CALCULUS II
Departments of Mathematics & Applied Mathematics
The University at Stony Brook
Homework # 3

Section 7.2

3. Compute L_{10} and R_{10} for $\int_1^2 x^2\,dx$, then compare these estimates with the exact value of the integral and check the bound on the magnitude of the approximation error.

   Answer. We must subdivide the interval $[1, 2]$ into 10 subintervals of equal length. Then the step size $\triangle x$ is given by

$$\triangle x = \frac{2-1}{10} = \frac{1}{10},$$

   and we insert the equally spaced points

$$1 = x_0,\ x_1 = 1 + \frac{1}{10},\ x_2 = 1 + \frac{2}{10},\ x_3 = 1 + \frac{3}{10},\ x_4 = 1 + \frac{4}{10},\ x_5 = 1 + \frac{5}{10},$$
$$x_6 = 1 + \frac{6}{10},\ x_7 = 1 + \frac{7}{10},\ x_8 = 1 + \frac{8}{10},\ x_9 = 1 + \frac{9}{10},\ x_{10} = 2$$

   to partition the interval $[1, 2]$.

   For the left-approximation, in each subinterval $[x_i, x_{i+1}]$ we consider the rectangle with height equal to the value of the function $f(x) = x^2$ to the left most point, fact which is illustrated in Fig. 1:

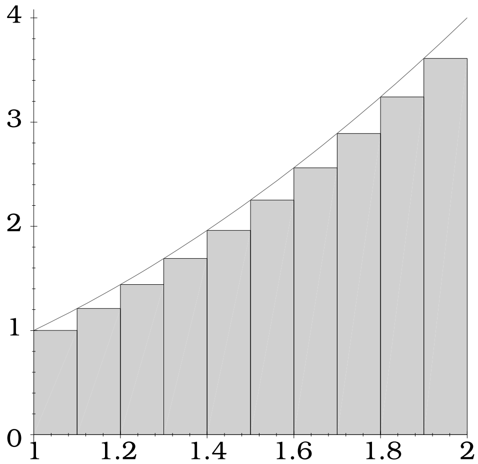

Fig. 1

We then get:

$$L_{10} = \sum_{i=0}^{9} f(x_i) \bigtriangleup x = \frac{1}{10} \sum_{i=0}^{9} \left(1 + \frac{i}{10}\right)^2 = 2.185\,.$$

For the right approximation, in each subinterval $[x_i, x_{i+1}]$ we consider the rectangle with height equal to the value of the function $f(x) = x^2$ to the rigth most point instead:

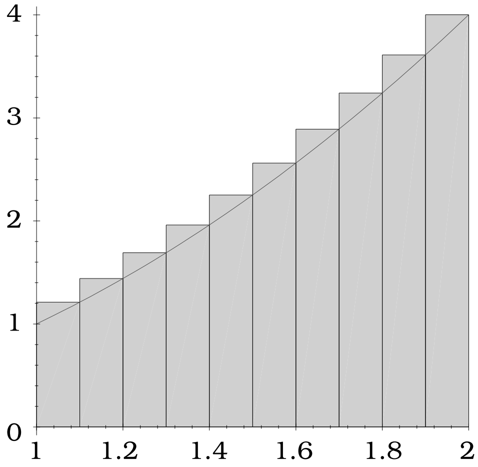

Fig. 2

We now get:

$$R_{10} = \sum_{i=1}^{10} f(x_i) \bigtriangleup x = \frac{1}{10} \sum_{i=1}^{10} \left(1 + \frac{i}{10}\right)^2 = 2.485\,.$$

The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{1}{3}x^3 \mid_{x=2} - \frac{1}{3}x^3 \mid_{x=1} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.333\ldots$$

We then see that L_{10} is smaller than the exact value of the integral in the amount $0.148333\ldots$, while R_{10} is larger than the exact value of the integral in the amount $0.151666\ldots$. That L_{10} and R_{10} are approximations to the integral of x^2 on $[1,2]$ which are smaller and larger than the actual value of the integral can be deduced from the fact that x^2 is an increasing function on the interval of integration.

