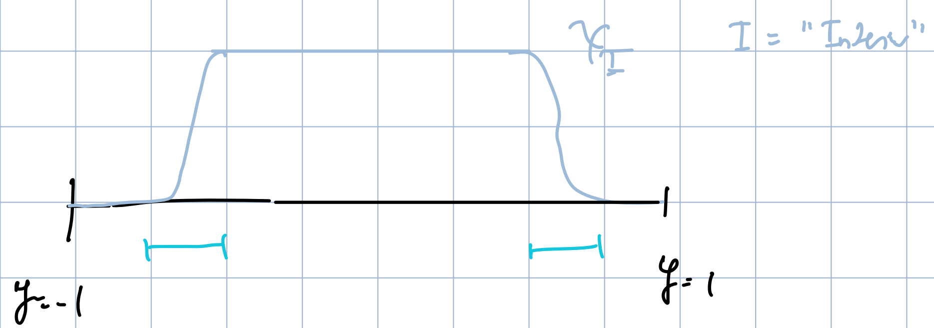
Lecture 4 - Pseudo - Gevrey Spaces

4.1 Motivation for Pseudo - Gevrey

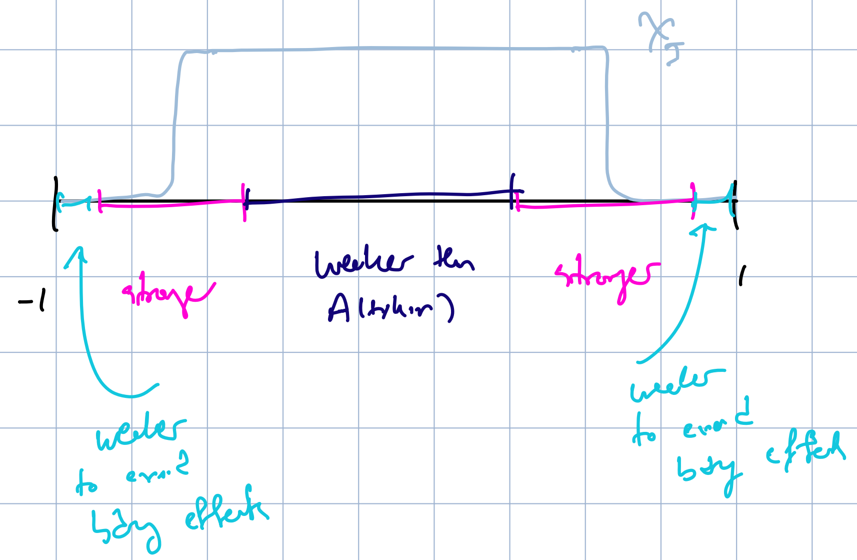


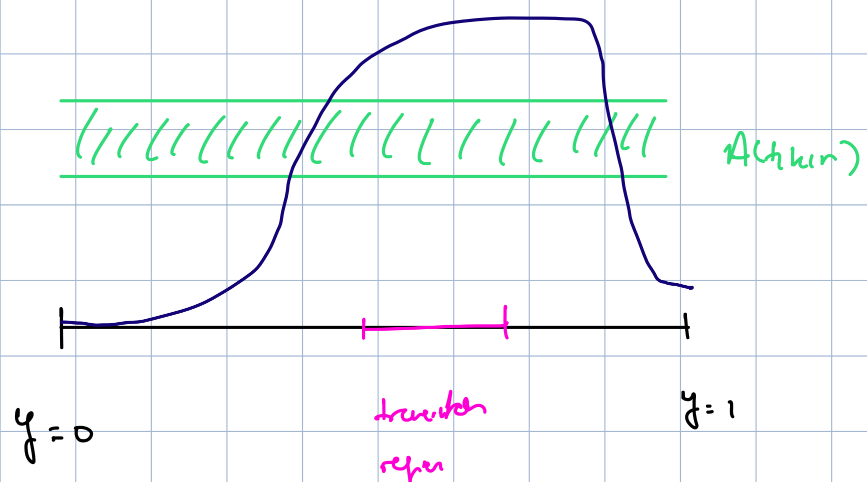
- Assume we $\omega^I = \omega \cdot \chi_I$, Interior Vorticity on which we plan to use nonlinear Inviscid Damping techniques (BM '15, IJ '20)

$$(\partial_t + y\partial_x - \nu\Delta)\,\omega^I = \underbrace{-\nu(\chi^I)''\omega}\;\underbrace{-\,2\nu(\chi^I)'\omega}$$

Commutators from Exterior, supported in transition region

. Assume we now want to study the global vorticity, ω.





