξ and the derived Chern-Simons form $\tilde{\chi}$, and a (pseudo)section of ξ over the 3–skeleton $\Lambda^{(3)}$ for which the calculation of the values $\hat{\chi}(\sigma)$ is as simple as possible. In order to state our result we need some preliminary notation and constructions.

Notation. We take $SU(2)$ as the unit sphere S^3 in the quaternionic space $\mathbf{H}$. For a quaternion $u = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, $(a, b, c, d \in \mathbf{R})$, we set as usual $\Re(u) = a$, $\Im(u) = b\mathbf{i} + c\mathbf{j}f + d\mathbf{k}$. The matrix trace of u becomes $2\Re(u)$.

For a product of transporters along consecutive 1-simplexes we use the abbreviation

$$u_{i_0 i_1 \ldots i_k} = u_{i_0 i_1} u_{i_1 i_2} \ldots u_{i_{k-1} i_k}.$$

The following constructions make extensive use of shortest geodesics in S^3 between various points constructed from $\mathbf{u}$, and certain families of such geodesics. For almost all $\mathbf{u}$, every such shortest geodesic or family is unique. Henceforth we shall assume that $\mathbf{u}$ is *generic* in this sense.

For every 1-simplex $< ij >$ of Λ, let $\mathbf{g}_{\alpha ij}$ be the geodesic in S^3 from $\mathbf{1}$ to u_{ij}, parametrised by $0 \le s_i \le 1$; for every 2-simplex $< ijk >$, let $\mathbf{g}_{ijk}$ be the geodesic from u_{ik} to u_{ijk}, parametrised by $0 \le s_j \le 1$.

Now let σ be a 3-simplex of Λ. We construct five 3-dimensional, oriented regions in S^3 associated to $\mathbf{u}$ and σ: four tetrahedra and what we have called a "pyramid" [14]. To simplify the notation, write σ as $< 0123 >$.

Let Q be the doubly ruled quadrilateral in S^3 defined by $Q(s_0, s_2) = \mathbf{g}_{\alpha 01}(s_0)\mathbf{g}_{123}(s_2), 0 \le s_0, s_2 \le 1$. Our *pyramid* P is the cone from $\mathbf{1}$ on Q; that is, P is the union of the shortest geodesics from $\mathbf{1}$ to $Q(s_0, s_2)$ for all $s_0, s_2 \in [0, 1]$. P is oriented by the following sequence of its vertices: $\mathbf{1}$, $u_{13} = Q(0, 0)$, $u_{013} = Q(1, 0)$, $u_{0123} = Q(1, 1)$.

This leaves the four tetrahedra. By a *tetrahedron* with vertices u,v,w and x we mean the geodesic convex hull of these four points, which we denote by $\mathrm{cx}(u, v, w, x)$. The tetrahedron is oriented by the given ordering of its vertices. Now set

$$\begin{aligned}
\Delta_1 &= \mathrm{cx}(\mathbf{1}, u_{03}, u_{013}, u_{0123}),\\
\Delta_2 &= \mathrm{cx}(\mathbf{1}, u_{03}, u_{023}, u_{0123}),\\
\Delta_3 &= \mathrm{cx}(\mathbf{1}, u_{23}, u_{023}, u_{0123}),\\
\Delta_4 &= \mathrm{cx}(\mathbf{1}, u_{23}, u_{123}, u_{0123}).
\end{aligned}$$

Theorem A Let $\mathbf{u}$ be a generic $SU(2)$-valued lattice gauge field on an ordered simplicial lattice Λ of dimension ≥ 3. Let ξ be the principal $SU(2)$-bundle assigned to $\mathbf{u}$ by our algorithm. Then there exist a piecewise smooth connection ω in ξ and a pseudo-section ψ of $\xi|_{\Lambda^{(3)}}$ such that the localisation of the Chern-Simons character, when applied to $\sigma =< 0123 >$, has the form

$$\hat{\chi}(\sigma) = \int_P dv + \sum_{i=1}^{4}(-1)^i \int_{\Delta_i} dv + \frac{1}{4\pi^2}\int_0^1 \Re[U_{01} \cdot \mathrm{Ad}_{\mathbf{g}_{\alpha 01}(s_0)} W_{123}] ds_0.$$

In this expression dv is the normalised volume form on S^3, so $\int_{S^3} dv = 1$, and $U_{01} \in \mathfrak{s}u_2$ is the matrix such that $\exp(U_{01}) = u_{01}$ (so that $\mathbf{g}_{\alpha 01}(s_0) = \exp(s_0 U_{01})$); similarly $U_{3123} \in \mathfrak{s}u_2$ is the matrix such that $\exp(U_{3123}) = u_{3123}$ (so that $\mathbf{g}_{123}(s_2) = u_{13}\exp(s_2 U_{3123})$, and $W_{123} = \mathrm{Ad}_{u_{13}} U_{3123}$.

The value of $\hat{\chi}(\sigma)$ will clearly change if $\mathbf{u}$ is replaced by a gauge-equivalent field. Nevertheless we have as much gauge-invariance as can be expected (compare Section 1.1):

Theorem B For any 3-cycle $Z = \sum_i a_i \sigma_i$, the sum $\chi(Z) = \sum_i a_i \hat{\chi}(\sigma_i)$, reduced mod $\mathbf{Z}$, is gauge-invariant.

Remarks on uniqueness. To what extent do these results depend on $\mathbf{u}$ alone, and not on details of the algorithm? If $\mathbf{u}$ is continuous, i.e. satisfies the following *continuity hypothesis*:

For every 2-simplex $< ijk > \in \Lambda$, $d(u_{ij}u_{jk}, u_{ik},) < \pi/n$,
where d is the unit-sphere metric on $SU(2)$ and $n = \dim \Lambda$,

then (Proposition 3.14) $\mathbf{u}$ determines a unique principal $SU(2)$-bundle $\xi = \xi(\mathbf{u})$. For a discussion of the relation between the lattice gauge field we start with, and the connection used to define its Chern=Simons character, see Section 4.4.

1.3 The Chern-Simons character and the calculation of topological charge

Here let $\dim \Lambda \geq 4$. Let $\mathbf{u}$ be a continuous, generic $SU(2)$-valued l.g.f. on Λ, and let ξ be the corresponding principal bundle. Let ω be the piecewise smooth connection in ξ constructed in Theorem A.

The Chern-Simons form in classical $SU(2)$ theory serves to connect the differential-geometric (Chern-Weil) and the obstruction-theoretic definitions of the second Chern class. The point of Theorem C is that the Chern-Simons character developed in this paper is related in almost exactly this way to the Chern form p pulled back from the universal example constructed from Dupont's connection in Milnor's classifying space and to the integral 4-cocycle N we constructed in [14] to compute the topological charge of an $SU(2)$-valued lattice gauge field. The "almost" is necessary because in [14] we augmented the lattice Λ by adding a new vertex ordered *above* the original ones, whereas in this paper it was more convenient to follow the opposite convention.

Let N be the 4-cocycle representative on Λ of the second Chern class of $\xi(\mathbf{u})$ constructed according to the algorithm of [14]. For any 4–simplex $\tau \in \Lambda$, by definition, $N(\tau)$ is the degree of a certain map from the boundary of a 4–cube into S^3.

On the other hand both $\int_\tau p$ and $\hat{\chi}(\partial\tau)$ give $\chi(\partial\tau)$ when reduced mod $\mathbf{Z}$, so their difference $\int_\tau p - \hat{\chi}(\partial\tau)$ is an integer.

Theorem C $\int_\tau p - \chi(\partial\tau)$ is equal to $\bar{N}(\bar{\tau})$, for every 4–simplex $\tau \in \Lambda$.

Here $\bar{N}(\bar{\tau})$ is the obstruction calculated as above, but when the order of the vertices is reversed. For details, see Section 6.4.

The rest of this paper: Section 3 summarizes material from Milnor [11, 12], Eilenberg-MacLane [5], and our own previous work [17]; Section 4 introduces a connection in Milnor's universal bundle and calculates the Chern-Simons character of this connection. The pseudosection associated to a lattice gauge field is constructed in Section 5. Section 6 contains the proofs of the main theorems; one computation is postponed to Section 7.

2 Milnor's universal bundle and the bar construction

2.1 Let $\xi_\Delta G = (\pi_\Delta : E_\Delta G \to B_\Delta G)$ be Milnor's model of a universal principal G–bundle [12]. Here $E_\Delta G = G_0 * G_1 * \cdots$, the join of countably many copies of G: a point of $E_\Delta G$ is $\sum_{i=0}^{\infty} t_i g_i$ where $g_i \in G_i$, $0 \le t_i \le 1$, all but finitely many t_i are 0, and $\sum t_i = 1$; it is to be understood that whenever $t_i = 0$, the term $t_i g_i$ is redundant. G acts on the left on $E_\Delta G$ by $L_g \sum t_i g_i = g \cdot (\sum t_i g_i) = \sum t_i g \cdot g_i$; and $B_\Delta G$ is the quotient space.

It is convenient to use the vectors $\mathbf{e}_0 = (1, 0, 0, \ldots)$, $\mathbf{e}_1 = (0, 1, 0, \ldots)$, etc. to index the various copies of G, by identifying $\mathbf{e}_i$ with the identity element in the i–th copy; thus we will write $\sum t_i g_i$ as $\sum t_i g_i \mathbf{e}_i$, where now all $g_i \in G$.

A feature that distinguishes $\xi_\Delta G$ from the more geometric Grassmannian models is the existence of *right*–automorphisms: if $\mathbf{h} = \{h_0, h_1, \ldots\}, h_i \in G$, set $R_{\mathbf{h}}(\sum t_i g_i) = \sum t_i g_i h_i$. Since left and right multiplication commute, $R_{\mathbf{h}}$ will take fibres to fibres, so there is a unique $r_{\mathbf{h}} : B_\Delta G \to B_\Delta G$ such that $r_{\mathbf{h}} \circ \pi_\Delta = \pi_\Delta \circ R_{\mathbf{h}}$.

There is a natural cell structure on $E_\Delta G$ which has been studied under the name of the *bar construction* [5].

Definition 2.2 Let $C_0, \ldots, C_k$ be cubes (of any dimension). The *geometric bar product* $C_0[C_1|\cdots|C_k]$ is defined by induction on k.

$C_0[\,]$ is just C_0; the brackets are included only to simplify certain formulas.

If $C_1[C_2|\cdots|C_k]$ has been defined, then

$$C_0[C_1|\cdots|C_k] = C_0 \times \mathrm{cone}(C_1[C_2|\cdots|C_k]).$$

We extend this linearly to define the geometric bar product of cubical chains.

Remark 2.3 **1.** If $C_0, \ldots, C_k$ are all 0–dimensional, then $C_0[C_1|\cdots|C_k]$ is just the simplex $< C_0, \ldots, C_k >$. Furthermore in general we have the following "barycentric" coordinates on $C_0[C_1|\cdots|C_k]$: it sits inside the r-fold join

$$C_0 * (C_0 \times C_1) * (C_0 \times C_1 \times C_2) * \cdots * (C_0 \times \cdots \times C_k);$$

and if $\vec{s}_i = (s_{i1}, \ldots, s_{ip_i})$ is a positively oriented coordinate system on C_i (so $0 \le s_{ij} \le 1$ and $p_i = \dim C_i$), then $C_0[C_1|\cdots|C_k]$ is the subset

$$\{t_0\vec{s}_0 + t_1(\vec{s}_0, \vec{s}_1) + t_2(\vec{s}_0, \vec{s}_1, \vec{s}_2) + \cdots + t_k(\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_k)\}$$

where $0 \le t_i \le 1$ and $\sum t_i = 1$, positively oriented (following Milgram [10]) by the coordinates $t_0, \ldots, t_k, s_{01}, \ldots, s_{0p_0}, \ldots, s_k, \ldots, s_{k,p_k}$.

2.

$$\dim C_0[C_1|\cdots|C_k] = k + \sum_{i=0}^{k} \dim C_i.$$

3.

$$\begin{aligned}
\partial(C_0[C_1|\cdots|C_k]) &= \partial C_0[C_1|\cdots|C_k] \\
&\quad + \sum_{i=1}^{k}(-1)^{k+\delta(i)} C_0[C_1|\cdots|\partial C_i|\cdots|C_k] \\
&\quad + (C_0 \times C_1)[C_2|\cdots|C_k] \\
&\quad + \sum_{i=1}^{k-1}(-1)^i C_0[C_1|\cdots|C_{i-1}|C_i \times C_{i+1}|C_{i+2}|\cdots|C_k] \\
&\quad + (-1)^k \epsilon(C_k) C_0[C_1|\cdots|C_{k-1}],
\end{aligned}$$

where $\delta(i) = \sum_{j<i} \dim C_j$, and $\epsilon(C_k) = 1$ if $\dim C_k = 0$, and $= 0$ otherwise (See Figure 1); note that our convention for orienting the boundary of X is that the outward normal to ∂X, followed by a positive basis for ∂X, gives a positive basis for X.

We extend these notions to certain singular cubes as follows.

Definition 2.4 Given $C_0, \ldots, C_k$ as before, together with indices $j(0) < j(1) < \cdots < j(k)$, and a collection of piecewise smooth maps $g_i : C_i \to G$, which we make into a collection of singular cubes $\gamma_i : C_i \to E_\Delta G$ by $\gamma_i(\vec{s_i}) = g_i(\vec{s_i}) \cdot \mathbf{e}_{j(i)}$, the *bar product*

$$\gamma_0[\gamma_1|\cdots|\gamma_k] : C_0[C_1|\cdots|C_k] \to E_\Delta G,$$

is given by

$$\gamma_0[\gamma_1|\cdots|\gamma_k](t_0\vec{s}_0 + t_1(\vec{s}_0, \vec{s}_1) + \cdots + t_k(\vec{s}_0, \ldots, \vec{s}_k))$$

$$= t_0(g_0(\vec{s}_0))\mathbf{e}_{j(0)} + t_1(g_0(\vec{s}_0) \cdot g_1(\vec{s}_1))\mathbf{e}_{j(1)} + \cdots + t_k(g_0(\vec{s}_0) \cdot \cdots \cdot g_k(\vec{s}_k))\mathbf{e}_{j(r)}.$$

Remark 2.5 **1.**

$$\dim \gamma_0[\gamma_1|\cdots|\gamma_k] = k + \sum_{j=0}^{k} \dim \gamma_j.$$

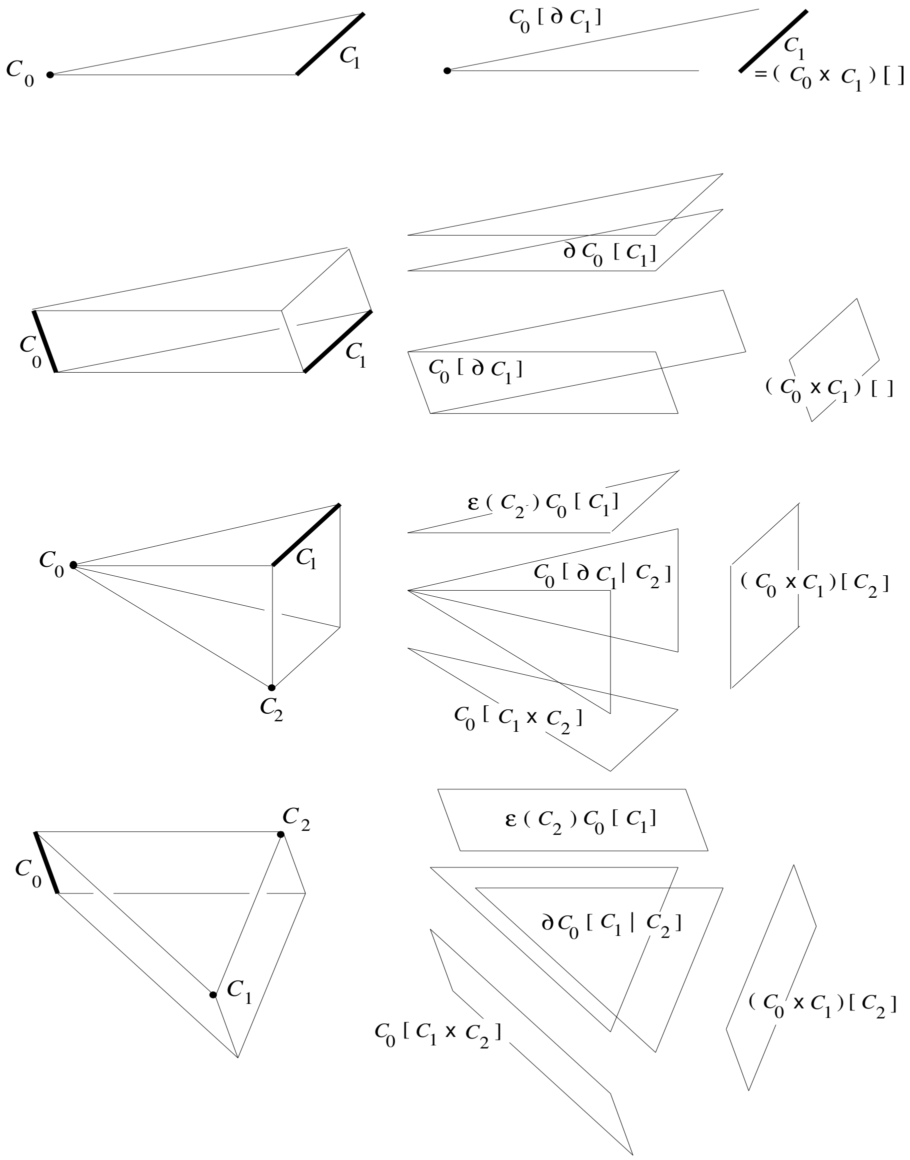

Figure 1: Some geometric bar-products and their boundaries. (a) $C_0[C_1]$, with $\dim C_0 = 0,\ \dim C_1 = 1$. (b) $C_0[C_1]$, with $\dim C_0 = 1,\ \dim C_1 = 1$. (c) $C_0[C_1|C_2]$, with $\dim C_0 = 0,\ \dim C_1 = 1,\ \dim C_2 = 0$. (d) $C_0[C_1|C_2]$, with $\dim C_0 = 1,\ \dim C_1 = 0,\ \dim C_2 = 0$.

