The goal of these few lectures will be

(1) to introduce the general area of <u>Hydrodynamic</u> <u>Stability</u>

(2) State some precise theorems, including Bedrossian-He-Iyer-Wang

(3) Focus on concrete energy methods motivated by linear / linearized dynamics in simpler settings

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<u>Lecture</u> 1 - <u>Overview</u> <u>of</u> <u>Three</u> <u>Physical</u> <u>Phenomena</u>

Goal: Two asymptotic problems observed in fluid dynamics

(1) $t \to \infty$: Large time asymptotic stability of various stationary solutions. Mechanisms:

Inviscid Damping

Enhanced Dissipation

(2) $\nu \to 0$: In the presence of boundaries, the

Inviscid Limit

becomes challenging on every time scale
due to boundary layers.

NS:

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p = \nu\Delta\vec{u}$$

$$\mathrm{div}(\vec{u}) = 0$$

$$\vec{u}\Big|_{t=0} = \vec{u}_{in}(x,y)$$

(BC's TBD)

$\mathbb{T}^1 \times [-1,1]$ or $\mathbb{T}^1 \times \mathbb{R}$

"Channel"

Velocity, $\vec{u} = \begin{pmatrix} u(t,x,y) \\ v(t,x,y) \end{pmatrix}$

Vorticity $\omega = \partial_y u - \partial_x v$

NS in Vorticity Formulation:

$$\partial_t \omega + \vec{u}\cdot\nabla\omega = \nu\Delta\omega$$

$$\omega\Big|_{t=0} = \omega_{in}(x,y)$$

$$u = \nabla^\perp f$$

$$\Delta f = \omega \quad \text{(Biot-Savart)}$$

BC's TBD

- Observe that $\vec{u}(x,y) = \begin{pmatrix} b(y) \\ 0 \end{pmatrix}$, shear flows, are stationary solutions to 2D Euler for any function $b(y)$.

- The specific choice $b(y) = y$ is called the Couette flow, which also solves NS.

- The aim is to study small perturbations of Couette,

$$\omega_{Total} = \underbrace{1}_{\text{Couette}} + \varepsilon \underbrace{\omega}_{\text{Perturbation}}$$

$$\partial_t \omega + y\, \partial_x \omega - \nu \Delta \omega = - u \cdot \nabla \omega$$

$$\omega\big|_{t=0} = \omega_{in}(y)$$

$$u = \nabla^{\perp} \Delta^{-1} \omega \qquad (\text{Biot-Savart})$$

BC's :

$\mathbb{T} \times [-1,1]$.

Asymptotics as $t \to \infty$:

$$\underbrace{\partial_t \omega + y\,\partial_x \omega}_{O(\varepsilon)} \; \underbrace{- \nu \Delta \omega}_{O(\nu\varepsilon)} = - \underbrace{u \cdot \nabla \omega}_{O(\varepsilon^2)}$$

$$\|\omega_{in}\|_X \le \varepsilon$$

(1) Uniform in ν: We can look for asymptotic stability for $\varepsilon \ll 1$, but independent of ν.
The relaxation mechanism has to then be inviscid.

(2) Threshold Theorems: We can assume $\varepsilon = O(\nu^p)$ for some power p. (Higher $p \Rightarrow$ easier thm).

Remark: There is value in both, as the space X may be different.

- Thm we state is uniform, but it relies on threshold as a substep ($t \ge \nu^{-1/3-\delta}$).

Boundary Conditions and $\omega \to 0$:

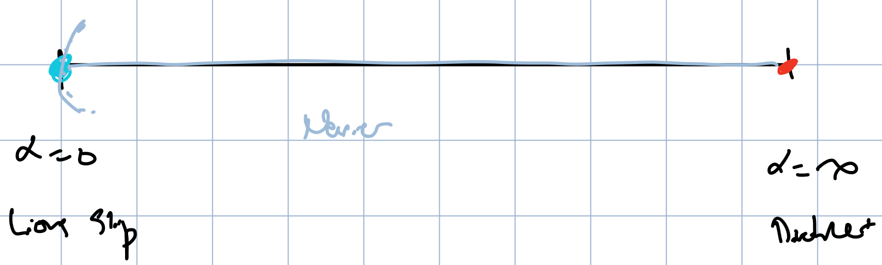
Navier BCs: $\omega\big|_{y=\pm 1} = 0, \quad \psi\big|_{y=\pm 1} = 0$

Creates weak BCs, in our case no BL.

