Lecture 2 : Hypocoercivity

Goal: To prove Enhanced Dissipation via energy methods

History: - Originally introduced by Villani, Guo for kinetic

- Bedrossian - Coti Zelati

- Bedrossian - He - I. Wang

Model pb:

$$\partial_t w + y\,\partial_x w - \nu\Delta w = 0$$

$$\partial_t w + U(y)\,\partial_x w - \nu\Delta w = 0, \qquad U \text{ is non-degenerate}$$

- Method can handle nonlinearities!

$$\partial_t w + y\,\partial_x w - \nu\Delta w = -u\cdot\nabla w,$$

and is a good tool to prove threshold thms.

* We will study the problem

$$\partial_t f + y\partial_x f - \nu \partial_y^2 f = 0$$

Go to Fourier in x: $\quad f_k(t,y) = \int_{\mathbb{T}}$

$$\partial_t f_k + iky f_k - \nu \partial_y^2 f_k = 0$$

Remark: (Scaling considerations)

$$iky \approx \nu \partial_y^2 \rightarrow (ik)^{1/3} \sim \nu^{1/3} \partial_y$$

· Define:

$$\Phi[f_k](t) = \gamma \|f_k\|^2 + 2\beta \nu^{1/3} \operatorname{Re} |k|^{-4/3} \langle ik f_k, \partial_y \overline{f_k} \rangle$$

$$+ \alpha \nu^{2/3} |k|^{-2/3} \|\partial_y f_k\|^2,$$

where $\langle f, g \rangle$ is the real inner product $= \int f(x) \overline{g(x)}\, dx.$

Prop'n: $\frac{d}{dt}\Phi[f_k](t)$

$$\leq -2\alpha\nu^{2/3}|k|^{-2/3}\|\nu^{1/2}\partial_y^2 f_k\|^2$$

$$-2\alpha\nu^{2/3}|k|^{-2/3}\langle ikf_k, \partial_y \bar{f}_k\rangle - 2\gamma\|\nu^{1/2}\partial_y f_k\|^2$$

$$-2\beta|k|^{2/3}\nu^{1/3}\|f_k\|_{L^2}^2$$

$$+4\beta\,\mathrm{Re}\,|k|^{-4/3}\nu^{1/3}\langle ik\nu\partial_y^2 f_k, \partial_y \bar{f}_k\rangle$$

Lemma: $\frac{d}{dt}\gamma\|f_k\|^2 = -2\gamma\|\nu^{1/2}\partial_y f_k\|^2$.

Pf:
$$\frac{d}{dt}\gamma\|f_k\|^2 = \frac{d}{dt}\gamma\langle f_k, \bar{f}_k\rangle$$

$$= \gamma\langle \tfrac{d}{dt}f_k, \bar{f}_k\rangle + \gamma\langle f_k, \tfrac{d}{dt}\bar{f}_k\rangle$$

$$= \gamma\langle \nu\partial_y^2 f_k, \bar{f}_k\rangle - \gamma\langle ikyf_k, \bar{f}_k\rangle$$

$$+\gamma\langle f_k, \nu\partial_y^2\bar{f}_k\rangle - \gamma\langle f_k, \overline{ikyf_k}\rangle$$

$$= -2\gamma\|\nu^{1/2}\partial_y f_k\|^2 - \gamma\langle ikyf_k, \bar{f}_k\rangle$$

$$+\gamma\langle f_k, iky\bar{f}_k\rangle$$

$$= -2\gamma \| \nu^{1/2} \partial_y f_k \|^2.$$

$\boxed{}$

Lemma:

$$\frac{d}{dt} \alpha \nu^{2/3} |k|^{-2/3} \| \partial_y f_k \|^2$$

$$= -2\alpha \nu^{2/3} |k|^{-2/3} \| \nu^{1/2} \partial_y^2 f_k \|^2$$

$$- 2\alpha \nu^{2/3} |k|^{-2/3} \langle ik f_k, \partial_y \overline{f_k} \rangle$$

Pf:

$$\frac{d}{dt} \alpha \nu^{1/3} |k|^{-2/3} \langle \partial_y f_k, \partial_y \overline{f_k} \rangle$$

(*) These should be changed from $\nu^{1/3} \mapsto \nu^{2/3}$?!