2.

$$\begin{array}{rcl}\partial(\gamma_0[\gamma_1|\cdots|\gamma_k]) & = & (\partial\gamma_0)[\gamma_1|\cdots|\,\gamma_k] \\ & & +\sum_{i=1}^{k}(-1)^{k+\delta(i)}\gamma_0[\gamma_1|\cdots|\partial\gamma_i|\cdots|\gamma_k] \\ & & +(g_0\cdot\gamma_1)[\gamma_2|\cdots|\gamma_k] \\ & & +\sum_{i=1}^{k-1}(-1)^{i}\gamma_0[\gamma_1|\cdots|\gamma_{i-1}|g_i\cdot\gamma_{i+1}|\gamma_{i+2}|\cdots|\gamma_k] \\ & & +(-1)^{k}\epsilon(\gamma_k)\gamma_0[\gamma_1|\cdots|\gamma_{k-1}],\end{array}$$

where $\delta(i)=\sum_{j<i}\dim\gamma_j$, and $\epsilon(\gamma_k)=1$ if $\dim\gamma_k=0$, and $=0$ otherwise.

So the set of bar products of singular cubical chains forms a chain complex of singular chains on $E_\Delta G$. It is called the bar construction, and denoted by $\mathcal{E}_*$.

Note that this construction is not quite the same as the bar construction $\mathcal{E}_*$ of [17]. The difference between the two chain complexes corresponds to that between Milnor's and Milgram's versions of the universal G–bundle. The important point for us is that (either) $\mathcal{E}_*$ gives a model for the topology of $E_\Delta G$. More precisely, there is a G-equivariant chain homotopy equivalence from $\mathcal{E}_*$ to the singular (simplicial) complex of $E_\Delta G$. See [18] for a discussion of the two models.

3 The classifying map of a lattice gauge field

We start with a finite simplicial complex Λ and a total ordering of its vertices. For notational convenience we will renumber the vertices 0,1,2,... accordingly. (We will be using the consequent identification of Λ with a subset of Δ^∞.) When we write a simplex of Λ as $< i_0, \ldots, i_r >$, it will always be understood that $i_0 < i_1 < \ldots < i_r$ in this ordering.

3.1 Simplexes, cubes, and bar-products

For each r–simplex $\sigma = < i_0, \ldots, i_r > \in \Lambda$, let $C_\sigma \subset \mathbf{R}^\infty$ be the r–cube given by $0 \leq s_{i_1} \leq 1, \ldots, 0 \leq s_{i_r} \leq 1$, with the other coordinates zero, and similarly c_σ the $(r-1)$–cube given by $0 \leq s_{i_1} \leq 1, \ldots, 0 \leq s_{i_{r-1}} \leq 1$. If $r = 0$, set $C_\sigma = (0, 0, \ldots) = \mathbf{0}$. Furthermore (for future reference) for $1 \leq j \leq r$, let $\partial^j C_\sigma = \{s_{i_j} = 1\}$, and $\partial^+ C_\sigma = \bigcup_{j=1}^r \partial^j C_\sigma$; and similarly define $\partial^j c_\sigma$ and $\partial^+ c_\sigma$.

To simplify notation in the rest of this section, suppose $\sigma =< 0, \ldots, r >$.

Definition 3.1 (See [15]) The *standard projection* $\pi_\sigma : C_\sigma \to \sigma$ is defined by induction on r. When $r = 0$, C_σ and σ are both points; π_σ takes the first to the second. Now assume that for $\tau =< 0, \ldots, r-1 >$, $\pi_\tau : C_\tau \to \tau$ has been constructed. Then set

$$\pi_\sigma(s_1, \ldots, s_r) = (1 - s_r)\pi_\tau(s_1, \ldots, s_{r-1}) + s_r < r > .$$

3.2 The *bar-product decomposition* of the r–cube C_σ is a useful way of writing it as a union of bar-products of some of its subcubes. (The case $r = 3$ is shown in Figure 2; the case $r = 2$ appears in Figure 4). We continue with the convention that $\sigma =< 0, \ldots, r >$.

For each pair of indices (i, j), with $0 \leq i < j \leq r$, let $C_\sigma(i, j)$ represent the $(j - i - 1)$-face of C_σ given by

$$\left\{ \begin{array}{ll} s_k = 1 & k \leq i \\ s_j = 1 & \\ s_k = 0 & k > j \end{array} \right\}$$

and let $C_\sigma(0)$ represent the point $(0, 0, \ldots) \in C_\sigma$ (this will be $C_\sigma(i_0)$ if i_0 is the first-ordered vertex of σ).

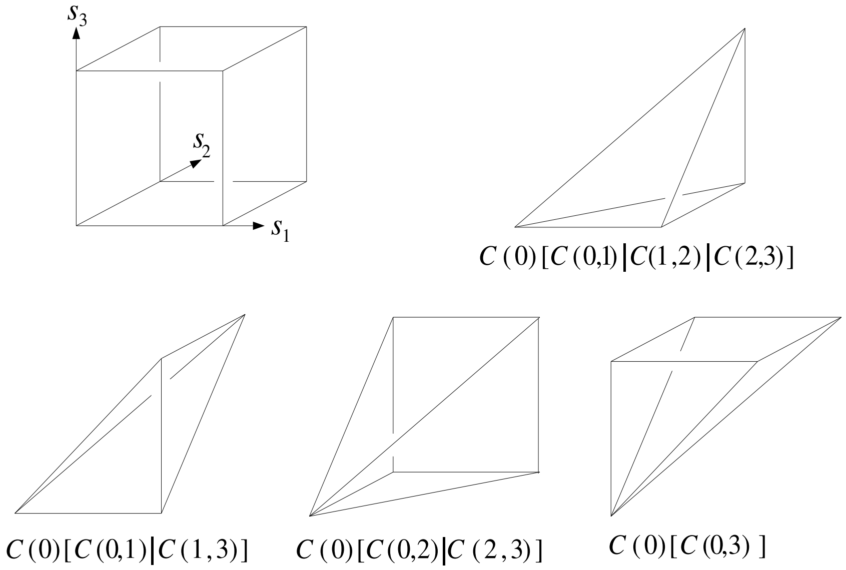

Figure 2: The bar-product decomposition of a 3–cube.

For each set of indices $J = \{0 = j_0 < j_1 < \cdots < j_p = r\}$ define $C_{\sigma,J}$ be the subpolyhedron of C_σ given by the inequalities

$$\begin{array}{ll} s_j \leq s_{j_{k+1}} & \text{for } j_k < j < j_{k+1} \quad k = 1, \dots, p-1 \\ s_{j_1} \geq \dots \geq s_{j_p}. & \end{array}$$

Proposition 3.3 (1) $C_{\sigma,J} = C_\sigma(0)[C_\sigma(0, j_1)|C_\sigma(j_1, j_2)|\cdots|C_\sigma(j_{p-1}, r)]$.
(2) As J varies, the $C_{\sigma,J}$ and their faces form a cellular decomposition of C_σ. More precisely, as oriented r-chains: $C_\sigma = \sum_J (-1)^{\theta_J} C_{\sigma,J}$, where $\theta_J = \sum_{i=1}^{p} (j_i - i)$.

Proof: (1) The construction begins with $C_\sigma(j_{p-2}, j_{p-1})[C_\sigma(j_{p-1}, r)]$. We embed this product in C_σ as follows. We construct the cone on $C_\sigma(j_{p-1}, r)$ from the vertex $o_p = \{s_j = 1 \text{ for } j \leq j_{p-1}, s_j = 0 \text{ for } j \geq j_p\}$. This cone and $C_\sigma(j_{p-2}, j_{p-1})$ have exactly the point o_p in common, so their cartesian product embeds in C_σ *via* the map $C_\sigma(j_{p-2}, j_{p-1}) \times \text{cone}\{C_\sigma(j_{p-1}, r)\} \to C_\sigma$ given by $(a, b) \to a + b - o_p$, with addition defined componentwise as usual.

For the next step, i.e. $C_\sigma(j_{p-3}, j_{p-3})[C_\sigma(j_{p-2}, j_{p-1})|C_\sigma(j_{p-1}, r)]$ we cone from $o_{p-1} = \{s_j = 1 \text{ for } j \leq j_{p-2}, s_j = 0 \text{ for } j \geq j_{p-1}\}$. This cone only has o_{p-1} in common with $C_\sigma(j_{p-3}, j_{p-3})$, so we can embed their cartesian product in C_σ as before.

The beginning of this construction may be made clearer by the following table, where we abbreviate $C_\sigma(j_{p-1}, r)$ by C_p, etc. Coordinates marked with

$*$ vary between 0 and 1. The coning variables are labelled $T_1,\dots,T_p$; for their relation to the "barycentric" coordinates of Remark 2.3 see Remark 3.4 below.

coordinates	$[1,j_{p-2}-1]$	j_{p-2}	$[j_{p-2}+1,j_{p-1}-1]$	j_{p-1}	$[j_{p-1}+1,r-1]$	r
C_p	1	1	1	1	$*$	1
o_p	1	1	1	1	0	0
cone$\{C_p\}$	1	1	1	1	T_p*	T_p
C_{p-1}	1	1	$*$	1	0	0
$C_{p-1}\times$ cone$\{C_p\}$	1	1	$*$	1	T_p*	T_p
o_{p-1}	1	1	0	0	0	0
cone$\{C_{p-1}[C_p]\}$	1	1	$T_{p-1}*$	T_{p-1}	$T_{p-1}T_p*$	$T_{p-1}T_p$

Finally we arrive at $C_\sigma(0)[C_\sigma(0,j_1)|C_\sigma(j_1,j_2)|\cdots|C_\sigma(j_{p-1},r)]$, where we have already embedded $C_\sigma(0,j_1)[C_\sigma(j_1,j_2)|\cdots|C_\sigma(j_{p-1},r)]$ as

$$\{(s_1,\dots,s_{j_1-1},1,T_2s_{j_1+1},\dots,T_2s_{j_2-1},T_2,\dots,\dots,T_2T_3\cdots T_p)\}.$$

We cone from $(0,\dots,0)=o_1$ but this time the cartesian product is with $C_\sigma(0)=o_1$ itself, so leaves the cone unchanged, and gives us the geometric bar-product $C_\sigma(0)[C_\sigma(0,j_1)|C_\sigma(j_1,j_2)|\cdots|C_\sigma(j_{p-1},r)]$ as

$$\{(T_1s_1,\dots,T_1s_{j_1-1},T_1,T_1T_2s_{j_1+1},\dots,T_1T_2s_{j_2-1},T_1T_2,\dots,\dots,T_1T_2T_3\cdots T_p)\}$$

which is clearly $C_{\sigma,J}$.

For the proof of (2) we refer to [17]; see Figure 2 for the cases $r=2,3$ which will be the ones of most importance for this work.□

Remark 3.4 A point $x\in C_\sigma(0)[C_\sigma(0,j_1)|C_\sigma(j_1,j_2)|\cdots|C_\sigma(j_{p-1},r)]$ has two sets of coordinates: the "barycentric" coordinates of Remark 2.3

$$x=t_0\vec{s}_0+t_1(\vec{s}_0,\vec{s}_1)+t_2(\vec{s}_0,\vec{s}_1,\vec{s}_2)+\cdots+t_p(\vec{s}_0,\vec{s}_1,\dots,\vec{s}_p)$$

(with $\sum t_i=1$) and the "interior" coordinates just calculated, which may be written

$$x=T_1\vec{s}_1,T_1,T_1T_2\vec{s}_2,T_1T_2,\dots,T_1\cdots T_p\vec{s}_p,T_1\cdots T_p.$$

For convenience in calculating the bar-product of singular cubes in Section 3.3 we note the relation

$$\begin{array}{rcl} t_0 &=& 1-T_1\\ t_1 &=& T_1(1-T_2)\\ t_2 &=& T_1T_2(1-T_3)\\ \vdots && \vdots\\ t_p &=& T_1T_2\cdots T_p. \end{array}$$

3.2 Parallel transport functions

Definition 3.5 A G-valued *parallel transport function* (p.t.f.; G a Lie group) over a locally ordered simplicial complex Λ consists of a family $\mathbf{V}$ of piecewise smooth maps of cubes into G, one $V_\sigma\colon c_\sigma \to G$ for each r-simplex σ of Λ, $r \geq 1$, such that, for every $\sigma =< i_0, \dots, i_r >$ the following compatibility conditions hold:

1. Cocycle condition

$$V_\sigma(s_{i_1}, ..., s_{i_{p-1}}, s_{i_p} = 1, s_{i_{p+1}}, ..., s_{i_{r-1}}) =$$
$$V_{<i_0,\dots i_p>}(s_{i_1}, ..., s_{i_{p-1}}) \cdot V_{<i_p,\dots i_r>}(s_{i_{p+1}}, ..., s_{i_{r-1}}),$$

2. Compatibility with faces

$$V_\sigma(s_{i_1}, ..., s_{i_{p-1}}, s_{i_p} = 0, s_{i_{p+1}}, ..., s_{i_{r-1}}) =$$
$$V_{<i_0,\dots i_{p-1}, i_{p+1},\dots i_{r-1}>}(s_{i_1}, ..., s_{i_{p-1}}, s_{i_{p+1}}, ..., s_{i_{r-1}}).$$

(See [15] for the relation of p.t.f.'s to coordinate bundles, etc.)

From a G-valued p.t.f. $\mathbf{V}$ over Λ we shall construct a map

$$f = f(\mathbf{V}) : |\Lambda| \to B_\Delta G;$$

this map induces a principal G–bundle $\xi = \xi(\mathbf{V})$ over $|\Lambda|$. We shall give an algorithm that from a generic lattice gauge field $\mathbf{u}$ over Λ constructs a p.t.f. $\mathbf{V}$; if in addition $\mathbf{u}$ satisfies a certain "continuity condition", the resulting bundle $\xi(\mathbf{V})$ will turn out to depend only on $\mathbf{u}$.

3.3 The classifying map of a p.t.f.

Let $\mathbf{V}$ be a G-valued p.t.f. on Λ, and consider an r–simplex $\sigma \in \Lambda$ which we will take again as $< 0, \dots, r >$. For each pair (i, j) with $0 \leq i < j \leq r$, set $\tau(i, j) =< i, i+1, \dots, j-1, j >$. We can identify $C_\sigma(i, j)$ with c_τ since both are parametrized by $(s_{i+1}, \dots, s_{j-1})$. In particular, since $\tau(0, r) = \sigma$, we identify the domain c_σ of V_σ with the face $C_\sigma(0, r) = \{s_r = 1\}$ of C_σ.

Define $\gamma_\sigma(i, j) : C_\sigma(i, j) \to G_j$ by

$$\gamma_\sigma(i, j)(s_{i+1}, \dots, s_{j-1}) = V_{\tau(i,j)}(s_{i+1}, \dots, s_{j-1}) \cdot \mathbf{e}_j,$$

where G_j is the jth factor of $E_\Delta G = G_0 * G_1 * \cdots$. Also, for the first-ordered vertex 0, define $\gamma_\sigma(0)$ to take the point $C_\sigma(0) = (0, 0, ...)$ to the identity element $\mathbf{e}_0$ in G_0.

Now (compare Proposition 3.3) for each $J = (0 = j_0 < j_1 < \cdots < j_p = r)$ set $\gamma_{\sigma,J} : C_{\sigma,J} \to E_\Delta G$ to be the bar product

$$\gamma_{\sigma,J} = \gamma_\sigma(0)[\gamma_\sigma(0,j_1)|\cdots|\gamma_\sigma(j_{p-1},r)].$$

Proposition 3.6 As J varies, the maps $\gamma_{\sigma,J}$ agree on the intersections of their domains; so they combine to define a map $\Gamma_\sigma : C_\sigma \to E_\Delta G$. $\square$

For examples derived from lattice gauge fields, see Figures 4 and 7.

Proposition 3.7 (1) There exists a map (necessarily unique) $f_\sigma : \sigma \to B_\Delta G$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & \Gamma_\sigma & & \\ & C_\sigma & \to & E_\Delta G & \\ \pi_\sigma & \downarrow & & \downarrow & \pi_\Delta. \\ & \sigma & \to & B_\Delta G & \\ & & f_\sigma & & \end{array}$$

(2) If τ is a face of σ, then $f_\sigma|_\tau = f_\tau$. Thus the $\{f_\sigma : \sigma \in \Lambda\}$ combine to give a map $f = f(\mathbf{V}) : |\Lambda| \to B_\Delta G$. Moreover f is an embedding.