Theorem 2 on page 433 of the textbook says that if K is such that $|f'(x)| \leq K$ for any x in $[a,b]$, then if I is the value of the integral, and n is the number of subdivisions of $[a,b]$, we have that

$$|I - L_n| \leq \frac{K(b-a)^2}{2n}\,.$$

In our case, the derivative of x^2 is $2x$. On the interval $[1,2]$ this function is bounded by 4. Hence, we get

$$|I - L_{10}| \leq \frac{4(2-1)^2}{20} = \frac{1}{5} = 0.20\,.$$

In our particular case, the estimate given by the *general* theorem is a bit larger than the actual error.

A similar argument works for the estimation of the error $|I - R_{10}|$.

18. Give an example of a function f such that R_n always underestimates the value of $\int_0^5 f(x)dx$ by the maximum amount allowed by Theorem 2.

    Answer. We know that if $f(x)$ is decreasing, then R_n underestimates the value of $\int_0^5 f(x)dx$. Choose $f(x) = 5 - x$. Then

$$\int_0^5 (5 - x)dx = 25 - \frac{25}{2} = \frac{25}{2},$$

while

$$R_n = \frac{5}{n}\sum_{i=1}^{n}(5 - \frac{5i}{n}) = \frac{25}{n^2}\sum_{i=1}^{n}(n - i) = \frac{25}{n^2}\sum_{j=1}^{n-1} j = \frac{25}{n^2}\frac{n(n-1)}{2} = \frac{25(n-1)}{2n}.$$

Again, Theorem 2 on page 433 of the textbook says that if K is such that $|f'(x)| \leq K$ for any x in $[a, b]$, then if I is the value of the integral, and n is the number of subdivisions of $[a, b]$, we have that

$$|I - R_n| \leq \frac{K(b-a)^2}{2n}.$$

Here we have $f'(x) = -1$ and therefore, we can choose $K = 1$. Thus,

$$|I - R_n| \leq \frac{(5-0)^2}{2n} = \frac{25}{2n}.$$

On the other hand, from the explicit calculations above, we see that

$$I - R_n = \frac{25}{2} - \frac{25(n-1)}{2n} = \frac{25}{2}\left(1 - \frac{n-1}{n}\right) = \frac{25}{2n}.$$

Thus, the error in this case is *as big* as the estimate given by Theorem 2.

20 c) For which of the first 6 integrals does T_{10} make no approximation error?

Answer. The trapezoidal approximation makes a larger error the more concave the graph of the function is, making no error at all if the graph is *flat*. This happens for linear functions. Thus, T_{10} makes no error for $\int_2^3 1\,dx$ and for $\int_1^3 x\,dx$.

20 d) Which integrals does T_{10} underestimate?

Answer. If the graph of f is concave-down in the domain of integration, T_n underestimates the integral of f on that interval. Since the concavity is determined by the sign of the second derivative, calculating the second derivatives of the integrands we see that T_{10} underestimates $\int_1^4 \sqrt{x}\,dx$ and $\int_2^3 \sin x\,dx$.

Section 7.3

1. Let $I = \int_0^1 e^{x^2}\,dx$.

   a) Compute M_2 and T_2 by hand.
   Answer. The function $f(x)$ is in this case $f(x) = e^{x^2}$. We subdivide the interval $[0,1]$ into $n = 2$ subintervals of equal length, $[x_0, x_1]$ and $[x_1, x_2]$ where $x_0 = 0$, $x_1 = 1/2$ and $x_2 = 1$, respectively. Then $\triangle x = (1-0)/2 = 1/2$ and

$$\begin{aligned} M_2 &= \textstyle\sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right)\triangle x \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^2 f\left(\frac{x_{i-1}+x_i}{2}\right) \\ &= \tfrac{1}{2}\left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right)\right] \\ &= \tfrac{1}{2}\left[f(\tfrac{1}{4}) + f(\tfrac{3}{4})\right] \\ &= \tfrac{1}{2}\left[e^{\frac{1}{16}} + e^{\frac{9}{16}}\right] \\ &= 0.5\,[1.064494459 + 1.755054657] \\ &= 1.409774558\,. \end{aligned}$$

