Lecture 3 - Nonlinear Results & Gevrey Spaces

Defn: $s>0$, $\lambda>0$, Gevrey class $1/s$ w/ radius λ is

$$\|f\|^2_{G_{\lambda,s}} := \int |\hat{f}(\xi)|^2 e^{2\lambda|\xi|^s} d\xi$$

Rmk: $s=1$, Analytic

Nonlinear Results

* We focus on Euler / NS incompressible, Dim 2, Near Couette, Uniform Results

		$\mathbb{T}\times\mathbb{R}$	$\mathbb{T}\times[-1,1]$
Euler	$\nu=0$	BM '13	IJ '20
NS	$\nu>0$	BMV '15	BHIW '24

* All results need Gevrey spaces in Fourier side (and subtle freq side weights).

Thm (Bedrossian - Masmoudi) $s > 1/2$, λ fixed $O(1)$.

If $\omega_{in} \in G_{\lambda,s}$, $\|\omega_{in}\|_{G_{\lambda,s}} := \varepsilon \ll 1$,

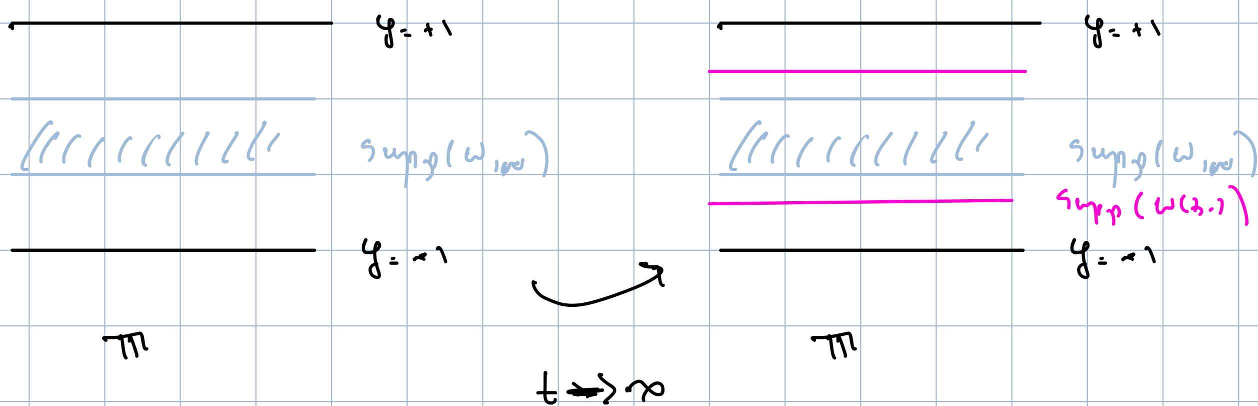
$$(1) \quad \|P_{\neq 0}\, \omega\|_{L^2} \lesssim \varepsilon/\langle t\rangle$$

$$(2) \quad \|v\|_{L^2} \lesssim \varepsilon/\langle t\rangle^2$$

(Discussion Postponed)

Thm (BMV) Same as BM'13 for $\nu \geq 0$, w/ enhanced dissipation effect.

Thm (IJ '20) Same as BM '13, w/ compact support ω_{in} away from $y = \pm 1$



Remark: Due to Inviscid Damping estimate, obtained

$$\partial_t \omega + (y + U)\partial_x \omega + \underbrace{v}_{\sim \frac{1}{\langle t\rangle^2}} \partial_y \omega = 0$$

* w/ Boundaries, we don't have access to Fourier transform near the bdry. → need Pseudo - Gevrey spaces

Fourier - Side Weights & Orr - Critical Time

(Following BM '13, 2D Euler)

$$\partial_t \omega + y\,\partial_x \omega = -\nabla^\perp f \cdot \nabla \omega$$

↳ $\partial_t \omega + (y + P_{=0}\, u(t,y))\,\partial_x \omega = -\nabla^\perp f_{\neq 0} \cdot \nabla \omega$

↳ $\partial_t f = -\nabla^\perp_{z,v} \phi_{\neq 0} \cdot \nabla_{z,v} f$ (cancellation here)

$$\approx -\partial_y \phi_{\neq 0}\, \partial_z f_{lo}$$

$$\partial_z^2 + (\partial_y - t\partial_z)^2 \phi = f$$

(Model Pb)

$$\hat{\phi}(t,k,\eta) = \frac{\hat{f}(t,k,\eta)}{k^2 + (\eta - kt)^2}$$

($t = \eta/k$ is called Orr - Critical Time).