Proof: Direct check, or see [17]. The fact that f is an embedding follows from the fact that

$$f_\sigma(\sum t_i < i >) = \pi_\Delta(\sum t_i g_i \mathbf{e}_i)$$

for suitable g_i. $\square$

Proposition 3.8 Continuing with the same notation, let $\xi = (\pi : E \to |\Lambda|)$ be the pull-back $f^*(\xi_\Delta G)$, and let $F = F(\mathbf{V})\colon E \to E_\Delta G$ be the bundle map covering f. Then ξ is uniquely determined by $\mathbf{V}$ up to isomorphism. Moreover, since F is an embedding, we may identify E with $F(E) = E_\Delta G|_{f(|\Lambda|)}$. $\square$

We write $\xi = \xi(\mathbf{V})$.

Definition 3.9 Let $\mathbf{V}$ and $\mathbf{V}'$ be G-valued p.t.f.'s on Λ. A *gauge-equivalence* of $\mathbf{V}$ with $\mathbf{V}'$ is a family $\mathbf{g} = \{g_i\}_{<i>\in\Lambda}$ of elements of G such that for every $\sigma =< i_0, \ldots, i_r >$ of dimension ≥ 1 and every $\vec{s} \in c_\sigma$,

$$V'_\sigma(\vec{s}\,) = (g_{i_0})^{-1} \cdot V_\sigma(\vec{s}\,) \cdot g_{i_r}.$$

We write $\mathbf{V}' = \mathrm{Ad}_{\mathbf{g}}(\mathbf{V})$.

Proposition 3.10 Suppose $\mathbf{V}' = \mathrm{Ad}_{\mathbf{g}}(\mathbf{V})$. Let $f, f' : X \to B_\Delta G$ be the classifying maps determined by $\mathbf{V}, \mathbf{V}'$ respectively, as in Prop. 3.7 and let $\xi(\mathbf{V}) = (\pi : E \to X)$, $\xi(\mathbf{V}') = (\pi' : E' \to X)$ be the corresponding bundles, with $X = |\Lambda|$ as usual. Set $\mathbf{h} = \{h_i\}, i = 0, 1, 2, \ldots$ to be the family of G–elements obtained by extending $\mathbf{g}$ by the identity for indices not corresponding to vertices of Λ, so $h_i = g_i$ if $< i > \in \Lambda$, and $h_i = e$ otherwise. Then the right-automorphism $R_{\mathbf{h}}$, restricted to $E = E_\Delta G|_{f(X)}$ gives a bundle isomorphism $\rho_{\mathbf{g}} : E \to E' = E_\Delta G|_{f'(X)}$. In particular, $\xi(\mathbf{V}) \approx \xi(\mathbf{V}')$.

Proof. For each $\sigma \in \Lambda$, let the $\gamma_{\sigma,J}$ and $\Gamma_\sigma : C_\sigma \to E_\Delta G$ be as in Proposition 3.6 for $\mathbf{V}$, and similarly let the $\gamma'_{\sigma,J}$ and Γ'_σ correspond to $\mathbf{V}'$. Write σ as $< 0, \ldots, r >$. By hypothesis, for every i and j with $0 \le i < j \le r$,

$$V'_{\tau(i,j)} = (g_i)^{-1} V_{\tau(i,j)} g_j,$$

which means that

$$\gamma'_\sigma(i,j) = (g_i)^{-1} \cdot \gamma_\sigma(i,j) \cdot g_j$$

where the dots refer to the global left and right actions of G on $E_\Delta G$. The corresponding relation for bar-products of the corresponding $\gamma_\sigma(i,j)$ is then given by

$$\gamma'_{\sigma,J} = (g_0)^{-1} \cdot R_{\mathbf{h}} \gamma_{\sigma,J},$$

where $J = (0 = j_0 < j_1 < \ldots < j_p = r)$. So

$$\pi_\Delta \circ \gamma'_{\sigma,J} = \pi_\Delta \circ R_{\mathbf{h}} \circ \gamma_{\sigma,J} = r_{\mathbf{h}} \circ \pi_\Delta \circ \gamma_{\sigma,J},$$

so $\pi_\Delta \circ \Gamma'_\sigma = r_{\mathbf{h}} \circ \pi_\Delta \circ \Gamma_\sigma$ and thus $f'_\sigma = r_{\mathbf{h}} \circ f_\sigma$, where $f_\sigma, f'_\sigma : \sigma \to B_\Delta G$ are as in Proposition 3.7 (1) for $\mathbf{V}$ and $\mathbf{V}'$ respectively. It follows that $f' = r_{\mathbf{h}} \circ f$, and that the automorphism $R_{\mathbf{h}} : E_\Delta G \to E_\Delta G$ carries $F(E)$ isomorphically onto $F'(E')$. Setting $\rho_{\mathbf{g}} = (F')^{-1} \circ R_{\mathbf{h}} \circ F$ completes the argument. $\square$

3.4 Lattice gauge fields and p.t.f.'s

Definition 3.11 A G-valued *lattice gauge field* (l.g.f.) on Λ is a collection $\mathbf{u} = \{u_{ij}\}$ of group elements, one for each 1–simplex $< ij >$ of Λ, subject to the condition $u_{ji} = u_{ij}^{-1}$. If $\mathbf{V}$ is a p.t.f. on Λ satisfying $u_{ij} = V_{<ij>}(0)$ for each 1–simplex $< ij >$ ($V_{<ij>}(0)$ is the image of $V_{<ij>}$ in this case), we will say $\mathbf{V}$ *is consistent with* $\mathbf{u}$.

Proposition 3.12 Let $\mathbf{u}$ be an $SU(2)$–valued l.g.f. on Λ. Let K be a subcomplex of Λ, and let a p.t.f. $\mathbf{V}$ on K be given which is consistent with $\mathbf{u}|_K$. Assume every simplex of Λ which is not in K has dimension ≤ 4. Then there exists a p.t.f. $\mathbf{V}'$ on Λ which is consistent with $\mathbf{u}$ and is such that $V'_\sigma = V_\sigma$ for every $\sigma \in K$.

Proof. We use induction on $r = \dim \sigma$ to construct $\{V'_\sigma\}$ for $\sigma \notin K$, $\dim \sigma \geq 1$. When $r = 1$, then V'_σ is determined by the consistency requirement. For the general step, suppose $\sigma =< 0, \ldots, r >$ and that V'_τ has been constructed for every proper face τ if σ, so as to satisfy the axioms for a p.t.f. and the requirements of the proposition. By those axioms, V'_σ is determined on ∂c_σ:

$$V'_\sigma = \begin{cases} V'_{<0\ldots\hat{\imath}\ldots r>} & \text{on } \{s_i = 0\}, i = 1, \ldots, r-1 \\ V'_{<0\ldots i>} V'_{<i\ldots r>} & \text{on } \{s_i = 1\}, i = 1, \ldots, r-1. \end{cases}$$

(We omit the verification that these formulae are consistent.) Now $r \leq 4$ implies $\dim(\partial c_\sigma) \leq 2$; and since $SU(2)$ is 2–connected, there is no obstruction to extending V'_σ to a map $V'_\sigma : c_\sigma \to SU(2)$. This completes the inductive step. □

We shall make use of this result in Section 5. Its weakness is that the isomorphism class of the bundle $\xi(\mathbf{V}')$ may depend on the choice of $\mathbf{V}'$, not just on $\mathbf{u}$ and $\mathbf{V}$. To obtain a bundle independent of choices, one must restrict the class of l.g.f.'s considered.

Definition 3.13 $\mathbf{u}$ is *continuous* if, for every 2-simplex $< ijk > \in \Lambda$,

$$d(u_{ij}u_{jk}, u_{ik}) < \pi/n$$

where $n = \dim \Lambda$, and the distance $d(\cdot, \cdot)$ is measured in the unit–sphere metric on $SU(2)$. (This condition is somewhat weaker than the ones used to define "continuous" in [14] and in [15]).

For the computations to follow, we shall need an explicit choice of $\mathbf{V}$, for a given continuous $\mathbf{u}$. This p.t.f. will be denoted $\mathbf{V}(\mathbf{u})$, and called the p.t.f. obtained from $\mathbf{u}$ by *iterated coning*.

Proposition 3.14 Let $\mathbf{u}$ be a continuous l.g.f. on Λ. Set

$$\delta(\mathbf{u}) = \max_{<ijk>\in\Lambda} \{d(u_{ij}u_{jk}, u_{ik})\}.$$

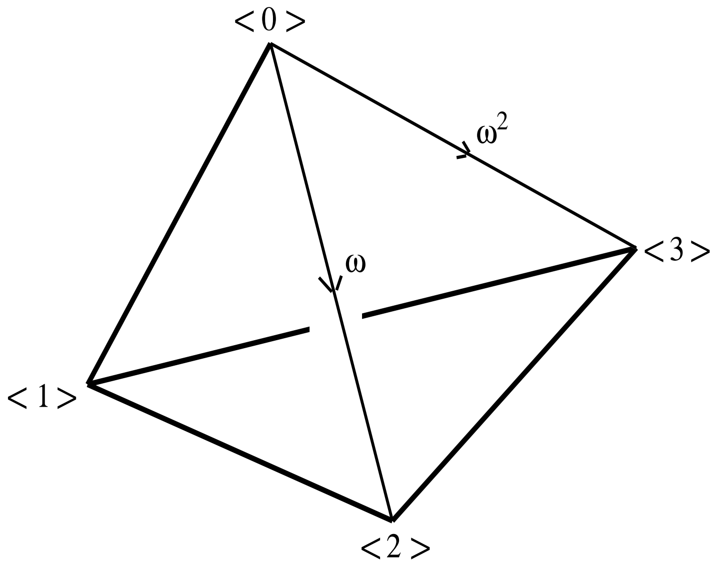

Figure 3: A $U(1)$-valued l.g.f. on the surface of a tetrahedron. ω and ω^2 are complex cube roots of 1. Unmarked bonds carry the transporter 1.

(1) The iterated coning procedure yields a p.t.f. $\mathbf{V}$ on Λ such that:
(i) $\mathbf{V}$ is consistent with $\mathbf{u}$
(ii) Whenever $\sigma = < i \dots j >$ has dimension $r \geq 2$, then

$$d(V_\sigma(\vec{s}), u_{ij}) \leq \delta(\mathbf{u})(r-1),$$

for every $\vec{s} \in c_\sigma^{r-1}$.
(2) Moreover if $\mathbf{V}'$ is another p.t.f. on Λ for which (i) and (ii) hold, then the bundles $\xi(\mathbf{V})$ and $\xi(\mathbf{V}')$ are isomorphic.

We write $\xi(\mathbf{u})$ for any $\xi(\mathbf{V})$ as in the proposition.

Figures 3, 4, 5 show how this procedure works in the more easily illustrated case $G = U(1)$. In these figures Λ is the surface of a tetrahedron $< 0123 >$, and $\mathbf{u}$ is made up of the third roots of unity $1, \omega, \omega^2$. Figure 6 gives a topological proof that the induced bundle is non-trivial: collapsing the face $< 123 >$ to a single vertex $< \infty >$ allows the use of $S^3 \to S^2$ as the universal bundle; the resulting map of Λ to S^2 is patently essential. It is instructive to repeat this exercise with the trivial l.g.f. $u_{ij} \equiv 1$. In that case the image of f is a single meridian of S^2.

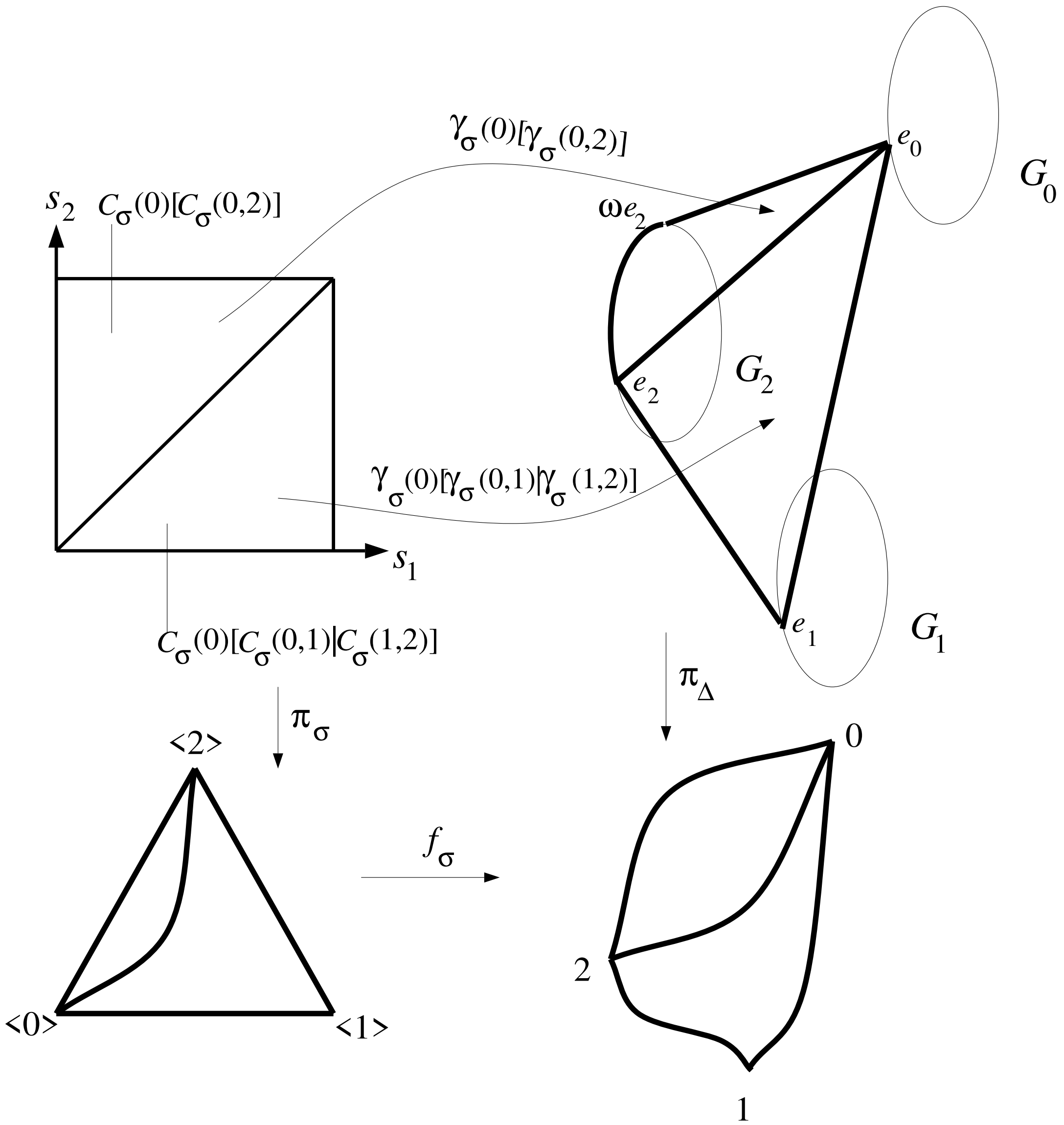

Figure 4: The maps Γ_σ and f_σ for $\sigma = <012>$.

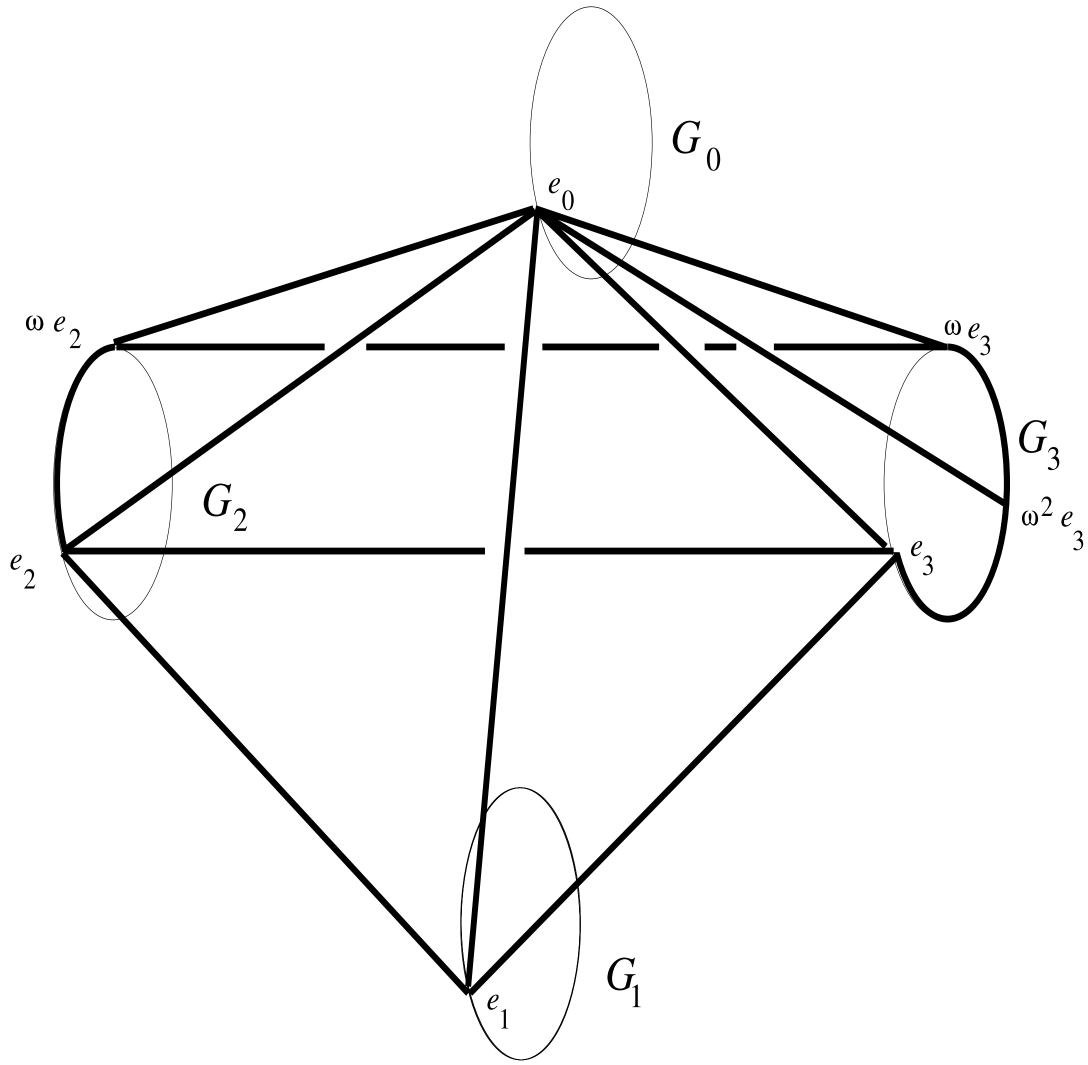

Figure 5: The four Γ_σ's; the horizontal lines in the rear of the figure are distinct, but have the same projection in $B_\triangle G$.