$$= \alpha \nu^{1/3} |k|^{-2/3} \langle \partial_y \partial_t f_k, \partial_y \overline{f_k} \rangle$$

$$+ \alpha \nu^{1/3} |k|^{-2/3} \langle \partial_y f_k, \partial_y \partial_t \overline{f_k} \rangle$$

$$= \alpha \nu^{1/3} |k|^{-2/3} \langle \partial_y (\nu \partial_y^2 f_k - iky f_k), \partial_y \overline{f_k} \rangle$$

$$+ \alpha \nu^{1/3} |k|^{-2/3} \langle \partial_y f_k, \partial_y (\nu \partial_y^2 \overline{f_k} + iky \overline{f_k}) \rangle$$

$$= -2\alpha \nu^{1/3} |k|^{-2/3} \| \nu^{1/2} \partial_y^2 f_k \|^2$$

$$- \alpha \nu^{1/3} |k|^{-2/3} \langle ik f_k, \partial_y \overline{f_k} \rangle$$

Commutators w/ ∂_y

$$\alpha \nu^{1/3} |k|^{-2/3} \langle \ldots$$

$$+ \, 2\alpha\nu^{1/3} |k|^{-2/3} \langle \partial_y f_k, ik f_k \rangle$$

$$= -2\alpha\nu^{1/3} |k|^{-2/3} \|\nu^{1/2} \partial_y^2 f_k\|^2$$

$$- 2\alpha\nu^{1/3} |k|^{-2/3} \langle ik f_k, \partial_y \bar{f}_k \rangle$$

$\boxtimes$

<u>Lemma</u>:
$$\frac{d}{dt} 2\beta \, \mathrm{Re} \, |k|^{-4/3} \nu^{1/3} \langle ik f_k, \partial_y \bar{f}_k \rangle$$
$$= -2\beta |k|^{2/3} \nu^{1/3} \|f_k\|^2$$
$$+ 4\beta \, \mathrm{Re} \, |k|^{-4/3} \nu^{1/3} \langle ik\nu \partial_y^2 f_k, \partial_y \bar{f}_k \rangle$$

<u>Pf</u>:
$$\frac{d}{dt} 2\beta \, \mathrm{Re} \, |k|^{-4/3} \nu^{1/3} \langle ik f_k, \partial_y \bar{f}_k \rangle$$

$$= 2\beta \, \mathrm{Re} \, |k|^{-4/3} \nu^{1/3} \langle ik \partial_t f_k, \partial_y \bar{f}_k \rangle$$
$$+ 2\beta \, \mathrm{Re} \, |k|^{-4/3} \nu^{1/3} \langle ik f_k, \partial_y \partial_t \bar{f}_k \rangle$$

$$= 2\beta \, \mathrm{Re} \, |k|^{-4/3} \nu^{1/3} \langle ik (-iky f_k + \nu \partial_y^2 f_k), \partial_y \bar{f}_k \rangle$$
$$+ 2\beta \, \mathrm{Re} \, |k|^{-4/3} \nu^{1/3} \langle ik f_k, \partial_y (iky \bar{f}_k + \nu \partial_y^2 \bar{f}_k) \rangle$$

$$= \cancel{2\beta \, \mathrm{Re} \, |k|^{-4/3} \nu^{1/3} \langle k^2 y f_k, \partial_y \bar{f}_k \rangle}$$

$$+ 2\beta \,\mathrm{Re}\, |k|^{-4/3} \nu^{1/3} \langle ik\nu\, \partial_y^2 f_k, \partial_y \bar{f}_k \rangle$$

$$+ 2\beta \,\mathrm{Re}\, |k|^{-4/3} \nu^{1/3} \langle -k^2 f_k, \bar{f}_k \rangle \qquad \text{(Main - term)}$$

$$\cancel{+ 2\beta \,\mathrm{Re}\, |k|^{-4/3} \nu^{2/3} \langle ik f_k, iky\, \partial_y \bar{f}_k \rangle}$$

$$- 2\beta \,\mathrm{Re}\, |k|^{-4/3} \nu^{2/3} \langle ik\, \partial_y f_k, \nu \partial_y^2 \bar{f}_k \rangle$$

$$= -2\beta |k|^{2/3} \nu^{1/3} \|f_k\|_{L^2}^2 + 4\beta \,\mathrm{Re}\, |k|^{-4/3} \nu^{1/3} \langle ik\nu\, \partial_y^2 f_k, \partial_y \bar{f}_k \rangle$$