Therefore:

Gevrey $\quad \| e^{\lambda |\xi|^{s}} \hat f \|_{L^2}$

Pseudo-Gevrey $\quad \| e^{\lambda |\xi|^{s(y)}} \hat f \|_{L^2}$ spatially dependent index.

The idea of assigning regularity depending on spatial location has to do w/ smoothing.

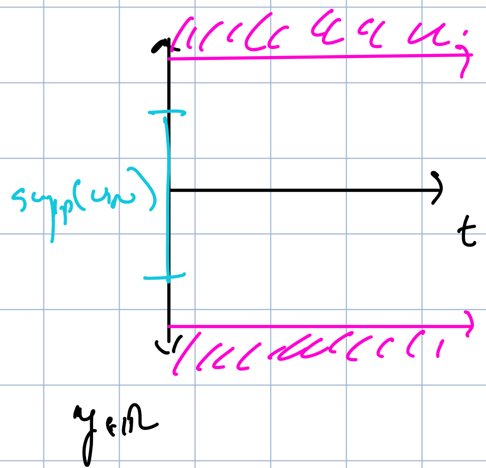
4.2 Basics of Parabolic Smoothing.

Q: Consider

$$\begin{cases} \partial_t u - \nu \partial_y^2 u = 0 & \text{on } y \in \mathbb{R} \\ u|_{t=0} = u_{in}(y) \in L^2 \end{cases}$$

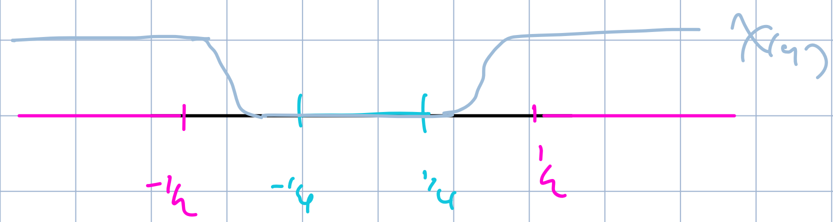
Assume $\operatorname{supp}(u_{in}) \subseteq [-1/4, 1/4]$

How do we show that u is H^1_y on $[-1/2, 1/2]^c$? (Use energy method!)



A: Fix a cutoff function $\chi(y)$ satisfying

$$\chi = 1 \text{ on } (-1/2, 1/2)^c, \quad \chi = 0 \text{ on } (-1/4, 1/4)$$



Step 1: $\langle \partial_t u - \nu \partial_y^2 u, u \rangle = 0$

$\hookrightarrow \frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|\partial_y u\|^2 = 0$

$\hookrightarrow \nu \|\partial_y u\|^2_{L^2_t L^2_y} \leq \|u_{in}\|^2_{L^2}$

Step 2: Commute vector-field $\frac{d}{dy}$ to the eqn.

$$(\partial_t - \nu \partial_y^2)\, \partial_y u = 0$$

Step 3: Localize by defining χ

Step 4: Multiplier of $\partial_y u\, \chi(y)^2$:

$$\langle (\partial_t - \nu \partial_y^2)\, \partial_y u, \partial_y u\, \chi(y)^2 \rangle$$

$\hookrightarrow$

$$\frac{1}{2} \frac{d}{dt} \|\partial_y u\, \chi\|^2_{L^2} + \nu \|\partial_y^2 u\, \chi\|^2$$

$$+ 2\nu \int \partial_y^2 u\, \partial_y u\, \chi(y) \chi'(y) = 0$$

$\hookrightarrow \frac{1}{2} \frac{d}{dt} \|\partial_y u \, x\|_{L^2}^2 + \nu \|\partial_y^2 u \, x\|^2$

$$- \nu \int |\partial_y u|^2 \left(x(y)\, x'(y)\right)' \, dy = 0$$

$\hookrightarrow \sup_t \frac{1}{2} \|\partial_y u \, x\|_{L^2}^2 + \nu \|\partial_y^2 u \, x\|_{L^2_t L^2_y}^2$