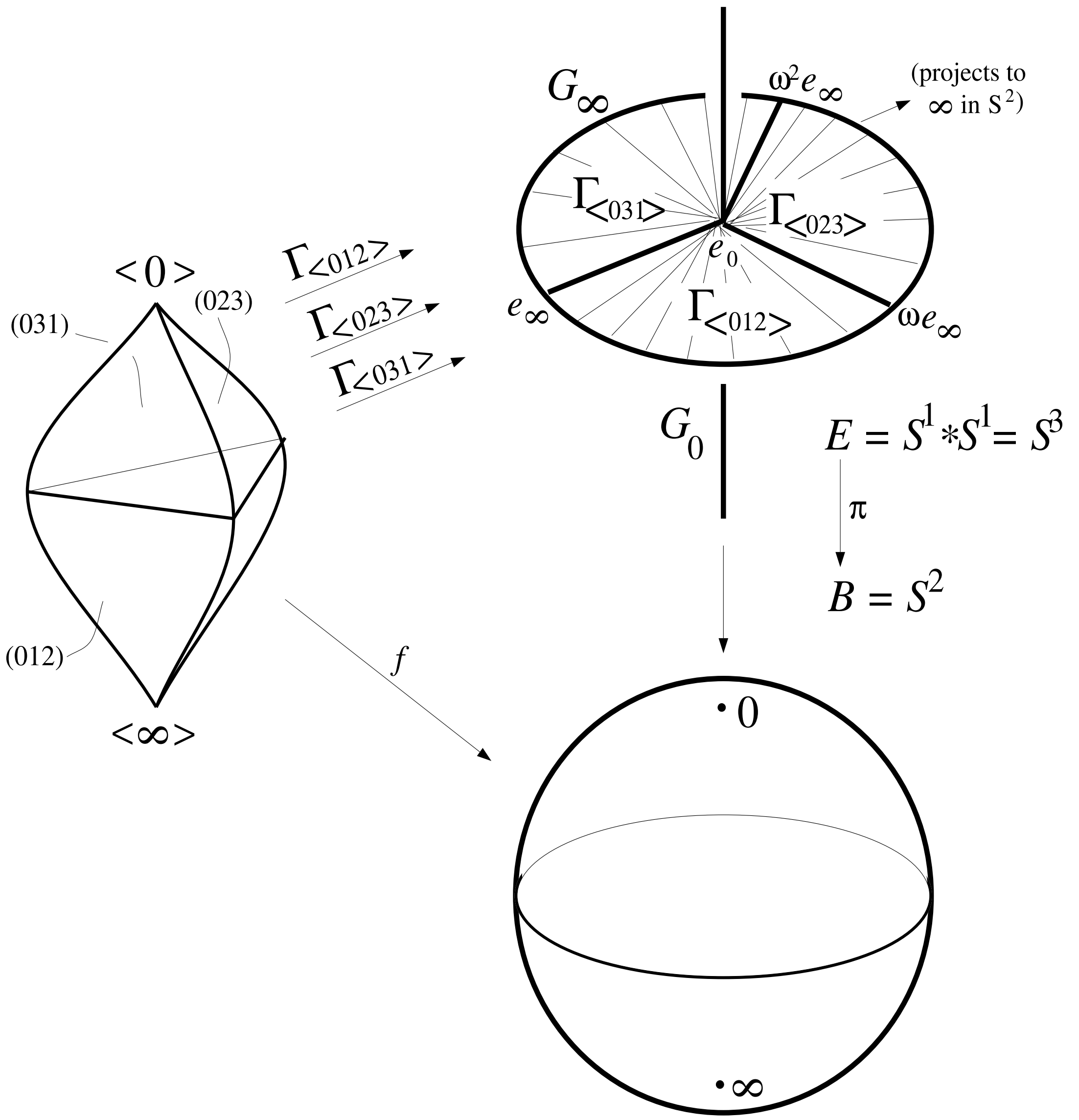

Figure 6: Collapse $< 123 >$ to a single vertex $< \infty >$. The Γ's now map to $G_0 * G_\infty = S^3 \subset E_\Delta G$. The induced $f\colon |\Lambda| \to S^2 = S^3/U(1)$ covers the sphere exactly once.

Proof of (1). V_σ is constructed by induction on $r = \dim \sigma$. For $r = 1$, V_σ is specified by (i). Now assume that $r \geq 2$ and that V_τ has been constructed for every proper subface τ of σ so as to satisfy (ii) and the axioms for a p.t.f. Say $\sigma =< 0 \dots r >$. With $\partial^+ c_\sigma$ defined as in Section 3.1, we can regard c_σ as the cone from $\vec{0}$ on $\partial^+ c_\sigma$, i.e. every $\vec{s} \in c_\sigma$ can be written in the form $\vec{s} = t\vec{s'}$ for some $t \in [0,1]$ and $\vec{s} \in \partial^+ c_\sigma$; and this expression is unique unless $\vec{s} = \vec{0}$ (in which case $t = 0$ and $\vec{s'}$ is arbitrary). Now $V_\sigma | \partial^+ c_\sigma$ is uniquely determined on $\{s_i = 1\}$ by the requirement $V_\sigma = V_{<0\dots i>} V_{<i\dots r>}$; moreover the p.t.f. axioms ensure that these requirements are consistent. We next check that (ii) holds for $\vec{s} \in \partial^+ c_\sigma$. Suppose $\vec{s}$ belongs to the face $\{s_i = 1\}$; then, writing $V_{<0\dots i>}(s_1, \dots, s_{i-1}) = V_1$ and $V_{<i\dots r>}(s_{i+1}, \dots, s_{r-1}) = V_2$,

$$\begin{aligned} d(V_\sigma(\vec{s}), u_{0r}) &= d(V_1 V_2, u_{0r}) \\ &\leq d(V_1 V_2, u_{0i} V_2) + d(u_{0i} V_2, u_{0i} u_{ir}) + d(u_{0i} u_{ir}, u_{0r}) \\ &= d(V_1, u_{0i}) + d(V_2, u_{ir}) + d(u_{0i} u_{ir}, u_{0r}) \\ &\leq \delta(\mathbf{u})(i-1) + \delta(\mathbf{u})(r-i-1) + \delta(\mathbf{u}) \\ &= \delta(\mathbf{u})(r-1), \end{aligned}$$

as required. (We have used the fact that V_2 and u_{0i} are unitary, and the definition of $\delta(\mathbf{u})$.)

Since $\mathbf{u}$ is continuous, $\delta(\mathbf{u}) < \pi/r$. Hence $V_\sigma(\partial^+ c_\sigma)$ lies in the ball of radius π about u_{0r} in $SU(2)$. Therefore there exists, for each $\vec{s'} \in \partial^+ c_\sigma$, a unique shortest geodesic in $SU(2)$, $\alpha_{\vec{s'}} : [0,1] \to S^3$, such that $\alpha_{\vec{s'}}(0) = u_{0r}$ and $\alpha_{\vec{s'}}(1) = V_\sigma(\vec{s'})$. Define $V_\sigma : c_\sigma \to SU(2)$ by $V_\sigma(0) = u_{0r}$ and $V_\sigma(\vec{s}) = \alpha_{\vec{s'}}(t)$, where $\vec{s} = t\vec{s'}$ with $\vec{s'} \in \partial^+ c_\sigma$. Then (ii) holds on all of c_σ. We omit the last verification: that $V_\sigma = V_{<0\dots\hat{\imath}\dots r>}$ on the face $\{s_i = 0\}$ of c_σ, for $i = 1, \dots, r$.

Proof of (2). We shall construct $\mathbf{W} = \{W_\sigma : c_\sigma \times I \to SU(2), \dim \sigma \geq 2\}$ in such a way that, setting $W_\sigma^t = W_\sigma|_{c_\sigma \times \{t\}}$,

- $W_\sigma^0 = V_\sigma$; $W_\sigma^1 = V'_\sigma$
- $\mathbf{W}^t = \{W_\sigma^t, \dim \sigma \geq 2\}$ is a p.t.f. on Λ which satisfies (i) and (ii) of (1).

Then for each t we get a principal $SU(2)$-bundle $\xi^t = \xi(\mathbf{W}^t)$ over $|\Lambda|$. It follows from the construction of the classifying map $f^t = f(\mathbf{W}^t) : |\Lambda| \to B_\Delta SU(2)$ that f^t varies continuously in t. Hence the ξ^t are all isomorphic bundles; in particular $\xi = \xi'$.

We construct W_σ by induction on $r = \dim \sigma$.

For $r = 1$, W_σ^t is determined by (i).

Now assume that $r \geq 2$ and that W_τ has been constructed for every proper face of σ so as to satisfy (ii) and the axioms of a p.t.f. Say $\sigma =< 0, \dots, r >$. The induction hypothesis determines W_σ on $\partial(c_\sigma \times I)$ in the following way:

(a) on $c_\sigma \times 0$ and $c_\sigma \times 1$ by V_σ and V'_σ, respectively;

(b) on $\{s_i = 0\}$, for $i = 1, \dots, r-1$ by $W_{\tau(i)}$, where $\tau(i) =< 0, \dots, \hat{i}, \dots, r >$;

(c) on $\{s_i = 1\}$, for $i = 1, \dots, r-1$ by $W_{<0,\dots,i>} \cdot W_{<i,\dots,r>}$.

It follows that $W_\sigma(\partial(c_\sigma \times I))$ is contained in the ball $B(u_{0r}, \delta(\mathbf{u})(r-1))$ about u_{0r} of radius $\delta(\mathbf{u})(r-1)$ in $SU(2)$: in case (a) this is the hypothesis of (2); in (b) it follows from the induction hypothesis and (ii); and in case (c) by the argument used in the proof of (1).

Since $\delta(\mathbf{u}) < \pi/n$ by the continuity of $\mathbf{u}$, it follows that $\delta(\mathbf{u})(r-1) < \pi$; so $B(u_{0r}, \delta(\mathbf{u})(r-1))$ is contractible in $SU(2)$, and the required extension W_σ thus exists.$\Box$

Remark 3.15 This algorithm can be performed, even if $\mathbf{u}$ is not continuous, on simplexes of dimension ≤ 4, provided $\mathbf{u}$ is "generic." (No uniqueness guarantees, however.)

Definition 3.16 Let $\mathbf{u}$ and $\mathbf{u}'$ be l.g.f.'s on Λ. They are called *gauge equivalent* if there exists $\mathbf{g} = \{g_i \in G\}_{<i>\in\Lambda}$ such that for every $< ij >\in \Lambda$, $u'_{ij} = g_i^{-1} u_{ij} g_j$. We write $\mathbf{u}' = \mathrm{Ad}_{\mathbf{g}}(\mathbf{u})$.

Proposition 3.17 Let $\mathbf{u}' = \mathrm{Ad}_{\mathbf{g}}(\mathbf{u})$. Then:
(1) If $\mathbf{u}$ is continuous, then so is $\mathbf{u}'$.
(2) The p.t.f.'s $\mathbf{V}(\mathbf{u})$ and $\mathbf{V}(\mathbf{u}')$ constructed by iterated coning are related by

$$\mathbf{V}(\mathbf{u}') = \mathrm{Ad}_{\mathbf{g}}(\mathbf{V}(\mathbf{u})).$$

In particular, $\xi(\mathbf{u}') \approx \xi(\mathbf{u})$. $\Box$

4 Dupont's canonical connection in Milnor's universal bundle

4.1 Dupont's connection

If G is a Lie group, the total space $E_\Delta G$ of Milnor's universal bundle is a stratified space. The strata can be labelled by sets of indices:

$$\Sigma_{i_0,\dots,i_m} = \{\sum t_i g_i \mathbf{e}_i | \begin{array}{l} t_i = 0 \text{ if } i \neq i_0,\dots,i_m \\ t_i \neq 0 \text{ if } i = i_0,\dots,i_m \end{array} \}.$$

These strata are invariant under the global left and right–actions of G.

Definition 4.1 The *canonical connection* ω_Δ on $E_\Delta G$ is defined as follows:

Let $x \in TE_\Delta G$ be a vector tangent to one of the strata, say $\Sigma_{i_0,\dots,i_m}$; i.e. let

$$x(s) = \sum_{j=0}^{m} t_{i_j}(s) g_{i_j}(s) \mathbf{e}_{i_j}$$

be a smooth curve in $\Sigma_{i_0,\dots,i_m}$, defined on some open interval containing 0, such that $x'(0) = x$.

Then for each j, $g_{i_j}(s) \cdot g_{i_j}^{-1}(0)$ is a curve through e in G. So

$$\omega_{i_j}(x) = \frac{d}{ds}[g_{i_j}(s) \cdot g_{i_j}^{-1}(0)]|_{s=0}$$

is an element of $TG|_e$, i.e. of the Lie Algebra $\mathfrak{g}$. Now set

$$\omega_\Delta(x) = \sum t_{i_j}(0) \omega_{i_j}(x).$$

Thus we may write $\omega_\Delta = \sum t_{i_j} dg_{i_j} g_{i_j}^{-1} = \sum t_{i_j} R_{g_{i_j}^{-1} *} dg_{i_j}$, using $R_g: G \to G$ for right-multiplication by g as usual. We leave the reader to check that the constructions of $\omega_{i_j}(x)$ and $\omega_\Delta(x)$ are independent of the choice of curve $x(s)$ satisfying (1) and (2). This connection is due to Dupont [4] who gives a general construction of connections in semi-simplicial bundles and specializes it to the Milgram model of the universal bundle. We will briefly develop its properties as they are manifested in $E_\Delta G$.

The next lemma shows that ω_Δ does indeed have the characteristic properties of a connection.

Proposition 4.2 ω_Δ satisfies the following conditions.

(1) Let $\mathsf{k} \in \mathfrak{g}$, and let $p \to \mathsf{k}_p$ be the corresponding fundamental vectorfield (see below) on $E_\Delta G$. Then, for any $p \in E_\Delta G$,

$$\omega_\Delta(\mathsf{k}_p) = \mathsf{k}.$$

(2) Let $g \in G$, and let $L_g : E_\Delta G \to E_\Delta G$ represent left-multiplication by g. Consider a vector $\mathsf{x} \in T_p E_\Delta G$. Then

$$\omega_\Delta(L_{g*}\mathsf{x}) = \mathrm{Ad}_g(\omega_\Delta(\mathsf{x})).$$

Proof: (1) Take k and $p \to \mathsf{k}_p$ as in the statement. Thus if $k(s)$ is a curve in G with $k(0) = e$ and $k'(0) = \mathsf{k}$ (for example, $k(s) = \exp(s\mathsf{k})$), then at $p = \sum t_i g_i \mathbf{e}_i \in E_\Delta G$ the vector k_p is given by

$$\mathsf{k}_p = \frac{d(k(s) \cdot p)}{ds}|_{s=0} = \frac{d \sum t_i k(s) g_i \mathbf{e}_i}{ds}|_{s=0},$$

the dot referring to the global left-action of G on $E_\Delta G$. It follows that

$$\omega_j(\mathsf{k}_p) = \frac{d}{ds}[k(s)g_i \cdot g_i^{-1}] = \mathsf{k}$$

$$\omega_\Delta(\mathsf{k}_p) = \sum_j t_j \mathsf{k} = \mathsf{k},$$

as required.

(2) To study $L_g^* \omega_\Delta$ we examine its value on a test vector $\mathsf{v} \in TE_\Delta G|_p$, say $\mathsf{v} = c'(0)$ where $c(s) = \sum t_i(s) g_i(s) \mathbf{e}_i$ is a curve in $E_\Delta G$ with $c(0) = p$. Then $L_{g*}(\mathsf{v})$ is the tangent vector to the curve $L_g \circ c(s) = \sum t_i(s) g g_i(s) \mathbf{e}_i$. Now

$$\omega_\Delta(\mathsf{v}) = \sum t_i(0) R^{-1}_{g_i(0)*} \frac{dg_i(s)}{ds}|_{s=0}$$

where $\frac{dg_i(s)}{ds}|_0 \in TG|_{g_i(0)}$ and the linear combination takes place in $TG|_e = \mathfrak{g}$. So

$$\omega_\Delta(L_{g*}(\mathsf{v})) = \sum t_i(0) R^{-1}_{gg_i(0)*} \frac{dgg_i(s)}{ds}|_{s=0} = \sum t_i(0) R^{-1}_{g*} R^{-1}_{g_i(0)*} L_{g*} \frac{dg_i(s)}{ds}|_{s=0}.$$

Since $R^{-1}_{g_i(0)}$ and L_g commute, this is

$$R^{-1}_{g*} L_{g*} \omega_\Delta(\mathsf{v}) = \mathrm{Ad}_g \omega_\Delta(\mathsf{v}),$$

as required. □

For $\mathbf{h} = \{h_0, h_1, \ldots\}, h_i \in G$, let $R_\mathbf{h} : E_\Delta G \to E_\Delta G$ be the right automorphism given in Paragraph 2.1.