$\square$

We now use this hypocoercivity estimate to prove our main result:

$$\frac{d}{dt} \Phi[f_k](t)$$

$$\leq -2\alpha \nu^{2/3} |k|^{-2/3} \|\nu^{1/2} \partial_y^2 f_k\|^2$$

$$- 2\alpha \nu^{2/3} |k|^{-2/3} \langle ik f_k, \partial_y \bar{f}_k \rangle - 2\gamma \|\nu^{1/2} \partial_y f_k\|^2$$

$$- 2\beta |k|^{2/3} \nu^{1/3} \|f_k\|_{L^2}^2$$

$$+ 4\beta \,\mathrm{Re}\, |k|^{-4/3} \nu^{1/3} \langle ik\nu\, \partial_y^2 f_k, \partial_y \bar{f}_k \rangle$$

$$\leq \; -2\alpha \nu^{2/3} |k|^{-2/3} \|\nu^{1/2} \partial_y^2 f_k\|^2 - 2\gamma \|\nu^{1/2} \partial_y f_k\|^2$$

$$-2\beta |k|^{2/3} \nu^{1/3} \|f_k\|_{L^2}^2$$

$$+ \; 2\alpha |k|^{1/3} \nu^{2/3} \|f_k\|_{L^2} \|\partial_y f_k\|_{L^2}$$

$$+ \; 4\beta |k|^{-1/3} \|\nu^{1/2 + 1/3} \partial_y^2 f_k\| \; \|\nu^{1/2} \partial_y f\|$$

$$\leq \; -2\alpha \nu^{2/3} |k|^{-2/3} \|\nu^{1/2} \partial_y^2 f_k\|^2 - 2\gamma \|\nu^{1/2} \partial_y f\|^2$$

$$-2\beta |k|^{2/3} \nu^{1/3} \|f\|^2$$

$$+ \; 2 \frac{\alpha}{\sqrt{\beta}} \left(\sqrt{\beta} |k|^{1/3} \nu^{1/6} \|f\|\right)\left(\|\nu^{1/2} \partial_y f\| \right)$$

$$+ \; 4 \frac{\beta}{\sqrt{\alpha}} \left(\sqrt{\alpha} |k|^{-1/3} \|\nu^{5/6} \partial_y^2 f\|\right)\left(\|\nu^{1/2} \partial_y f\|\right)$$

$$\leq \; -\alpha \nu^{2/3} |k|^{-2/3} \|\nu^{1/2} \partial_y^2 f\|^2$$

$$-\left(2\gamma - C\frac{\alpha^2}{\beta} - C\frac{\beta^2}{\alpha}\right) \|\nu^{1/2} \partial_y f\|^2$$

$$-\beta |k|^{2/3} \nu^{1/3} \|f\|^2$$

$$\leq \quad -\gamma \, \|\nu^{1/2} f_y\|^2 - \beta\, |k|^{2/3} \nu^{1/3} \|f\|^2$$

If we take $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha} \ll \gamma$

Lemma: Assume $\beta^2/\alpha \ll \gamma$. Then

$$\Phi \leq \frac{\gamma}{2} \|f_k\|^2 + \frac{\alpha}{2} \nu^{2/3} |k|^{-2/3} \|\partial_y f\|^2$$

Pf:

$$\Phi = \gamma \|f_k\|^2 + 2\beta \nu^{1/3} \operatorname{Re} |k|^{-4/3} \langle ik f_k, \partial_y f_k \rangle$$

$$+ \alpha \nu^{2/3} |k|^{-2/3} \|\partial_y f_k\|^2$$

$$\leq \frac{\gamma}{2} \|f_k\|^2 + \frac{\alpha}{2} \nu^{2/3} |k|^{-2/3} \|\partial_y f\|^2$$

$$\frac{2\beta}{\sqrt{\alpha}\sqrt{\gamma}} \left(\sqrt{\gamma} \, \|f_k\| \right) \left(\sqrt{\alpha} \|\nu^{1/3} \partial_y f_k\| \, |k|^{-1/3} \right)$$

$$\leq C^4 \frac{\beta^2}{\alpha\gamma} \gamma \|f_k\|^2 + \frac{\alpha}{2} \|\nu^{1/3} |k|^{-1/3} \partial_y f\|^2$$

$$\Rightarrow \qquad 4C \frac{\beta^2}{\alpha\gamma} < \frac{1}{2}$$

$\boxtimes$

To finish,

$$\frac{d}{dt}\Phi \leq -\gamma \|\nu^{1/2} f_y\|^2 - \beta |k|^{2/3}\nu^{1/3}\|f\|^2$$

$$\leq -C(\gamma,\alpha,\beta)\, \nu^{1/3}|k|^{2/3}\,\Phi$$

$$\hookrightarrow \qquad \Phi \lesssim \exp\left(-\delta\, \nu^{1/3}|k|^{2/3} t\right)\Phi_0$$

The final step is to show

$$\Phi \gtrsim \|f_k\|^2.$$

$\blacksquare$