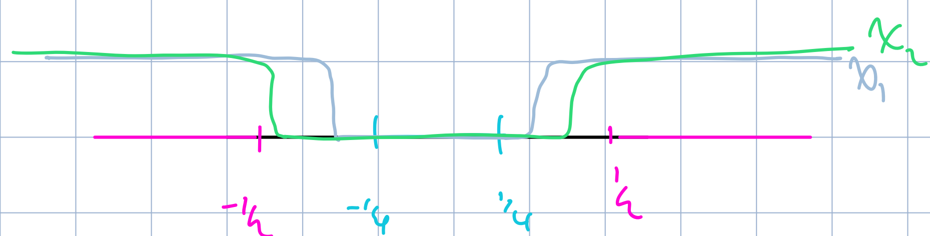
$$= \int_t \int \nu |\partial_y u|^2 \left(x(y)\, x'(y)\right)' \quad \leq \|\nabla_y u\|^2$$

$$+ \frac{1}{2} \underbrace{\int |\partial_y u_{in}|^2 x(y)^2}_{=0}$$

$\boxed{=}$

Q: How about H^2?

A:



4.3 Parabolic Smoothing in Gevrey Spaces

Q: What about $G_{1,s}$?

A:

infinite cascade of cutoffs.

Define $x_0 = 1/4$

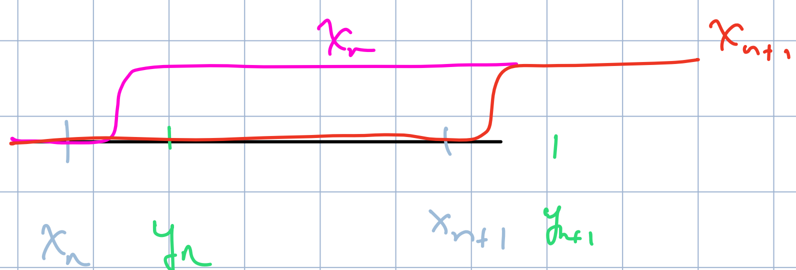
$$x_{n+1} = x_n + \frac{c_\sigma}{n^{1+\sigma}}$$

$$y_n = x_n + \frac{c_\sigma}{100\, n^{1+\sigma}}$$

$0 < \sigma < s-1$

Construction does not work for $s = 1$ analytically.

Then x_n is defined



By defn. we chose c_σ so that all the x_n's, y_n's are contained before $1/2$.

Claim: ω is instantaneously $G_{1,s}$, $s<1$, on $(-\frac{1}{4},\frac{1}{4})^c$

Pf: We induct on n.

Step 1: Commute ∂_y^{n+1}:

$$(\partial_t - \nu\, \partial_y^2)\, \partial_y^{n+1} \omega = 0$$

Step 2: Inner-product w/ $\partial_y^{n+1}\omega\, \chi_{n+1}^2$

$$\langle (\partial_t - \nu \partial_y^2)\, \partial_y^{n+1}\omega, \partial_y^{n+1}\omega\, \chi_{n+1}^2 \rangle$$

$$\hookrightarrow \quad \frac{\partial_t}{2} \| \partial_y^{n+1}\omega\, \chi_{n+1} \|^2 + \nu \| \partial_y \partial_y^{n+1}\omega\, \chi_{n+1} \|^2$$

$$= \frac{\nu}{2} \int |\partial_y^{n+1}\omega|^2\, \partial_y^2(\chi_{n+1}^2)$$

Step 3: Commutator Estimate

$$O\left(\nu \left(\int |\partial_y^{n+1}\omega|^2\, \chi_n^2\right) (n+1)^{2(1+\sigma)}\right)$$

Step 4: Multiply by Gevrey scaling factor of $\left(\frac{1}{(n+1)!}\right)^{2s}$

$$\frac{\partial_t}{2}\left(\frac{1}{(n+1)!}\right)^{2s}\|\partial_y^{n+1}\omega\,\chi_{n+1}\|^2 + \nu\left(\frac{1}{(n+1)!}\right)^{2s}\|\partial_y\partial_y^{n+1}\omega\,\chi_{n+1}\|^2$$

$$\lesssim \nu\,\frac{(n+1)^{2(1+\sigma)}}{((n+1)!)^{2s}}\,\|\partial_y\partial_y^{n}\omega\,\chi_n\|^2$$

$$= \underbrace{\frac{(n+1)^{2(1+\sigma)}}{(n+1)^{2s}}}_{\lesssim 1.}\;\underbrace{\frac{\nu}{(n!)^{2s}}\,\|\partial_y\partial_y^{n}\omega\,\chi_n\|^2}_{\lesssim \|\omega_{in}\|^2}$$

$\square$

Remarks:

(1) In the real implementation ∂_y is replaced w/ a commuting vector field, Γ

(2) Spatial weights are implemented which quantify rate of spreading.

(3) Open how to do this for $s=1$ (Analytic).