Proposition 4.3 The automorphism $R_{\mathbf{h}}$ preserves ω_Δ.

Proof: The identity $R_{\mathbf{h}}^*\omega_\Delta = \omega_\Delta$ is established in the same way as (2) of Proposition 4.2.□

4.2 Its relation to the classical standard connection when $G = SU(2)$

Dupont shows how the curvature and Chern-Simons forms may be defined for connections in general semi-simplicial bundles. In the interest of explicitness, we will show how, when $G = SU(2)$ = the group of unit quaternions, and $\mathfrak{g}$ = the vector space of pure imaginary quaternions, these forms may also be defined directly, in terms of the standard, classical representatives.

The standard, classical model of the universal $SU(2)$–bundle is $(E \to B) = \lim_k (E_k \to B_k)$ where $E_k = S^{4k+3}$, the unit sphere in quaternionic $(k+1)$–space, B_k is quaternionic projective k–space, and the quotient is by the action of $SU(2)$ as the group of unit quaternions.

There is a standard connection in this bundle. We will define it for a fixed k but it clearly is compatible with the inclusions $E_k \subset E_{k+1} \subset \cdots$ and so extends to the limit. Consider $S^{4k+3} \subset R^{4(k+1)} = H^{k+1}$, so a point in S^{4k+3} can be labelled by its quaternionic coordinates $p_0, \ldots, p_k$, where $p_i = x_i + y_i\mathbf{i} + z_i\mathbf{j} + w_i\mathbf{k}$ with $\sum_0^k (x_i^2 + y_i^2 + z_i^2 + w_i^2) = 1$. Similarly a tangent vector to S^{4k+3} at $(p_0, \ldots, p_k)$ is given by a quaternionic k–tple $(q_0, \ldots, q_k)$ where $q_i = \kappa_i + \lambda_i\mathbf{i} + \mu_i\mathbf{j} + \nu_i\mathbf{k}$ with

$$(*) \quad \sum_0^k \kappa_i x_i + \lambda_i y_i + \mu_i z_i + \nu_i w_i = 0.$$

With this notation, the standard connection form ω assigns to the vector $(q_0, \ldots, q_k)$ at $(p_0, \ldots, p_k)$ the pure imaginary quaternion

$$\omega_{(p_0,\ldots,p_k)}(q_0, \ldots, q_k) = \Im \sum q_i \overline{p}_i.$$

Proposition 4.4 The homeomorphism $\Phi : E_\Delta G \to E$ defined by

$$\Phi(t_0 g_0 + t_1 g_1 + \cdots) = (\sqrt{t}_0 g_0, \sqrt{t}_1 g_1, \ldots)$$

is smooth on strata and, restricted to any stratum, pulls back the standard connection ω to the canonical connection ω_Δ .

Proof: Smoothness on strata is clear. Suppose for simplicity we are in the top–dimensional stratum of the $(k+1)$–fold join, at the point $t_0 g_0 + \cdots + t_k g_k$. The tangent space to the stratum at that point is spanned by $\partial/\partial t_0, \partial/\partial t_1, \ldots, \partial/\partial t_k$ (note that $\sum \partial/\partial t_i = 0$) and by the tangent spaces to $G_0, G_1, \ldots, G_k$, with total dimension $k + 3(k+1) = 4k+3$.

Consider first a vector in the span of $\partial/\partial t_0, \partial/\partial t_1, \ldots, \partial/\partial t_k$, say $\mathbf{v} = \sum_0^k a_i \partial/\partial t_i$, with $\sum a_i = 0$. This vector is automatically in the kernel of ω_Δ, since ω_Δ does not involve any of the dt_i's. To calculate $\Phi_*(\mathbf{v})$, consider the curve $c(s) = (t_0 + a_0 s)g_0 + \cdots (t_k + a_k s)g_k$ which has $c'(0) = \mathbf{v}$. Substituting we find $\Phi \circ c(s) = (\sqrt{t_0 + a_0 s} g_0, \ldots, \sqrt{t_k + a_k s} g_k) \in S^{4k+3}$, and $\Phi_*(\mathbf{v}) = (\Phi \circ c)'(0) = \frac{a_0}{2\sqrt{t_0}} g_0 + \cdots + \frac{a_k}{2\sqrt{t_k}} g_k$. Now this vector is at $(\sqrt{t_0} g_0, \sqrt{t_1} g_1, \ldots)$, a point with conjugate coordinates $(\sqrt{t_0} g_0^{-1}, \ldots, \sqrt{t_k} g_k^{-1})$. Applying ω gives $(1/2)(a_0 + \cdots + a_k) = 0$.

Now suppose the test vector is in the complement of the span of the $\partial/\partial t_i$'s. This would be the tangent vector to a curve $c(s) = t_0 g_0(s) + \cdots + t_k g_k(s)$, and $\omega_\Delta(c'(0))$ would be

$$(**) \quad t_0 g_0'(0) g_0(0)^{-1} + \cdots + t_k g_k'(0) g_k(0)^{-1}.$$

Proceeding as above, we calculate $\Phi \circ c(s) = (\sqrt{t_0} g_0(s), \ldots, \sqrt{t_k} g_k(s)) \in S^{4k+3}$ and $\Phi_*(c'(0)) = (\Phi \circ c)'(0) = \sqrt{t_0} g_0'(0) + \cdots + \sqrt{t_k} g_k'(0)$. Now the calculation of $\omega(\Phi_*(c'(0)))$ gives $\sum_i \sqrt{t_i} g_i'(0) \sqrt{t_i} g_i^{-1}(0)$, which is exactly(**).$\square$

The standard connection has curvature form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, with Chern form $\tilde{p}$ and Chern-Simons $\tilde{\chi}$ form defined as in Section 1.1. (In quaternionic notation, we have $\tilde{\chi} = \frac{1}{4\pi^2}\mathrm{Re}(\omega \wedge \Omega - \frac{1}{3}\omega \wedge \omega \wedge \omega)$.) These last two are related by $\tilde{p} = d\tilde{\chi}$.

On any stratum of $E_\Delta G$ we can use Proposition 4.4 to define $\Omega_\Delta, \tilde{p}_\Delta$ and $\tilde{\chi}_\Delta$ as Φ^* of the corresponding standard forms.

Theorem 4.5 $\tilde{\chi}_\Delta$ is invariant under the automorphisms $R_{\mathbf{h}}$ of $\xi_\Delta G$.

Proof: this follows directly from the invariance of ω_Δ.$\square$

4.3 The connection and Chern-Simons forms of a p.t.f.

Let $\mathbf{V}$ be a G-valued p.t.f. on an ordered simplicial complex Λ with underlying space $|\Lambda| = X$. We assume that each V_σ is piecewise smooth. Following Prop. 3.7 and Prop. 3.8 we construct $\xi = \xi(\mathbf{V})$, the principal bundle corresponding to $\mathbf{V}$; say $\xi = (\pi : E \to X)$, with classifying map $f = f(\mathbf{V}) : X \to B_\Delta G$; let $F : E \to E_\Delta G$ be the corresponding bundle map.

Set

$$\omega = \omega_{\mathbf{V}} = F^*\omega_\Delta,$$
$$\tilde{\chi} = \tilde{\chi}_{\mathbf{V}} = F^*\tilde{\chi}_\Delta.$$

Proposition 4.6 $\omega_{\mathbf{V}}$ is a piecewise smooth, $\mathfrak{g}$-valued 1-form on E which has the two characteristic properties of a connection (see Proposition 4.2). □

Definition 4.7 $\tilde{\chi}_{\mathbf{V}}$ is the *Chern-Simons form* of $\mathbf{V}$.

When $\mathbf{V}$ is constructed from a lattice gauge field $\mathbf{u}$ as in Proposition 3.14, we will write these forms as $\omega_{\mathbf{u}}$ and $\tilde{\chi}_{\mathbf{u}}$.

4.4 The relation between $\omega_{\mathbf{u}}$ and $\mathbf{u}$.

The construction of $\omega_{\mathbf{u}}$ from $\mathbf{u}$ proceeds in a roundabout way: $\mathbf{u}$ is used to construct a bundle $\xi_{\mathbf{u}}$ (*which we will think of here as a subbundle of* $\xi_\Delta G$, see Proposition 4.4), and $\omega_{\mathbf{u}}$ is taken to be the restriction to $\xi_{\mathbf{u}}$ of the canonical connection ω_Δ. Nevertheless the two are very closely related.

Lemma 4.8 Any join-line in $E_\Delta G = G_0 * G_1 * \cdots$, i.e. any line of the type $t \mapsto (1-t)x_i + tx_j$, where $x_i \in G_i$ and $x_j \in G_j$, is horizontal with respect to ω_Δ.

Proof: Immediate since $\omega_\Delta = \sum t_i dg_i \cdot g_i^{-1}$ does not involve the differentials dt_i of the join variables.□

Proposition 4.9 Parallel transport by $\omega_{\mathbf{u}}$ around the edges of a plaquette $< 012 >$ of Λ takes an element x in the fiber over vertex $< 2 >$ back to xu_{2012}.

Proof: We identify $< 012 >= \sigma$ with its image in $B_\Delta G$ under the classifying map constructed from $\mathbf{u}$ (compare Figures 4 and 5). That image is covered by $\Gamma_\sigma\colon C_\sigma \to E_\Delta G$, and in particular the boundary of σ is covered by three of the four edges of Γ_σ: the path $< 2 > \to < 0 > \to < 1 > \to < 2 >$ in $\partial\sigma$ is covered by the path $\Gamma_\sigma(s_1 = 0, s_2), 1 \geq s_2 \geq 0$ followed by $\Gamma_\sigma(s_1, s_2 = 0), 0 \leq s_1 \leq 1$ followed by $\Gamma_\sigma(s_1 = 1, s_2), 0 \leq s_2 \leq 1$. These paths are, respectively, $(1-s_2)\mathbf{e}_0 + s_2 u_{02}\mathbf{e}_2$ leading from $u_{02}\mathbf{e}_2$ (when $s_2 = 1$) to $\mathbf{e}_0$, $(1-s_1)\mathbf{e}_0 + s_1 u_{01}\mathbf{e}_1$ leading from there to $u_{01}\mathbf{e}_1$, and $(1-s_2)u_{01}\mathbf{e}_1 + s_2 u_{012}\mathbf{e}_2$, ending at $u_{012}\mathbf{e}_2$. By Lemma 4.8 to ω_Δ, and so $u_{012}\mathbf{e}_2$ is the result of parallel transport of $u_{02}\mathbf{e}_2$ around $\partial\sigma$. The corresponding holonomy element is $u_{02}^{-1}u_{012}$, which is precisely the "plaquette" (2–simplex) product u_{2012}. By left-invariance of parallel transport any $x \in G_2$ will end up at $x \cdot u_{2012}$. $\square$

This means that $\mathbf{u}$ essentially determines all the curvature of $\omega_{\mathbf{u}}$: any nullhomotopic loop in Λ, when deformed into the 1–skeleton, can be written (by a standard argument) as a product of paths of the form $c \cdot \partial\sigma \cdot c^{-1}$; so the holonomy around that loop is a product of plaquette products and images of plaquette products under conjugation in $SU(2)$.

5 Construction of the pseudosection ψ

On an ordered simplicial complex Λ, with underlying space $X = |\Lambda|$, let $\mathbf{V}$ be a parallel transport function with values in $G = SU(2)$, and $f : X \to B_\Delta G$ the classifying map constructed from $\mathbf{V}$ in Proposition 3.7. the standard explicit form of the bundle $f^*\xi_\Delta G$: it has total space $E = \{(e, x) \in E_\Delta G \times X : \pi_\Delta e = f(x)\}$; the projection $\pi : E \to X$ and the bundle map $F : E \to E_\Delta G$ covering f are given by $\pi(e, x) = x$ and $F(e, x) = e$.

The Chern-Simons form of $\mathbf{V}$ on E is $\tilde{\chi} = F^*\tilde{\chi}_\Delta$. To calculate the value of the Chern-Simons character on a 3-cycle $Z \subset \Lambda$ we will construct from $\mathbf{V}$ in E over Z a pseudo-section on which $\tilde{\chi}$ can be integrated.

This construction has several steps, and can be summarized in this diagram:

$$\begin{tikzcd}
\widehat{E} \arrow[r, "P"] \arrow[d] & E \arrow[r, "F"] \arrow[d] & E_\Delta G \arrow[d, "\pi_\Delta"] \\
\widehat{Z} \arrow[r, "p"'] \arrow[ur, "\psi"] \arrow[urr, "\Psi"] \arrow[u, bend left=60, "\widehat{\psi}"] & Z \subset X \arrow[r, "f"'] & B_\Delta G
\end{tikzcd}$$

5.1 A partial cone on Λ.

Suppose for simplicity that Λ is a complex of dimension 3 or 4. Let Λ^α represent the union of Λ with the cone on the 3-skeleton $\Lambda^{(3)}$ from a new vertex α. For $\sigma = < 0, \dots, r > \in \Lambda^{(3)}$, set $\sigma^\alpha = < \alpha, 0, \dots, r >$, so $\Lambda^\alpha = \Lambda \cup \bigcup_{\sigma \in \Lambda^{(3)}} \sigma^\alpha$. Order the vertices of Λ^α by ranking α below all the vertices of Λ, and let $\mathbf{u}^\alpha$ be the lattice gauge field on Λ^α defined by

$$\mathbf{u}^\alpha_{ij} = V_{<ij>} \in G \quad \text{for } < i, j > \in \Lambda$$

$$\mathbf{u}^\alpha_{\alpha i} = \text{ identity} \quad \text{for } < i > \in \Lambda.$$

Following Proposition 3.12, $\mathbf{V}$ can be extended to a p.t.f. $\mathbf{V}^\alpha$ defined on Λ^α and consistent with $\mathbf{u}^\alpha$. For any simplex $\lambda \in \Lambda^\alpha$ let $\Gamma_\lambda : C_\lambda \to E_\Delta G$ be defined from $\mathbf{V}^\alpha$ as in Proposition 3.6.

5.2 A cycle $\widehat{Z}$ covering Z.

Set

$$\widehat{Z} = \bigcup_{\sigma \in Z} \partial^+ C_{\sigma^\alpha},$$

using the notation of Section 3.1.

Proposition 5.1

$$\partial\partial^+ C_{\sigma^\alpha} = \sum (-1)^i \partial^+ C_{\sigma_i^\alpha},$$

where σ_i is the i-th face of σ.□

It follows that $\widehat{Z}$ is again a cycle. Moreover, $\widehat{Z}$ admits a projection map

$$p : \widehat{Z} \to Z$$

defined by

$$p|_{\partial^+ C_{\sigma^\alpha}} = \pi_{\sigma^\alpha}|_{\partial^+ C_{\sigma^\alpha}},$$

where $\pi_{\sigma^\alpha} : C_{\sigma^\alpha} \to \sigma^\alpha$ is the standard projection of Section 3.1. This map is clearly compatible with the the face relation among simplexes of Z, and therefore extends to all of $\widehat{Z}$ as a well-defined, continuous map. To see that p actually maps $\partial^+ C_{\sigma^\alpha}$ to $\sigma \subset \sigma^\alpha$ note that for $I = \{\alpha, 0, \ldots, r\}$ the cell

$$C_{\sigma^\alpha, I} = C_{\sigma^\alpha}(\alpha)[C_{\sigma^\alpha}(\alpha, 0)|C_{\sigma^\alpha}(0,1)|\cdots|C_{\sigma^\alpha}(r-1, r)] \subset C_{\sigma^\alpha}$$

contributes $C_{\sigma^\alpha}(\alpha, 0)[C_{\sigma^\alpha}(0,1)|\cdots|C_{\sigma^\alpha}(r-1,r)]$ to $\partial^+ C_{\sigma^\alpha}$; this face is an r-simplex which is mapped by p onto σ. The other faces of $\partial^+ C_{\sigma^\alpha}$ are collapsed onto various faces of σ.

5.3 A section over $\widehat{Z}$ ...

Use $f \circ p$ to pull back $\xi_\Delta G$ to a bundle $\widehat{\xi}$ over $\widehat{Z}$. The total space is $\widehat{E} = \{(e, y) \in E_\Delta G \times \widehat{Z} : \pi_\Delta e = f(p(y))\}$. Define a map $\Psi = \Psi(\mathbf{V}) : \widehat{Z} \to E_\Delta G$ by

$$\Psi|\partial^+ C_{\sigma^\alpha} = \Gamma_{\sigma^\alpha}|\partial^+ C_{\sigma^\alpha}.$$

Then it is straightforward to check that $\pi_\Delta \circ \Psi = f \circ p$ and that therefore $y \mapsto (\Psi(y), y) \in \widehat{E}$ defines a section $\widehat{\psi}$ in $\widehat{\xi}$.

