

**Stony Brook University**  
**MAT 320**  
**Introduction to Analysis**  
**Final Examination with Solutions**

December 21, 2007

WORK ANY FIVE PROBLEMS. TELL US CLEARLY WHICH ONES YOU HAVE CHOSEN.

1. Show how the Bolzano-Weierstrass Theorem (Every bounded sequence contains a convergent subsequence) follows from the Least Upper Bound Axiom (Every bounded set of real numbers has a least upper bound).

- (Solution) This is from the book. You need to show first that every sequence contains a monotone subsequence. Then show that a bounded monotone sequence converges.

2. Prove: If  $f: \mathbf{R} \rightarrow \mathbf{R}$  is positive, bounded away from zero (i.e.  $\exists K > 0$  such that  $|f(x)| \geq K$  for all  $x$ ) and differentiable, then  $g(x) = \sqrt{f(x)}$  is also differentiable.

- (Solution) The Chain rule tells us that if  $f$  is differentiable at  $x$  and if  $H$  is differentiable at  $f(x)$  then  $H \circ f$  is differentiable at  $x$  (and in fact  $(H \circ f)'(x) = H'(f(x))f'(x)$ ). Here  $H$  is the square-root function, which is differentiable at all  $x > 0$ . (“Bounded away from zero” was not necessary for this argument.)

3. Prove: if  $f'$  is continuous and  $f'(x) > 0$ , then  $f$  is increasing on a neighborhood of  $x$ . Show by example that the hypothesis “ $f'$  is continuous” is necessary.

- (Solution) If  $f'$  is and  $f'(x) > 0$ , then there exists a  $\delta$ -neighborhood  $V$  of  $x$  where  $f' > 0$ . Let  $y_1$  and  $y_2$  be any two points in that neighborhood, with  $y_1 < y_2$ . The Mean Value Theorem tells us that  $f(y_2) - f(y_1) = (y_2 - y_1)f'(c)$  for some  $c$ ,  $y_1 < c < y_2$ . Since  $y_1$  and  $y_2$  are in  $V$ , so is  $c$ . Since  $y_2 - y_1$  and  $f'(c)$  are both positive, so is  $f(y_2) - f(y_1)$ . *It is not enough to show that  $x$  is above points to its left and below points to its right.*

The example given in the book  $f(x) = x + 2x^2 \cos(\frac{1}{x})$  and  $f(0) = 0$  has

$$f'(0) = \lim_{h \rightarrow 0} (h + 2h^2 \cos(\frac{1}{h})) / h = \lim_{h \rightarrow 0} (1 + 2h \cos(\frac{1}{h})) = 1$$

but  $f'(x) = 1 + 2 \sin(\frac{1}{x}) + 4x \cos(\frac{1}{x})$  which takes on positive and negative values arbitrarily near 0.

4. Use the Squeeze Theorem (see below) to show that the function defined on  $[0, 1]$  by

$$f(x) = \begin{cases} 1 & x = \frac{1}{n}, n \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable with integral 0.

- (Solution) There was a similar homework problem with  $\sin(\frac{1}{x})$ . Given  $\epsilon > 0$ , define  $\alpha_\epsilon$  to be 0, and  $\omega_\epsilon(x) = 1$  for  $x \leq \epsilon/2$ . Now  $1/n > \epsilon/2$  when  $n < 2/\epsilon$ . Let  $N$  be the smallest integer  $> 2/\epsilon$ . There are at most  $N - 1$  points  $1, 1/2, \dots, 1/(N - 1)$  outside  $[0, \epsilon/2]$  where  $f(x) \neq 0$ . Let  $I_j$  be the interval  $[1/j - \epsilon/4N, 1/j + \epsilon/4N]$  of length  $\epsilon/2N$  about  $1/j$ , and let  $\omega_\epsilon(x) = 1$  on the union of the  $I_j$ ,  $1 \leq j \leq N - 1$ . Now  $\omega_\epsilon$  is a step function on  $[0, 1]$  and hence integrable; the total length of the intervals where it is non-zero is less than or equal to  $\epsilon/2 + (N - 1)\epsilon/2N < \epsilon$ . This  $\alpha_\epsilon$  and  $\omega_\epsilon$  satisfy the hypotheses of the squeeze theorem, and this can be done for any  $\epsilon$ . *Note that in general for a step function to be Riemann integrable it can only have a finite number of discontinuities.*