From here, we can derive a Toy Problem

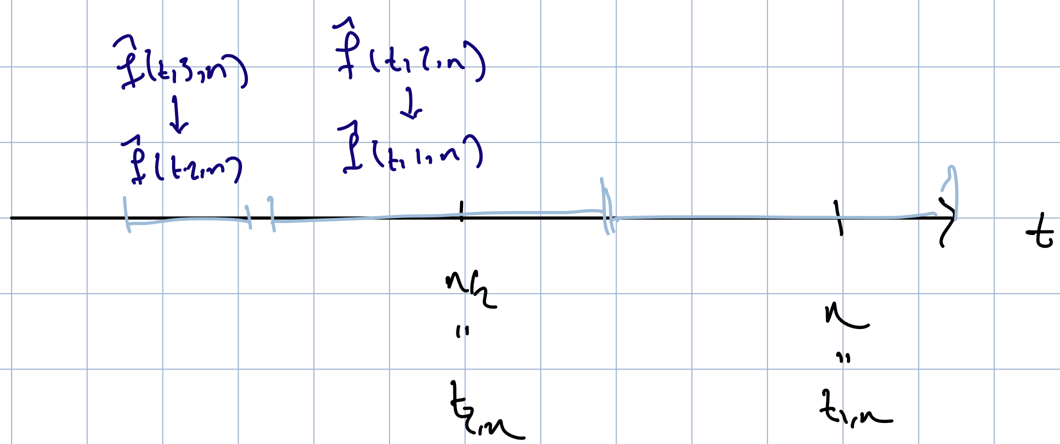
$$\partial_t f(t,k;n) = \sum_{l\neq 0}\int_{\xi} \frac{\xi\,(k-l)}{l^2+(\xi-lt)^2}\,\hat f(t,l,\xi)\,\hat f_{lo}(t,k-l,n-\xi)$$

$$\simeq \varepsilon \sum_{l\neq 0} \frac{n\,(k-l)}{l^2+(n-lt)^2}\,\hat f(t,l,n)$$

$$\simeq \varepsilon\, \frac{n}{(k+1)^2+(n-(k+1)t)^2}\,\hat f(t,k+1,n)$$

We think of a fixed, system of ODEs infinitely coupled in $k\in\mathbb{Z}$.

· Our Critical Times: For given (n,k), $t_{k,n} = n/k$



· This leads to the defn. of a freq. weight function $A(t,k,n)^{-1}$ which mimics the worst possible growth of $\hat f(t,k,n)$

· $E[f] = \|A\hat f\|_{L^2}^2$

Thm (Bedrossian - He - I - Wang
3 papers)

Consider

$$\left.\begin{array}{l}\partial_t \omega + y\partial_x \omega - \nu\Delta\omega = -\nabla^{\perp}\phi \cdot \nabla\omega \\ \omega|_{t=0} = \omega_{in}(x,y) \\ \omega|_{y=\pm 1} = 0 \qquad \text{(Navier BCs)}\end{array}\right| \quad \mathbb{T} \times [-1,1]$$

Assume

$$\left|\begin{array}{l}\|\omega_{in}\|_{G_{2,\lambda}} := \varepsilon \ll 1 \\ \text{supp}(\omega_{in}) \subseteq (-\tfrac{1}{4}, \tfrac{1}{4})\end{array}\right.$$

Then

(1) $\|P_{\neq 0}\, u\|_{L^2} \lesssim \frac{\varepsilon}{\langle t\rangle} \exp(-c\,\nu^{1/3} t)$

$\|v\|_{L^2} \lesssim \frac{\varepsilon}{\langle t\rangle^2} \exp(-c\,\nu^{1/3} t)$ (Inviscid Damping)

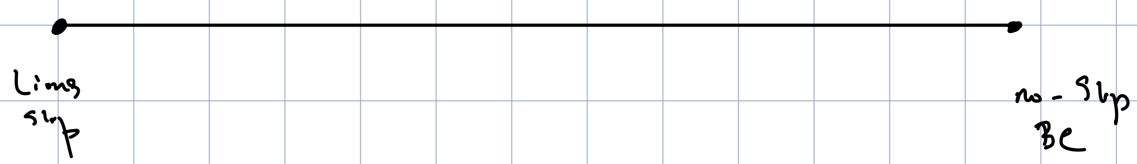
(2) $\|P_{\neq 0}\, \omega\|_{L^2} \lesssim \varepsilon \exp(-c\,\nu^{1/3} t)$ (Enhanced Dissipation)

(3) $\|\omega^{(\nu)} - \omega^{(0)}\|_{L^\infty} \lesssim \varepsilon\, \nu\, t^{3+n}$ (Inviscid Limit)

on $0 \le t \le \nu^{-\frac{1}{3+n}}$, $n > 0$

Boundary

Navier BC: $\omega\big|_{y=\pm1} = 0$



$$\omega + \alpha u\big|_{y=\pm1} = 0$$

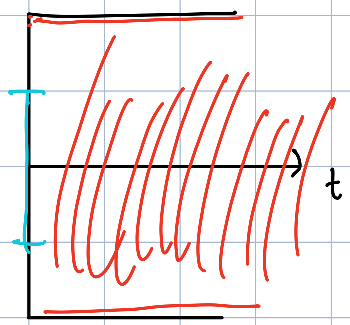
Infinite Speed of Propagation

$\partial_t \omega - \nu \partial_y^2 \omega = 0 \quad$ on $[-1,1]$

$\omega\big|_{y=\pm1} = 0$

$\mathrm{supp}(\omega_{in}) \subseteq (-1/4, 1/4).$

Initially $\partial_y^j \omega\big|_{y=\pm1} = 0$, but $\partial_y^j \omega\big|_{y=\pm1} \neq 0$ for any $t > 0$.



↳ Convert to the Fourier transform.

<u>Physical Side Gevrey Spaces</u>

Defn:
$$\|f\|^2_{H_{\lambda,s}} = \sum_{n=0}^{\infty} \left(\frac{\lambda^n}{n!}\right)^{2/s} \|\partial_x^n f\|^2_{L^2}$$

Remark: Also L^∞ based Gevrey spaces, etc...

Prop'n: $\|f\|_{G_{\lambda,s}} \sim \|f\|_{H_{\lambda,s}}$