5.4 ... becomes a pseudosection over Z.

Let $P : \widehat{E} \to E$ be given by $P((e, y)) = (e, p(y))$. The map $\psi = \psi(\mathbf{V})$ given by

$$\psi = P \circ \widehat{\psi} : \widehat{Z} \to E$$

is what we will call *the pseudosection associated to* $\mathbf{V}$. Since $\pi(\psi(y)) = p(y)$, it follows that ψ of the face $C_{\sigma^\alpha}(\alpha, 0)[C_{\sigma^\alpha}(0,1)|\cdots|C_{\sigma^\alpha}(r-1,r)] \subset \partial^+ C_{\sigma^\alpha}$ lies "over" σ. This justifies the name. See Figure 7, which illustrates the

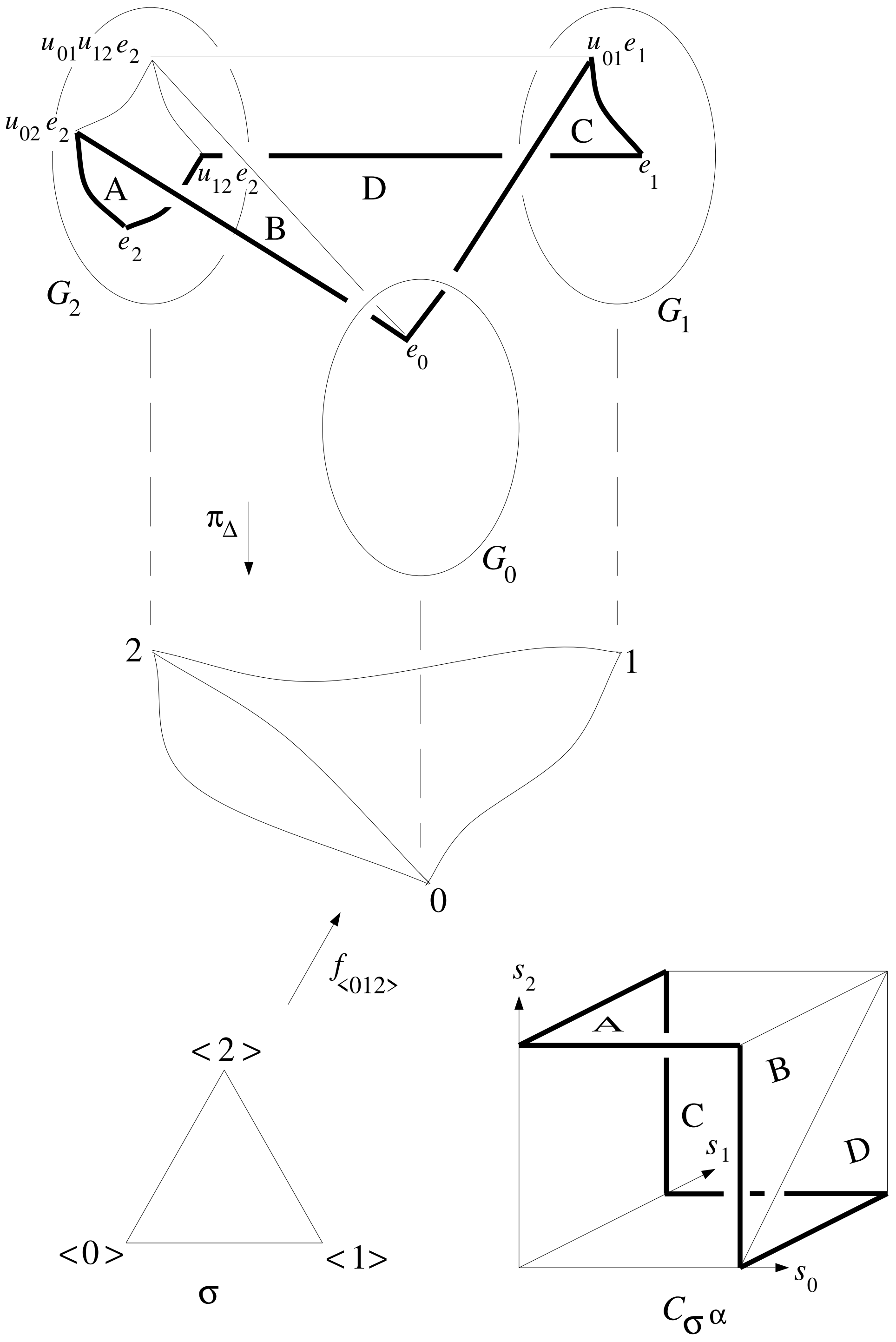

Figure 7: The pseudosection ψ constructed from a l.g.f. $\mathbf{u}$, shown for a 2-simplex $< 012 >$. Note that parts B and D make up the map $\Gamma_{<012>}$.

construction over a 2–simplex $\sigma =< 012 >$. In this figure, the cells marked $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ make up $\partial^+ C_{\sigma^\alpha}$. The top of the figure represents their image, under Ψ, in $E_\Delta G$.

Definition 5.2 For a 3–cycle $Z \subset \Lambda$, the value of the Chern-Simons character on Z is defined to be

$$\chi(Z) = \sum_{\sigma \in Z} \hat{\chi}_{\mathbf{u}}(\sigma) = \sum_{\sigma \in Z} \int_{\psi(\partial^+ C_{\sigma^\alpha})} F^* \tilde{\chi}_\Delta = \sum_{\sigma \in Z} \int_{\Gamma_{\sigma^\alpha}(\partial^+ C_{\sigma^\alpha})} \tilde{\chi}_\Delta,$$

reduced mod $\mathbf{Z}$.

Remark 5.3 In the case of a simplex of dimension 3, say $\sigma =< 0, 1, 2, 3 >$, then $\sigma^\alpha =< \alpha, 0, 1, 2, 3 >$, and

$$\begin{aligned} \Gamma_{\sigma^\alpha}(\partial^+ C_{\sigma^\alpha}) \quad = \quad & \gamma_{\sigma^\alpha}(\alpha, 0)[\gamma_\sigma(0,1) \mid \gamma_\sigma(1,2) \mid \gamma_\sigma(2,3)] \\ & + \gamma_{\sigma^\alpha}(\alpha, 0)[\gamma_\sigma(0,2) \mid \gamma_\sigma(2,3)] \\ & - \gamma_{\sigma^\alpha}(\alpha, 0)[\gamma_\sigma(0,1) \mid \gamma_\sigma(1,3)] \\ & + \gamma_{\sigma^\alpha}(\alpha, 0)[\gamma_\sigma(0,3)] \\ & - \gamma_{\sigma^\alpha}(\alpha, 1)[\gamma_\sigma(1,2) \mid \gamma_\sigma(2,3)] \\ & - \gamma_{\sigma^\alpha}(\alpha, 1)[\gamma_\sigma(1,3)] \\ & + \gamma_{\sigma^\alpha}(\alpha, 2)[\gamma_\sigma(2,3)] \\ & - \gamma_{\sigma^\alpha}(\alpha, 3)[\ ], \end{aligned}$$

continuing with the notation of Proposition 3.3 and Remark 2.3.

5.5 Preparation for the proof of Theorem C.

In the proof of Theorem C we shall need an extension of the construction of $\psi_{\mathbf{V}}$. Let $\tau = < 01234 >$ be a 4-simplex of Λ, and $\tau^\alpha = < \alpha 01234 >$. Continuing with the notation of Section 3.1, set $W(\tau^\alpha)$ to be the chain $\partial^+ C_{\tau^\alpha} - \partial^4 C_{\tau^\alpha}$, so that, as a subset of C_{τ^α}, $W(\tau^\alpha) = \{\vec{s} \in C^5_{\tau^\alpha} : s_j = 1 \text{ for some } j \neq 4\}$. Note that as before the standard projection $\pi_{\tau^\alpha} : C_{\tau^\alpha} \to \tau^\alpha$ restricts to a surjective map $p: W(\tau^\alpha) \to \tau$.

The obstruction to defining Γ_{τ^α} stems from the impossibility in general of extending $V_{\tau^\alpha} : c_{\tau^\alpha} \to SU(2)$ over the interior of the cell, since ∂c_{τ^α} is a 3-sphere. Examining the definition of Γ_{τ^α} as a union of singular bar-products (see Section 3.3) shows that V_{τ^α} only occurs in $\gamma_{\tau^\alpha}(\alpha)[\gamma_{\tau^\alpha}(\alpha, 4)]$ *via* the singular cell $\gamma_{\tau^\alpha}(\alpha,4)\colon C_{\tau^\alpha}(\alpha,4) \to SU(2)$. It follows that Γ_{τ^α} may be defined on all of C_{τ^α} except the bar-cell $C_{\tau^\alpha}(\alpha)[C_{\tau^\alpha}(\alpha,4)]$, and in particular, since $C_{\tau^\alpha}(\alpha,4) = \{s_4 = 1\}$, on all the bar-cells of $W(\tau^\alpha)$. Define a map

$$\Psi^{\Box}_{\tau^\alpha}\colon W(\tau^\alpha) \to E_\Delta G$$

by

$$\Psi^{\Box}_{\tau^\alpha} = \Gamma_{\tau^\alpha}|W(\tau^\alpha).$$

As before, we pull back $\xi_\Delta G$ by $f_\tau \circ p$ to a bundle $\widehat{E}_\tau$ over $W(\tau^\alpha)$, and we interpret $\Psi^{\Box}_{\tau^\alpha}$ as a section in that bundle, which we project to a pseudosection $\psi^{\Box}(\tau)$ in $E|_\tau$, where E is the bundle defined by $\mathbf{V}$ over Λ.

Lemma 5.4 (1) $\pi_*\psi^{\Box}(\tau) = \tau$ as a 4-chain.
(2) $\partial\psi^{\Box}(\tau) = \psi(\partial\tau) + \psi^{\Box}_{\tau^\alpha} \mid \partial^4 C_{\tau^\alpha}$.
(3) Moreover the last term is a map of a 3-sphere into the single fibre $E|_{<4>}$.

Proof: Statement (1) was handled above. For statement (2), observe that

$$\begin{aligned} \partial W(\tau^\alpha) &= \partial\partial^+ C_{\tau^\alpha} - \partial\partial^4 C_{\tau^\alpha} \\ &= \textstyle\sum(-1)^i \partial^+ C_{\tau^\alpha_i} - \partial\partial^4 C_{\tau^\alpha}, \end{aligned}$$

(using Proposition 5.1) where $\partial\tau = \sum(-1)^i\tau_i$. Statement (3) follows from the definition of Γ_{τ^α}. $\Box$

6 Proofs of Main Theorems

We now turn to the computation of $\hat{\chi}_{\mathbf{V}}$ for a p.t.f. $\mathbf{V}$ obtained from a generic l.g.f. $\mathbf{u}$ by iterated coning. We write $\hat{\chi}_{\mathbf{u}}$ for $\hat{\chi}_{\mathbf{V}_{\mathbf{u}}}$ in this case.

6.1 Integration over a bar-product

For $i = 0, \ldots, r$ let C_i be a cube of dimension $p(i)$, parametrised by $\vec{s^i} = (s^i_1, \ldots, s^i_{p(i)})$, and let $\mathbf{C} = C_0[C_1 \mid \ldots \mid C_r]$ be their bar-product. Set $p = \dim \mathbf{C} = r + \sum_{i=0}^r p(i)$. We identify the p-cube C^p with $C^r \times (\prod_{i=0}^r C_i)$, parametrised by $(s_1, \ldots, s_r, \vec{s^0}, \ldots, \vec{s^r})$.

Definition 6.1 The *standard bar-projection* $\boldsymbol{\pi}^C : C^p \to \mathbf{C}$ is given by

$$\boldsymbol{\pi}^C(s_1, \ldots, s_r, \vec{s^0}, \ldots, \vec{s^r}) = \\ t_0\vec{s^0} + t_1(\vec{s^0}, \vec{s^1}) + \ldots + t_r(\vec{s^0}, \vec{s^1}, \ldots, \vec{s^r}),$$

where $(t_0, \ldots, t_r)$ is the image of $(s_1, \ldots, s_r)$ under the standard projection $\pi_{<0,\ldots,r>} : C^r \to < 0, \ldots, r >$ of Section 3.1.

Now let $\boldsymbol{\gamma} : \mathbf{C} \to E_\Delta G$ be the bar-product of singular cubes $\gamma_i : C_i \to G_{j(i)}$, for $i = 0, \ldots, r$; so $\boldsymbol{\gamma}$ is a generator of $\mathcal{E}_*$. Here as in Definition 2.4 $\gamma_i(\vec{s^i}) = g_i(\vec{s^1})\mathbf{e}_{j(i)}$, with g_i mapping C_i to G. Set $\boldsymbol{\gamma}^C = \boldsymbol{\gamma} \circ \boldsymbol{\pi}^C : C^p \to E_\Delta$, and define

$$\int_{\boldsymbol{\gamma}} \lambda = \int_{C^p} (\boldsymbol{\gamma}^C)^* \lambda$$

for any p-form λ on E_Δ.

In the next section we shall prove the following result.

Proposition 6.2 Let $\boldsymbol{\gamma} \colon \mathbf{C} \to E_\Delta$ be as above, with dimension $p = 3$. Then $\int_{\boldsymbol{\gamma}} \tilde{\chi}_\Delta = 0$ except in these two cases
(1) If $\mathbf{C} = C_0[\,]$ with $\dim C_0 = 3$, then $\int_{\boldsymbol{\gamma}} \tilde{\chi}_\Delta = \int_{\gamma_0} dv$, where dv is the bi-invariant volume form on the unit sphere in quaternionic space, normalised so that $\int_{S^3} dv = 1$.
(2) If $\mathbf{C} = C_0[C_1]$ with $\dim C_0 = \dim C_1 = 1$, then γ_0 and γ_1 are curves, and

$$\int_{\boldsymbol{\gamma}} \tilde{\chi}_\Delta = -\frac{1}{4\pi^2} \int_0^1 \int_0^1 \Re(g_0'(s^1_0) g_0(s^1_0)^{-1} \cdot \mathrm{Ad}_{g_0(s^1_0)}(g_1'(s^1_1) g_1(s^1_1)^{-1}) ds^1_0 ds^1_1.$$

6.2 Proof of Theorem A

We begin by restating the theorem.

Theorem 6.3 Let $\mathbf{u}$ be a generic $SU(2)$–valued l.g.f. on Λ, with Chern-Simons character $\hat{\chi}_{\mathbf{u}}$ given as above. Then for every 3-simplex $\sigma =< 0123 >\in \Lambda$,

$$\hat{\chi}_{\mathbf{u}}(\sigma) = \int_P dv + \sum_{i=1}^{4}(-1)^i \int_{\Delta_i} dv + \frac{1}{4\pi^2}\int_0^1 \Re[U_{01} \cdot \mathrm{Ad}_{\mathbf{g}_{\alpha 01}(s_0)} W_{123}] ds_0$$

where

$$\begin{aligned} P &= \text{cone } (1, \mathbf{g}_{\alpha 01} \cdot \mathbf{g}_{123}), \\ \Delta_1 &= \mathrm{cx}(1, u_{03}, u_{013}, u_{0123}), \\ \Delta_2 &= \mathrm{cx}(1, u_{03}, u_{023}, u_{0123}), \\ \Delta_3 &= \mathrm{cx}(1, u_{23}, u_{023}, u_{0123}), \\ \Delta_4 &= \mathrm{cx}(1, u_{23}, u_{123}, u_{0123}), \end{aligned}$$

where $\mathbf{g}_{\alpha 01}(s_0) = \exp(s_0 U_{01})$ is the shortest geodesic from $\mathbf{1}$ to u_{01}, where $\mathbf{g}_{123}(s_2) = u_{13}\exp(s_2 U_{3123})$ is the shortest geodesic from u_{13} to $u_{12}u_{23}$, and $W_{123} = \mathrm{Ad}_{u_{13}} U_{3123}$.

Proof. By Proposition 6.2, Remark 5.3 and the definition of $\hat{\chi}$,

$$\hat{\chi}(\sigma) = -\int_{\gamma_{\sigma^\alpha}(\alpha, 3)} dv - \int_{\gamma_{\sigma^\alpha}(\alpha, 1)[\gamma_\sigma(1,3)]} \chi_\Delta, \tag{1}$$

and the second term on the right-hand side of this equation equals

$$+\frac{1}{4\pi^2}\int_0^1\int_0^1 \Re(\mathbf{g}'_{\alpha 01}(s_0)\mathbf{g}_{\alpha 01}(s_0)^{-1} \cdot \mathrm{Ad}_{\mathbf{g}_{\alpha 01}(s_0)}(\mathbf{g}'_{123}(s_2)\mathbf{g}_{123}(s_2)^{-1}) ds_0 ds_2.$$

The expressions $\mathbf{g}_{\alpha 01}(s_0) = \exp(s_0 U_{01})$ and $\mathbf{g}_{123}(s_2) = u_{13}\exp(s_2 U_{3123})$ introduced above reduce $\mathbf{g}'_{\alpha 01}(s_0)\mathbf{g}_{\alpha 01}(s_0)^{-1}$ to U_{01} and $\mathbf{g}'_{123}(s_2)\mathbf{g}_{123}(s_2)^{-1}$ to $\mathrm{Ad}_{u_{13}} U_{3123}$. The variable s_2, which no longer appears in the integrand, integrates out to 1; the integral now matches the last term in the statement of the theorem.