5. If  $f: \mathbf{R} \rightarrow \mathbf{R}$  is continuous and  $c > 0$ , define  $g: \mathbf{R} \rightarrow \mathbf{R}$  by  $g(x) = \int_{x-c}^{x+c} f(t)dt$ . Show that  $g$  is differentiable on  $\mathbf{R}$  and calculate  $g'(x)$ .

- (Solution) We need to calculate

$$\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{[\int_{x+h-c}^{x+h+c} - \int_{x-c}^{x+c}] f(x) dx}{h}.$$

Using additivity we can add and subtract  $\int_{x+c}^{x+h-c}$  make this

$$\lim_{h \rightarrow 0} \frac{[\int_{x-c+h}^{x-c} - \int_{x+c+h}^{x+c}] f(x) dx}{h}.$$

Let  $J$  be an interval large enough to contain all of the points in question. Then  $f$ , being continuous, is in  $\mathcal{R}(J)$  and there has an anti-derivative  $F$ . Our difference quotient becomes

$$\lim_{h \rightarrow 0} \left[ \frac{F(x-c) - F(x-c+h)}{h} - \frac{F(x+c) - F(x+c+h)}{h} \right] = -f(x-c) + f(x+c)$$

by the Fundamental Theorem II.

6. Prove: A uniform limit of continuous functions is continuous.

- (Solution) Suppose  $(f_n)$  is converging uniformly to  $f$ , and consider a point  $x$  in their common domain. Given  $\epsilon$ , we can find an  $N$  such that if  $n \geq N$ , then  $|f_n(z) - f(z)| < \epsilon/3$  for all  $z$  (since the convergence is uniform). Look at  $f_N$  itself. Since it is continuous at  $x$  we can find  $\delta$  so that  $|h| < \delta$  implies  $|f_N(x+h) - f_N(x)| < \epsilon/3$ . Then if  $|h| < \delta$ ,

$$\begin{aligned} |f(x+h) - f(x)| &\leq |f(x+h) - f_N(x+h)| + |f_N(x+h) - f_N(x)| + |f_N(x) - f(x)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

7. (a) Use what we know about power series to prove that if  $|x| < 1$ , then

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \ln(1+x).$$

- (b) Now show that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln(2).$$

- (Solution) (a) The geometric series  $\sum_{n=0}^{\infty} x^n$  has interval of convergence  $(-1, 1)$  and converges to  $\frac{1}{1-x}$  on that interval. Using  $-x$  for  $x$ , the series  $\sum_{n=0}^{\infty} (-1)^n x^n$  converges to  $\frac{1}{1+x}$  on that same interval. We know that the term-by-term integration (from, say, 0) of the series yields a series with the same radius of convergence, which converges to the integral of the limit function:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{has limit} \quad \ln(1+x)$$

for  $|x| < 1$ .

- (b) The series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$  has interval of convergence  $(-1, 1]$  by the above plus the Alternating Series Test. We know that on any closed interval contained in the interval of convergence, the series converges uniformly to its limit. Therefore the function it defines is continuous. Since it is equal to  $\ln(1+x)$  on  $(-1, 1)$ , it must be equal to  $\ln(1+x)$  at 1, so the sum there is  $\ln 2$ .

[The Squeeze Theorem states that  $f$  is Riemann integrable on  $[a, b]$  if and only if  $\forall \epsilon > 0$  there exist functions  $\alpha_\epsilon$  and  $\omega_\epsilon$  such that

- $\alpha_\epsilon$  and  $\omega_\epsilon$  are Riemann integrable on  $[a, b]$
- $\alpha_\epsilon(x) \leq f(x) \leq \omega_\epsilon(x)$  for all  $x \in [a, b]$
- $\int_a^b (\omega_\epsilon - \alpha_\epsilon) < \epsilon$  ]