Dirichlet BCs (No slip): $u\big|_{y=\pm 1} = 0, \quad \psi\big|_{y=\pm 1} = 0$

Creates strong boundary layers, too difficult for now.

Navier: $(\omega + \alpha u)\big|_{y=\pm 1} = 0$



Linear Inviscid Damping

- $\partial_t \omega + y \partial_x \omega = 0$
  $\Delta \psi = \omega$ | $\mathbb{T} \times \mathbb{R}$

- Method of Characteristics:

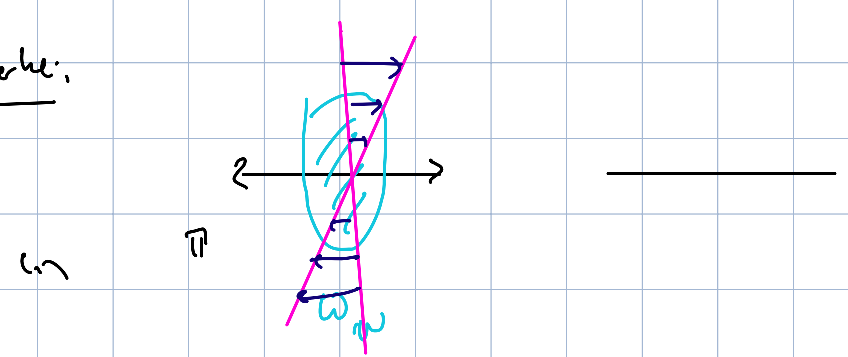
$$z = x - ty, \qquad \omega(t,x,y) = f(t,z,y) = f(t, x-ty, y)$$

$$\partial_t \omega = f_t - y f_z$$
$$y \partial_x \omega = y \partial_z f$$
$$\partial_t f(t,z,y) = 0$$

- $\omega(t,x,y) = f(t,z,y) = f_{in}(z,y) = \omega_{in}(x-ty, y)$

↳ $\omega(t,x,y) = \omega_{in}(x-ty, y)$

Remarks:



Physical Interpretation is "Shearing" by the Background Flow

(2) ω itself does not decay in amplitude!

ω is transported.

(3) However, we will see that ω exhibits

<u>weak</u> <u>convergence</u>.

- $\hat{\omega}(t, k, \eta) =$

Fourier variable of $x \in \mathbb{T}$, so $k \in \mathbb{Z}$.

Fourier variable of $y \in \mathbb{R}$, so $\eta \in \mathbb{R}$.

- $\partial_t \hat{\omega}(t,k,\eta) - k \partial_\eta \hat{\omega}(t,k,\eta) = 0$ (Transport in freq. space)

$\hat{\omega}(t,k,\eta) = \hat{\omega}_{IN}(k, \eta + kt)$

- Remarks:

(4) Physical Interpretation is linear in time transfer of enstrophy to higher frequencies

This explain weak convergence: and derivatives will gain $\langle t\rangle$.

$z = x - ty$, $\quad f(t,z,y) = \omega(t,x,y)$, $\quad \phi(t,z,y) = \psi(t,x,y)$

$$\begin{cases} \partial_t f = 0 \\ \left[\partial_z^2 + (\partial_y - t\partial_z)^2\right]\phi = f \end{cases}$$

$$\hat{\phi}(t,k,\eta) = \frac{\hat{\omega}_{in}(k,\eta)}{k^2 + (\eta - kt)^2}$$

$\frac{1}{t^2}$ decay

Prop'n: (Linear Inviscid Damping):

$$\| P_{\neq 0}\, \phi \|_{L^2} \lesssim \frac{1}{\langle t\rangle^2} \| \omega_{in} \|_{H^2}$$

$$\| P_{\neq 0}\, u \|_{L^2} \lesssim \frac{1}{\langle t\rangle} \| \omega_{in} \|_{H^3}$$

$$\| P_{\neq 0}\, v \|_{L^2} \lesssim \frac{1}{\langle t\rangle^2} \| \omega_{in} \|_{H^2}$$