Turning now to the first integral on the right hand side of Equation 1, we see that the proof of the theorem will be complete once we show that, as oriented 3-chains,

$$V^\alpha_{\sigma^\alpha}(\alpha, 3) = -P - \sum_{i=1}^{4}(-1)^i \Delta_i. \tag{2}$$

Now $V^\alpha_{\sigma^\alpha}(\alpha, 3)$ is the image of the fundamental chain on $c^3_{\sigma^\alpha}$ under the map $V^\alpha_{\sigma^\alpha} : c^3_{\sigma^\alpha} \to G$. And $V^\alpha_{\sigma^\alpha}$ was constructed so that line segments from $\vec{0}$ to points of $\partial^+ c^3_{\sigma^\alpha}$ are mapped into geodesics from $\mathbf{1}$ to points of $V^\alpha_{\sigma^\alpha}(\partial^+ c^3_{\sigma^\alpha})$. Thus it suffices to show that, as oriented 2-chains,

$$\begin{aligned} V^\alpha_{\sigma^\alpha}(\partial^+ c^3_{\sigma^\alpha}) &= -V^\alpha_{\alpha 01} \cdot V_{123} \\ &\quad +\mathrm{cx}(u_{03}, u_{013}, u_{0123}) \\ &\quad -\mathrm{cx}(u_{03}, u_{023}, u_{0123}) \\ &\quad +\mathrm{cx}(u_{23}, u_{023}, u_{0123}) \\ &\quad -\mathrm{cx}(u_{23}, u_{123}, u_{0123}). \end{aligned}$$

On $\partial^+ c^3_{\sigma^\alpha}$, $V^\alpha_{\sigma^\alpha}$ is defined *as a function* by:

$$V^\alpha_{\sigma^\alpha}(s_0, s_1, s_2) = \begin{cases} V_\sigma(s_1, s_2), & \text{if } s_0 = 1 \\ V^\alpha_{\alpha 01}(s_0) \cdot V_{123}(s_2), & \text{if } s_1 = 1 \\ V^\alpha_{\alpha 012}(s_0, s_1) \cdot u_{23}, & \text{if } s_2 = 1 \end{cases}$$

(since $V^\alpha_{\alpha 0}$ and V_{23} map 0-cubes into $\mathbf{1}$ and u_{23} respectively). Allowing for orientations, this gives that, *as oriented 2-chains*,

$$V^\alpha_{\sigma^\alpha}(\partial^+ c^3_{\sigma^\alpha}) = V_\sigma - V^\alpha_{\alpha 01} \cdot V_{123} + V^\alpha_{\alpha 012} \cdot u_{23}.$$

But $V_\sigma(c^2_\sigma)$ is the 2-chain $\mathrm{cx}(u_{03}, u_{013}, u_{0123}) - \mathrm{cx}(u_{03}, u_{023}, u_{0123})$; and $V^\alpha_{\alpha 012}(c^2_{<012>^\alpha})$ is the 2-chain $\mathrm{cx}(\mathbf{1}, u_{02}, u_{012}) - \mathrm{cx}(\mathbf{1}, u_{12}, u_{012})$. Now Equation 2 follows, and with it, the theorem. □

6.3 Proof of Theorem B

Suppose on a lattice Λ we are given two $SU(2)$–valued lattice gauge fields $\mathbf{u}$ and $\mathbf{u}'$ which are related by a change of gauge: there exists a family (see Definition 3.16) $\mathbf{g} = \{g_0, g_1, \ldots\}$ of elements of $G = SU(2)$ such that $\mathbf{u}' = \mathrm{Ad}_{\mathbf{g}}\mathbf{u}$.

Suppose $\mathbf{u}$ and $\mathbf{u}'$ are generic, so that they may be extended by iterated coning (see Remark 3.15) to parallel transport functions $\mathbf{V}$ and $\mathbf{V}'$; let $f, f' \colon |\Lambda| \to B_\Delta G$ be the corresponding classifying maps, constructed as in Section 3.3; they pull back $\xi_\Delta G$ to bundles ξ, ξ' for which we will use the explicit form described in Section 5.

Given a 3–cycle $Z \subset \Lambda$, let $\chi_{\mathbf{u}}$ and $\chi_{\mathbf{u}'}$ be the values on Z of the Chern-Simons characters defined from $\mathbf{u}$ and $\mathbf{u}'$ as in Section 5. With this notation, Theorem B is equivalent to the following statement.

Theorem 6.4 $\chi_{\mathbf{u}}(Z) \equiv \chi_{\mathbf{u}'}(Z) \bmod \mathbf{Z}$.

Proof: We adapt to this context an argument of Cheeger and Simons [2], continuing with the notation from Section 5: $\xi = (\pi: E \to X)$, $F: E \to E_\Delta G$, $\widehat{\xi} = (f \circ p)^* \xi_\Delta G = (\widehat{\pi}: \widehat{E} \to \widehat{Z})$, $P: \widehat{E} \to E$, the section $\widehat{\psi}: \widehat{Z} \to \widehat{E}$, the pseudosection $\psi = P \circ \widehat{\psi}: \widehat{Z} \to E$, as well as $E', F', \ldots, \psi'$ the corresponding elements constructed from $\mathbf{u}'$. Note that the bundles ξ, ξ' inherit connections pulled back from the canonical connection in $\xi_\Delta G$.

Let $\mathbf{h} = \{h_0, h_1, \ldots\}$ extend $\mathbf{g}$ to the rest of the join-factors of $E_\Delta G$. Right multiplication by $\mathbf{h}$ defines an automorphism $R_{\mathbf{h}}: E_\Delta G \to E_\Delta G$ (see Paragraph 2.1) and the corresponding $r_{\mathbf{h}}: B_\Delta G \to B_\Delta G$. From Proposition 3.10 we know that $f' = r_{\mathbf{h}} \circ f$, leading to the following commutative diagram.

$$\begin{tikzcd}
 & E' \arrow[rr, "F'"] \arrow[dd, "\pi'"'] & & E_\Delta G \arrow[dd, "\pi_\Delta"] \\
E \arrow[ur, "\Phi"] \arrow[rr, "F"] \arrow[dd, "\pi"'] & & E_\Delta G \arrow[ur, "R_{\mathbf{h}}"'] \arrow[dd, "\pi_\Delta"] & \\
 & X \arrow[rr, "f'"] & & B_\Delta G \\
X \arrow[ur, "\mathrm{id}"] \arrow[rr, "f"] & & B_\Delta G \arrow[ur, "r_{\mathbf{h}}"'] &
\end{tikzcd}$$

It follows that the right-automorphism $R_{\mathbf{h}}$ induces an isomorphism $\Phi: \xi \to \xi'$. On the total space E it is defined by $\Phi((e, x)) = (R_{\mathbf{h}} e, x)$. To check that $\Phi(E) = E'$ note that if $\pi_\Delta e = f(x)$ then $\pi_\Delta R_{\mathbf{h}} e = r_{\mathbf{h}} \pi_\Delta e = r_{\mathbf{h}} f(x) = f'(x)$. Since $R_{\mathbf{h}}$ preserves the canonical connection (Proposition 4.3) it follows that Φ is an isomorphism of bundles with connections.

Similarly, moving up to $\widehat{Z}$, this Φ, or $R_{\mathbf{h}}$, induce an isomorphism $\widehat{\Phi}: \widehat{E} \to \widehat{E}'$ of bundles with connections, and extending the diagram above, and the diagram at the beginning of Section 5, as shown below.

Furthermore $f \circ p$ and $f' \circ p$ are homotopic, since $r_{\mathbf{h}}$ is homotopic to the identity. If the homotopy between them is $H: \widehat{Z} \times [0,1] \to B_\Delta G$, then $H^* \xi_\Delta G$ is a bundle with connection which restricts to the bundles with connection $\widehat{\xi}$ over $\widehat{Z} \times \{0\}$ and $\widehat{\xi}'$ over $\widehat{Z} \times \{1\}$. Say $H^* \xi_\Delta G = (\tilde{\pi}: \tilde{E} \to \widehat{Z} \times [0,1])$. The fiber is $SU(2) = S^3$.

The cell decomposition of $\widehat{Z} \times \{0\}$ and $\widehat{Z} \times \{1\}$ extends to a cell decomposition of $\widehat{Z} \times [0,1]$, a 4-manifold with boundary, with 4-cells $c_1, \ldots, c_\ell$. The sections $\widehat{\psi}$ and $\widehat{\psi}'$ defined over the two ends extend over the 3-skeleton of this decomposition, and may be radially extended to a section $\widehat{\Psi}$ defined on the complement of one point b_i in the interior of each c_i, $i = 1, \ldots, \ell$.

$$
\begin{tikzcd}
\widehat{E} \arrow[rrrrrr, "\widehat{\Phi}"] \arrow[dr, "P"] \arrow[ddddd] & & & & & & \widehat{E} \arrow[dl, "P'"'] \arrow[ddddd] \\
& E \arrow[rrrr, "\Phi"] \arrow[dr, "F"] \arrow[dddd, "\pi"] & & & & E' \arrow[dl, "F'"'] \arrow[dddd, "\pi'"'] & \\
& & E_\Delta G \arrow[rr, "R_{\mathbf{h}}"] \arrow[d, "\pi_\Delta"'] & & E_\Delta G \arrow[d, "\pi_\Delta"] & & \\
& & B_\Delta G \arrow[rr, "r_{\mathbf{h}}"'] & & B_\Delta G & & \\
& Z \subset X \arrow[ur, "f"] \arrow[rrrr, "\mathrm{id}"] & & & & Z \subset X \arrow[ul, "f'"'] & \\
\widehat{Z} \arrow[uuuuu, bend left=60, "\widehat{\psi}"] \arrow[ur, "p"'] \arrow[uuuur, "\psi"] \arrow[rrrrrr] & & & & & & \widehat{Z} \arrow[uuuuu, bend right=60, "\widehat{\psi}'"'] \arrow[ul, "p"] \arrow[uuuul, "\psi'"']
\end{tikzcd}
$$

Puting in the radial limits makes $\widehat{\Psi}$ into a piecewise smooth map from $\widehat{Z} \times [0,1] - b_1 - \cdots - b_\ell \cup S^3_1 \cup \cdots \cup S^3_\ell$ into $\tilde{E}$ with the property that each S^3_i is mapped into the fiber over the corresponding b_i.

Now let us use $\widehat{\Phi}$ to glue $\tilde{E}|_{\widehat{Z}\times\{0\}} = \widehat{E}$ to $\tilde{E}|_{\widehat{Z}\times\{1\}} = \widehat{E}'$. Since $\widehat{\Phi}$ is an isomorphism of bundles with connection, this gives a bundle-with-connection $\overline{\xi} = (\overline{\pi}\colon \overline{E} \to \widehat{Z} \times S^1)$, with base a 4-manifold. The "section" $\widehat{\Psi}$ gets wrapped around so that over $\widehat{Z} \times \{\text{the basepoint}\}$ we see the sections $\widehat{\psi}'$ and $\widehat{\Phi} \circ \widehat{\psi}$

Let $\overline{p}$ and $\overline{\chi}$ be the Chern and Chern-Simons forms in $\overline{E}$, and consider $\int_{\widehat{\Psi}} \overline{p}$. By Stokes' Theorem this integral is equal to

$$\int_{\partial\widehat{\Psi}} \overline{\chi} = \int_{\widehat{\psi}'} \overline{\chi} - \int_{\widehat{\Phi}\circ\widehat{\psi}} \overline{\chi} + \sum_i \int_{\widehat{\Psi}(S^3_i)} \overline{\chi}.$$

The following three observations finish the argument.

• Each $\widehat{\Psi}(S^3_i)$ is a map of S^3 into a fiber, where $\overline{\chi}$ restricts to the v¡olume form; so $\int_{\widehat{\Psi}(S^3_i)} \overline{\chi}$ is the winding number of $\widehat{\Psi}(S^3_i)$, an integer.

• $\displaystyle\int_{\widehat{\Psi}} \overline{p} = \int_{\widehat{\Psi}} \overline{\pi}^* p = \int_{\widehat{Z}\times S^1} p$
where p is the "downstairs" Chern form, is a Chern number of the bundle $\overline{\xi}$ and therefore an integer.

• $\displaystyle\int_{\widehat{\Phi}\circ\widehat{\psi}} \overline{\chi} = \int_{\widehat{\Phi}\circ\widehat{\psi}} (F' \circ P')^* \widetilde{\chi}_\Delta = \int_{\widehat{\psi}} \widehat{\Phi}^*(P')^*(F')^* \widetilde{\chi}_\Delta = \int_{\widehat{\psi}} P^* F^* R^*_{\mathbf{h}} \widetilde{\chi}_\Delta = \int_{\widehat{\psi}} P^* F^* \widetilde{\chi}_\Delta$

using Theorem 4.5

$$\int_{\widehat{\psi}} P^*F^*\widetilde{\chi}_\Delta = \int_\psi F^*\widetilde{\chi}_\Delta = \chi_{\mathbf{u}}(Z),$$

whereas

$$\int_{\widehat{\psi'}} \overline{\chi} = \int_{\widehat{\psi'}} (F' \circ P')^*\widetilde{\chi}_\Delta = \int_{\psi'} (F')^*\widetilde{\chi}_\Delta = \chi_{\mathbf{u}'}(Z).$$

6.4 Proof of Theorem C

Refering to Section 1.3 for motivation, let $\bar{\Lambda}$ and $\bar{\Lambda}^\alpha$ denote Λ and Λ^α (respectively) with the order of their vertices reversed. We shall denote vertex $< i >$ in the reverse ordering by $< -i >$, $< 0 >$ by $< -0 >$ and $< \alpha >$ by $< -\alpha >$. For $\sigma =< i_0, \ldots, i_r >$, set $\bar{\sigma} =< -i_r, \ldots, -i_0 >$, $\bar{\sigma}_\alpha =< -i_r, \ldots, -i_0, -\alpha >$.

Given a l.g.f. $\mathbf{u}$ on Λ, let $\bar{\mathbf{u}}$ denote the same l.g.f., but regarded as being on $\bar{\Lambda}$. Then $\bar{\mathbf{u}}$ is continuous if and only if $\mathbf{u}$ is; and in this case, $\delta(\bar{\mathbf{u}}) = \delta(\mathbf{u})$. By Proposition 3.14, Part 2, we have $\xi(\mathbf{u}) = \xi(\bar{\mathbf{u}})$.

Let $\bar{N}$ be the 4-cocycle representative on $\bar{\Lambda}$ of the second Chern class of $\xi(\bar{\mathbf{u}}) \approx \xi(\mathbf{u})$ constructed according to the algorithm of [14]. For any 4–simplex $\tau =< 0, \ldots, 4 > \in \Lambda$, by definition, $\bar{N}(\bar{\tau})$ is the degree of a certain map, which we shall here denote by $\bar{V}_{\bar{\tau}_\alpha}\colon \partial c^4_{\bar{\tau}_\alpha} \to S^3$, where $c^4_{\bar{\tau}_\alpha} = \{(s_{-3}, s_{-2}, s_{-1}, s_{-0}) : 0 \le s_{-i} \le 1\}$.

Theorem C may be restated as follows:

Theorem 6.5 For every 4-simplex τ of Λ, the value of the Chern-Simons character $\chi = \chi_{\mathbf{u}}$ on the 3–cycle $\partial\tau$ is given by

$$\chi(\partial\tau) = \int_\tau p - \bar{N}(\bar{\tau}).$$

Proof. Let $\mathbf{u}^\alpha$, Λ^α and $\mathbf{V}^\alpha$ be as at the beginning of this section. Let

$$\Psi^\square_{\tau^\alpha}\colon W(\tau^\alpha) \to E_\Delta G;$$

and let $\psi^\square_{\mathbf{V}}(\tau)$ be as in Lemma 5.4. Then

$$\begin{aligned} \chi(\partial\tau) &= \int_{\psi(\partial\tau)} F^*\tilde{\chi}_\Delta \\ &= \int_{\partial\psi^\square(\tau)} F^*\tilde{\chi}_\Delta - \int_{\Psi^\square_{\tau^\alpha}|\partial\partial^4 C_{\tau^\alpha}} F^*\tilde{\chi}_\Delta. \end{aligned}$$

The first integral, by Stokes' Theorem, naturality and Lemma 5.4 (1), is equal to $\int_\tau p$. The second, since $\tilde{\chi}_\Delta$ restricts to the normalized volume form on fibers, is equal to the degree of $\Psi^{\square}_{\tau^\alpha}|\partial\partial^4 C_{\tau^\alpha}$, which is the degree of $V_{\tau^\alpha}|\partial^4 c_{\tau^\alpha}$ (see Section 5.5). To compare this integer with the degree of $\bar{V}_{\bar{\tau}_\alpha}\colon \partial c_{\bar{\tau}_\alpha} \to S^3$, we use the orientation-preserving homeomorphism $h\colon c_{\tau^\alpha} \to c_{\bar{\tau}_\alpha}$ given by $h(s_0, s_1, s_2, s_3) = (s_3, s_2, s_1, s_0)$. Comparing the construction of V_{τ^α} and $\bar{V}_{\bar{\tau}_\alpha}$ (the latter in [15]) shows that $\bar{V}_{\bar{\tau}_\alpha} \circ h = V_{\tau^\alpha}$, so the degrees are the same. $\square$

7 Which cells contribute to the integral of $\tilde{\chi}$? Proof of Proposition 6.2.

Proof of Proposition 6.2. Continuing with the notation from Section 6.1, let

$$\partial_i = (\boldsymbol{\gamma}^C)_* \frac{\partial}{\partial s_i} \quad \text{and} \quad \partial_j^i = (\boldsymbol{\gamma}^C)_* \frac{\partial}{\partial s_j^i},$$

$i = 1, \ldots, r$, $j = 1, \ldots, p(i)$, denote the images in $TE_\Delta G$ of the coordinate vector fields on C^3. The proposition will follow from Lemmas 7.1, 7.2 and 7.3.