   Similarly,

$$\begin{aligned} T_2 &= \textstyle\sum_{i=1}^n \frac{f(x_{i-1})+f(x_i)}{2}\triangle x \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^2 \frac{f(x_{i-1})+f(x_i)}{2} \\ &= \tfrac{1}{4}\left[f(x_0) + 2f(x_1) + f(x_2)\right] \\ &= \tfrac{1}{4}\left[1.000000 + 2\cdot 1.284025417 + 2.718281828\right] \\ &= 0.25\cdot 6.286332662 \\ &= 1.571583166\,. \end{aligned}$$

   b) Compute L_{10}, R_{10}, M_{10}, and T_{10}. Which of these underestimates the exact value of I.
   Answer. Since the function $f(x) = e^{x^2}$ increases and it is concave-up on $[0,1]$, both L_{10} and M_{10} underestimate the exact value of the integral. Similarly, R_{10} and T_{10} overestimate the value. We have:

$$\begin{aligned} L_{10} &= \tfrac{1}{10}\textstyle\sum_{i=0}^9 e^{(\frac{i}{10})^2} = \tfrac{1}{10}\sum_{i=0}^9 e^{\frac{i^2}{100}} = 1.381260601\,, \\ R_{10} &= \tfrac{1}{10}\textstyle\sum_{i=1}^{10} e^{(\frac{i}{10})^2} = \tfrac{1}{10}\sum_{i=1}^{10} e^{\frac{i^2}{100}} = 1.553088784\,, \\ T_{10} &= \tfrac{1}{20}\textstyle\sum_{i=1}^n \left(e^{\frac{(i-1)^2}{100}} + e^{\frac{i^2}{100}}\right) = 1.467174693\,, \\ M_{10} &= \tfrac{1}{10}\textstyle\sum_{i=1}^n e^{\frac{i}{10}-\frac{1}{20}} = 1.460393091 \end{aligned}$$

   c) Use Theorem 3 to find n such that the error $|I - M_n| \le 0.0005$.
   Answer. Since $f''(x) = (4x^2+2)e^{x^2}$, on $[0,1]$ this is bounded by its value at $x = 1$, that is to say, by $6e$. Then,

$$|I - M_n| \le \frac{6e(1-0)^3}{24n^2} = \frac{e}{4n^2}\,.$$

Since we want the error to be less than 0.0005, we must choose n such that

$$\frac{e}{4n^2} \leq 0.0005\,.$$

Solving for n, we obtain

$$n \geq \sqrt{e/(4 \cdot 0.0005)} > 36.866\,.$$

Hence, we just need to choose $n = 37$ or greater to ensure the desired accuracy.

3. Suppose that $f''(x) = \frac{e^x \cos x}{1+x^2}$. Find an integer n such that T_n approximates $\int_0^5 f(x)dx$ within 0.001.

   Answer. According to Theorem 3 on page 440 of the textbook, if K is an upper bound for $|f''(x)|$ on the interval $[a,b]$, then if I is the value of the integral and T_n is the trapezoidal approximation with n subdivisions of $[a,b]$, we have that

$$|I - T_n| \leq \frac{K(b-a)^3}{12n^2}\,.$$

   In this problem, we know that $f''(x) = \frac{e^x \cos x}{1+x^2}$. The absolute value of this function has a maximum in between $x = 3$ and $x = 4$, which is not greater than 2.5 (in fact, it is not greater than 2.4). Hence,

$$|f''(x)| \leq 2.5\,,$$

   and we have that

$$|I - T_n| \leq \frac{2.5(5-0)^3}{12n^2}\,.$$

   We want the error to be no larger than 0.001. So we choose n such that

$$\frac{2.5(5-0)^3}{12n^2} = \frac{26.41666}{n^2} \leq 0.001\,.$$

   Solving for n, we see that

$$n \geq 161.37\,.$$

   Thus, we may choose $n = 162$ or greater to ensure an accuracy better than 0.001.

14. Using left, right, midpoint and trapezoidal rules, we obtained approximations of $\int_a^b f(x)dx$ using the same number of subdivisions of the interval $[a,b]$. These approximations were 8.52974, 9.71090, 9.74890 and 11.04407. Which rule produce which estimate?

    Answer. The function depicted in the graph is increasing and concave-up. Hence, both left and midpoint approximation underestimates the interval, and right and trapezoidal overestimate it. But we know that the midpoint approximation is a better approximation to the integral than the left approximation, while the trapezoidal is better than the right approximation. Hence, we have

$$L_n < M_n < I < T_n < R_n\,.$$

    Thus, L_n is the smallest of the given approximations, while R_n is the largest. M_n is the second smallest. So we get:

$$L_n = 8.52974\,, \quad M_n = 9.71090\,, \quad T_n = 9.74890\,, \quad R_n = 11.04407\,.$$