Pf:

$$\| \phi \|_{L^2} \leq \left\| \frac{\omega_{in}}{k^2 + (\eta - kt)^2} \right\|_{L^2}$$

$$\frac{1}{h^2+(n-ht)^2} \lesssim \frac{\langle n\rangle^2}{(ht)^2}$$

If $\quad |n| < \frac{1}{2}|ht| \quad$ or $\quad |n| > 2|ht|$

then $\quad (n-ht)^2 \geq C\,|ht|^2$

$$\Rightarrow \frac{1}{h^2+(n-ht)^2} \lesssim \frac{1}{(ht)^2}$$

If $\quad \frac{1}{2}|ht| < |n| < 2|ht|, \quad$ then

$$\frac{1}{h^2+(n-ht)^2} \lesssim \frac{1}{h^2} = \frac{|n|^2}{|n|^2 h^2} \lesssim \frac{\langle n\rangle^2}{(ht)^2}$$

$\square$

Remarks:

(1) Elliptic Regularity usually gives stream fn. is two deriv smoother than vorticity. However, here we lose two derivatives to gain decay $\frac{1}{\langle t\rangle^2}$.

"Decay costs Regularity"

(2) $u = \partial_y f$ has non-integrable decay $\frac{1}{\langle t\rangle}$.

$$\partial_y f = \partial_y \phi(t, x - ty, y)$$

$$= (\partial_y - t\,\partial_z)\phi$$

Each ∂_y derivative costs $\langle t\rangle$.

(3) Commuting vector field:

- $\partial_t \omega + y\,\partial_x \omega = 0$

∂_y is NOT a commuting vector field.

$\Gamma = \partial_y - t\,\partial_x$ is a commuting vector field.

(4) The role of $P \neq 0$: For $P = 0$, no shearing effect $(k=0)$.

Linear Enhanced Dissipation

- $\partial_t \omega + y\,\partial_x \omega - \nu \Delta \omega = 0$
  $\Delta \psi = \omega$ $\Big|$ $\mathbb{T} \times \mathbb{R}$

- Heat Eqn Timescales:

$$\partial_t \omega - \nu \partial_y^2 \omega = 0 \quad \Big| \quad y \in \mathbb{R} \quad \text{or} \quad y \in \mathbb{T}$$

$$\frac{\partial_t}{2} \|\omega\|^2 \leq - \nu \|\nabla \omega\|^2 \leq -C\nu \|\omega\|^2$$

Poincaré type ineq.

↳ $\|\omega\|^2 \leq e^{-\nu t} \|\omega_{in}\|^2$

↳ Half-life $\approx$ $\boxed{t = 1/\nu}$.

- Oscillation $\longmapsto$ Decay:

$$(\partial_t - \partial_y^2)\,\omega_1 = 0 \quad , \quad (\partial_t - \partial_y^2)\,\omega_n = 0$$

$$\omega_1\big|_{t=0} = \sin(x) \qquad \omega_{100}\big|_{t=0} = \sin(10x)$$

$$w_1(t,x) = e^{-t}\sin(x)$$

$$w_{100}(t,x) = e^{-100t}\sin(10x)$$

(Fourier energy strength high Fourier modes => better Poincaré Ineq'l)

. Idea of Enhanced Dissipation:

$$(\underbrace{\partial_t + y\partial_x}_{} - \underbrace{\nu\Delta}_{})\, w = 0$$

Sends info. to higher freqs

Responds to higher oscillations by increasing decay.

. We follow Kelvin (1887)

$$\partial_t f = \nu\partial_z^2 f + \nu(\partial_y - t\partial_z)^2 f$$

$$\partial_t \hat{f} + \nu(k^2 + (n-tk)^2)\hat{f} = 0$$

$\hookrightarrow$

Integrating Factor

$$\hat{f}(t,k,n) = \hat{\omega}_{in}(k,n)\exp\left(-\nu\int_{s=0}^{t} |k|^2 + |n-ks|^2\, ds\right)$$

$$\|\hat{f}(t,k,\cdot)\|_{L^2_\eta} \lesssim e^{-c\nu t^3}$$