The first lemma is a compilation of useful formulae.

Lemma 7.1 (A) $\tilde{\chi}_\Delta = \frac{1}{4\pi^2}\Re(\omega_\Delta \wedge \Omega_\Delta - \frac{1}{3}\omega_\Delta \wedge \omega_\Delta \wedge \omega_\Delta)$.
(B) $\tilde{\chi}_\Delta = \frac{1}{4\pi^2}\Re(\omega_\Delta \wedge d\omega_\Delta + \frac{2}{3}\omega_\Delta \wedge \omega_\Delta \wedge \omega_\Delta)$.
(C) $\int_{\boldsymbol{\gamma}} \tilde{\chi}_\Delta = \int_{C^3} \tilde{\chi}_\Delta(\ldots \partial_i \ldots \partial_j^i \ldots) \mid_{(\ldots s_i \ldots s_j^i \ldots)} \ldots ds_i \ldots ds_j^i \ldots$
(where $(\ldots s_i \ldots s_j^i \ldots)$ are the parameters of C^3).
(D) ∂_i is horizontal with respect to each ω_j, and hence also w.r.t. ω_Δ.
(E) ∂_j^0 is vertical, i.e. in the kernel of $\pi_{\Delta *}$, for $j = 1, \ldots, p(0)$.
(F) $\omega_\Delta \wedge \omega_\Delta \wedge \omega_\Delta(\ldots \partial_i \ldots \partial_j^i \ldots) \equiv 0$ except when $\mathbf{C} = C_0[\,]$ ($\dim C_0 = 3$).
(G) $\omega_\Delta \wedge \Omega_\Delta(\ldots \partial_i \ldots \partial_j^i \ldots) \equiv 0$ if $\dim C_0 \geq 2$.
(H) $\omega_\Delta \wedge d\omega_\Delta = \sum_{k,l=0}^{\infty} t_k t_l \omega_k \wedge d\omega_l - \sum_{k,l=0}^{\infty} t_k dt_l \wedge \omega_k \wedge \omega_l$.
(J) $\omega_k \wedge d\omega_l(\ldots \partial_i \ldots \partial_j^i \ldots) \equiv 0$ if $r \geq 1$ (r the number of bar-factors in γ).
(K) $dt_l \wedge \omega_k \wedge \omega_l(\ldots \partial_i \ldots \partial_j^i \ldots) \equiv 0$ if $r \geq 2$.

Proof: (A), (B), and (C) are definitions given earlier. (D) is clear since ω_j ignores the t_i and consequently the s_i also. (E) follows from π_Δ being constant on left-equivalence classes, and γ_0 appearing on the left. (F) follows from (D); (G) follows from (E) and the vanishing of the curvature form when one of its vector arguments is vertical; (H) is elementary.

Proof of (J), in case $r = 1$ (the other cases are similar). The images under $(\boldsymbol{\gamma}^C)_*$ of the coordinate vector fields on C^3 must be of types ∂_i, $\partial_{j'}^{i'}$, $\partial_{j''}^{i''}$. Now $\omega_k \wedge d\omega_l(\partial_i,\ \partial_{j'}^{i'},\ \partial_{j''}^{i''}) = \sum_\alpha \{(-1)^\alpha \omega_k(\alpha\partial_i) d\omega_l(\alpha\partial_{j''}^{i''}, \alpha\partial_{j'}^{i'})$: α is a permutation of $\{\partial_i,\ \partial_{j''}^{i''},\ \partial_{j'}^{i'}\}\}$. The first factor is 0 in those terms in which $\alpha\partial_{j'}^{i'} = \partial_i$, by (D). In the remaining terms, the second factor is, up to sign, of the form $d\omega_l(\partial_{j'}^{i'}, \partial_i) = \partial_i(\omega_l(\partial_{j'}^{i'}))$ (again by (D)). But $\omega_l(\partial_{j'}^{i'})$, as a function of s_i, $s_{j'}^{i'}$, and $s_{j''}^{i''}$, depends on only the last two variables; so when this function is acted upon by the vector field ∂_i, the result is 0.

Proof of (K), in case $r = 2$ (the case $r = 3$ is similar). The images under $(\boldsymbol{\gamma}^C)_*$ of the coordinate vector fields on C^3 must be of types $\partial_{i'}$, $\partial_{i''}$

and ∂^i_j. Now $dt_l \wedge \omega_k \wedge \omega_l(\partial_{i'}, \partial_{i''}, \partial^i_j) = \sum_\alpha \{(-1)^\alpha dt_l(\alpha\partial_{i'})\omega_k(\alpha\partial_{i''})\omega_l(\alpha\partial^i_j) :$ α is a permutation of $\{\partial_{i'}, \partial_{i''}, \partial^i_j\}\}$. Since in each term one of $\alpha\partial_{i'}$ or $\alpha\partial_{i''}$ must be either $\partial_{i'}$ or $\partial_{i''}$, each term vanishes, by (D). $\Box$

The combinatorial types of 3-dimensional bar-product cells $\mathbf{C}$ are the following:

a. $\mathbf{C} = C_0[\,]$, with $\dim C_0 = 3$.

b. $\mathbf{C} = C_0[C_1]$, with

  i. $\dim C_0 = 2, \dim C_1 = 0$

  ii. $\dim C_0 = 1 = \dim C_1$

  iii. $\dim C_0 = 0, \dim C_2 = 2$.

c. $\mathbf{C} = C_0[C_1 \mid C_2]$, with

  i. $\dim C_0 = 1, \dim C_1 = \dim C_2 = 0$

  ii. $\dim C_1 = 1, \dim C_0 = \dim C_2 = 0$

  iii. $\dim C_2 = 1, \dim C_0 = \dim C_1 = 0$.

d. $\mathbf{C} = C_0[C_1 \mid C_2 \mid C_3]$, with $\dim C_i = 0, i = 0, \ldots, 3$.

Lemma 7.2 $\int_{\boldsymbol{\gamma}} \tilde{\chi}_\Delta = 0$ in cases $(b)(i), (c)((i), (ii)$ and $(iii))$, and (d).

Proof. This follows from (A), (F) and (G) in case $(b)(i)$; and from (B), (F), (H), (J) and (K) in the other cases. $\Box$

Analysis of the remaining cases.

Case *(a)*: $\mathbf{C} = C_0[\,]$, $\dim C_0 = 3$.

By (A) and (G), $\int_{\boldsymbol{\gamma}}^C \tilde{\chi}_\Delta = -\frac{1}{12\pi^2}\int_{C^3} \Re(\omega_\Delta \wedge \omega_\Delta \wedge \omega_\Delta(\partial^0_1, \partial^0_2, \partial^0_3))ds^0_1 ds^0_2 ds^0_3$. Say $\gamma_0 : C_0 \to G_{j(0)}$; on $G_{j(0)}$, ω_Δ reduces to the canonical left-invariant connection ω^G. Proposition 6.2(1) now follows from the next lemma.

Lemma 7.3 Let ω^G denote the canonical left-invariant connection on the group of unit quaternions, S^3, and let dv be the bi-invariant volume form on S^3 for which $\int_{S^3} dv = 1$. Then $\Re(\omega^G \wedge \omega^G \wedge \omega^G) = -12\pi^2 dv$.

Proof. Let $\varphi : S^3 \to \mathbf{R}$ be the function such that $\Re(\omega^G \wedge \omega^G \wedge \omega^G) \mid_x = \varphi(x) dv \mid_x$. By left-invariance of ω^G and dv, φ is constant. To evaluate φ, we set $x = 1$ and evaluate both $\Re(\omega^G \wedge \omega^G \wedge \omega^G) \mid_1$ and $dv \mid_1$ on the orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of $T_1 S^3$. Now $dv \mid_1 (\mathbf{i}, \mathbf{j}, \mathbf{k}) = (1/\mathrm{vol}\, S^3) = (1/2\pi^2)$. And

$$\begin{aligned}
(\omega^G \wedge \omega^G \wedge \omega^G) \mid_1 (\mathbf{i}, \mathbf{j}, \mathbf{k}) &= \textstyle\sum_\alpha \{(-1)^\alpha \omega^G(\alpha\mathbf{i}) \omega^G(\alpha\mathbf{j}) \omega^G(\alpha\mathbf{k}) : \\
&\qquad \alpha \text{ is a permutation of } (\mathbf{i}, \mathbf{j}, \mathbf{k})\} \\
&= \mathbf{ijk} - \mathbf{jik} + \ldots \\
&= -6.
\end{aligned}$$

Hence $\Re(\omega^G \wedge \omega^G \wedge \omega^G) \mid_1 = -6$; and the lemma follows. $\square$

Case*(b)(ii)*: $\mathbf{C} = C_0[C_1]$, with $\dim C_0 = 1 = \dim C_1$.

The parameters on C^3 are (s_1, s_1^0, s_1^1). Say $\gamma_i = g_i \mathbf{e}_{j(i)} : C_i \to G_{j(i)}$ for $i = 0, 1$. Then

$$\boldsymbol{\gamma}^C(s_1, s_1^0, s_1^1) = (1 - s_1) g_0(s_1^0) \mathbf{e}_{j(0)} + s_1 g_0(s_1^0) g_1(s_1^1) \mathbf{e}_{j(1)}. \tag{3}$$

It follows from (B), (F), (H), and (J) that in this case

$$\int_{\boldsymbol{\gamma}} \tilde{\chi}_\Delta = -\frac{1}{4\pi^2} \sum_{k,l=0}^{\infty} \int \Re(t_k dt_l \wedge \omega_k \wedge \omega_l)(\partial_1, \partial_1^0, \partial_1^1) ds_1 ds_1^0 ds_1^1.$$

In this sum, the only non-zero terms are those in which each of k and l equals either $j(0)$ or $j(1)$. Moreover, the real part of any commutator of quaternions is zero; this reduces the integrand to

$$\Re(t_{j(0)} dt_{j(1)} \wedge \omega_{j(0)} \wedge \omega_{j(1)} + t_{j(1)} dt_{j(0)} \wedge \omega_{j(1)} \wedge \omega_{j(0)})(\partial_1, \partial_1^0, \partial_1^1).$$

Expanding the definitions of $\partial_1, \partial_1^0, \partial_1^1$ in terms of the decomposition (valid off a set of measure zero) of $T(G_{j(0)} * G_{j(1)})$ as $\mathbf{R} \oplus TG_{j(0)} \oplus TG_{j(1)}$ we have

$\partial_1 = (1, 0, 0)$;

$\partial_1^0 = (0, (1 - s_1) g_0'(s_1^0), s_1 g_0'(s_1^0) \cdot g_1(s_1^1))$;

$\partial_1^1 = (0, 0, g_0(s_1^0) \cdot g_1'(s_1^1))$.

In particular $\omega_{j(0)}(\partial_1) = \omega_{j(1)}(\partial_1) = \omega_{j(0)}(\partial_1^1) = 0$. Also, since $t_{j(0)} = (1 - s_1)$ and $t_{j(1)} = s_1$, it follows that $dt_{j(0)}(\partial_1) = -1$ and $dt_{j(1)}(\partial_1) = 1$, whereas clearly $dt_{j(i)}(\partial_1^k) = 0$ for either choice of i or k. The integrand thus reduces to

$$\Re[(1 - s_1)\omega_{j(0)} \wedge \omega_{j(1)}(\partial_1^0, \partial_1^1) - s_1 \omega_{j(1)} \wedge \omega_{j(0)}(\partial_1^0, \partial_1^1)] = \Re[\omega_{j(0)}(\partial_1^0) \omega_{j(1)}(\partial_1^1)].$$

Applying $\omega_{j(0)} = dg_{j(0)} \cdot g_{j(0)}^{-1}$ to ∂_1^0 gives, following Definition 4.1

$$\omega_{j(0)}(\partial_1^0) = g_0'(s_1^0) \cdot g_0(s_1^0)^{-1},$$

and similarly

$$\omega_{j(1)}(\partial_1^1) = \mathrm{Ad}_{g_0(s_1^0)}(g_1'(s_1^1) \cdot g_1(s_1^1)^{-1}),$$

and finally, since s_1 integrates out to 1,

$$\int_{\boldsymbol{\gamma}} \tilde{\chi}_\Delta = -\frac{1}{4\pi^2} \int_0^1 \int_0^1 \Re(g_0'(s_0^1) g_0(s_0^1)^{-1} \cdot \mathrm{Ad}_{g_0(s_0^1)}(g_1'(s_1^1) g_1(s_1^1)^{-1}) ds_0^1 ds_1^1,$$

which is Proposition 6.2(2).

Case*(b)(iii)*: $\mathbf{C} = C_0[C_1]$ with $\dim C_0 = 0$, $\dim C_1 = 2$.

In this case the parameters on C^3 are (s_1, s_1^1, s_2^1), with

$$\boldsymbol{\gamma}^C(s_1, s_1^1, s_2^1) = (1 - s_1)\gamma_0 \vec{e}_{j(0)} + s_1 \gamma_0 \gamma_1(s_1^1, s_2^1) \vec{e}_{j(1)}.$$

The proof that in this case $\int_{\gamma^C} \chi = 0$ is similar to the argument in the previous Case*(b)(ii)*, replacing ∂_1^0 and ∂_1^1 by ∂_1^1 and ∂_2^1 respectively. The only difference appears in analysing the terms on the right-hand side of the following equation.

$$\begin{aligned} \sum_{k,l=j(0),j(1)} (t_k dt_l \wedge \omega_k \wedge \omega_l)(\partial_i, \partial_1^1, \partial_2^1) = \\ (1 - s_1)\{\omega_{j(0)}(\partial_1^1)\omega_{j(1)}(\partial_2^1) - \omega_{j(0)}(\partial_2^1)\omega_{j(1)}(\partial_1^1)\} \\ -s_1\{\omega_{j(1)}(\partial_1^1)\omega_{j(0)}(\partial_2^1) - \omega_{j(1)}(\partial_2^1)\omega_{j(0)}(\partial_1^1)\}. \end{aligned}$$

Here $\omega_{j(0)}(\partial_1^1)$ and $\omega_{j(0)}(\partial_2^1)$ both vanish identically for the reason given in the previous argument. Thus the sum is zero. This completes the proof of Proposition 6.2. □

References

[1] J. Böhm, E. Hertel, Polyedergeometrie in n-dimensionaler Räumen konstanter Krümmung, *Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften* Vol. 70, Birkhäuser, Basel 1981

[2] J. Cheeger, J. Simons, Differential characters and geometric invariants, in Geometry and Topology, *Lecture Notes in Math.* Vol. 1167, Springer, NY 1985

[3] S.-S. Chern, J. Simons, Characteristic forms and geometric invariants, *Ann. of Math.* **99** (1974) 48-69

[4] J. Dupont, Curvature and Characteristic Classes, *Lecture Notes in Math.* Vol. 640, Springer, NY 1978

[5] S. Eilenberg, S. MacLane, On the groups $H(\pi, n)$, I, *Ann. of Math.* **51** (1953) 55-106

[6] M. Göckeler, A. Kronfeld, G. Schierholz, U.-J. Wiese, Continuum gauge fields from lattice gauge fields, HLRZ preprint HLRZ-92-34

[7] F. Karsch, M. Laursen, B. Plache, T. Neuhaus, U.-J. Wiese, Chern-Simons term in the 4-dimensional SU(2) Higgs model, HLRZ preprint HLRZ-92-55

[8] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry I, Interscience, New York 1963

[9] M. Laursen, Chern-Simons term and topological charge on the lattice, HLRZ preprint HLRZ-92-61

[10] R. Milgram, The bar construction and abelian H-spaces, *Ill. J. Math.* **11** (1967) 242-250

[11] J. Milnor, Construction of universal bundles I, *Ann. of Math.*(2) **63** (1956) 272-284

[12] J. Milnor, Construction of universal bundles II, *Ann. of Math.*(2) **63** (1956) 430-436

[13] A. Phillips, Characteristic numbers of U_1–valued lattice gauge fields, *Ann. Phys.* **161** (1985) 399-422

[14] A. Phillips, D. Stone, Lattice gauge fields, principal bundles, and the calculation of topological charge, *Commun. Math. Phys.* **103** (1986) 599-636

[15] A. Phillips, D. Stone, The computation of characteristic classes of lattice gauge fields, *Commun. Math. Phys.* **131** (1990) 255-282

[16] A. Phillips, D. Stone, Chern-Simons on a lattice, *Nucl. Phys.* **B** (Proc. Suppl.) **20** (1991) 28-31

[17] A. Phillips, D. Stone, A topological Chern-Weil theory, to appear in *Memoirs Amer. Math. Soc.*

[18] G. Segal, Classifying spaces and spectral sequences, *Publ. Math. I.H.E.S.* **34** (1968) 105-112

[19] K. Wilson, Confinement of Quarks, *Phys. Rev.* **D 10** (1974) 2445-2459

[20] E. Witten, Quantum field theory and the Jones polynomial, *Commun. Math. Phys.* **121** (1989) 351-399

[21] P. Woit, Chern-Simons numbers and universal bundles on the lattice, Stony Brook preprint ITP-SB-87-56