

Geometric and Dynamical aspects of fluid motion

Outline: Lecture 1 : The Euler equations

Lecture 2 : Long time dynamics of two-dimensional inviscid fluids

Lecture 3: Transition to turbulence and a problem of Kolmogorov

Lecture 4 : Phenomenology and Mathematics of three-dimensional turbulence

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Transition (Dynamical progression) to Turbulence ①

"A. N. Kolmogorov considered his work on turbulence to be non-mathematical. He wanted to explain observed phenomena from first principles, i.e. from the Navier-Stokes equations. This dream was one of the sources of his interest in the theory of dynamical systems, in which he perceived a set of tools for the description of turbulent chaos." – V.I. Arnol'd [22]

Consider the ODE

$$\frac{d}{dt} u(t) = F(u(t))$$
$$u(0) = u_0$$

The variable $u = u(t)$ belongs to a linear space H called the **phase space** and $F: H \rightarrow H$.

Two cases:

- finite dimensional $H = \mathbb{R}^d$
- infinite dimensional $H =$ a Hilbert space.

2D fluids fall into the latter category with $H = L^2_\sigma$
or $H = H^1_\sigma$ (depending on regularity of external force)

$$L^2_\sigma = \{ u \in L^2(\mathbb{T}^2) : \nabla \cdot u = 0 \}$$

$$H^1_\sigma = \{ u \in H^1(\mathbb{T}^2) : \nabla \cdot u = 0 \}$$

$$\frac{d}{dt} u(t) = F(u(t)), \quad u(0) = u_0$$

(2)

- We are interested in the $t \rightarrow \infty$ behavior.
- Generally $F(u) = F_\mu(u)$ for some parameter μ .
Long time behavior will depend strongly on μ .
The role of μ will depend on details of system.
Here is a schematic of STAGES: (Landau)

① For μ small, $\mu < \mu_1$, there exists a unique stationary solution $u = u_1^s$, i.e.

$$(*) \quad F_\mu(u_1^s) = 0.$$

This solution is stable and attracts all orbits

$$u(t) \rightarrow u_1^s \quad \text{as } t \rightarrow \infty.$$

② For larger μ , $\mu_1 < \mu < \mu_2$, other solutions of (*) appear, u_2^s, u_3^s, \dots and u_1^s loses stability. We say a bifurcation has occurred at $\mu = \mu_1$.

The new solutions are typically stable and each have a basin so $u(t) \rightarrow u_i^s$ for some $i = 2, 3, \dots$.

③ When μ is further increased, $\mu_2 < \mu < \mu_3$, a **Hopf bifurcation** can occur. In this case the flow doesn't become stationary but rather

$$u(t) - \phi(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

where $\phi(t)$ is a **time-periodic solution**

(*)
$$\frac{d}{dt} \phi(t) = F(\phi(t)) \quad t \in \mathbb{R}$$

$$\phi(t+T) = \phi(t).$$

A priori, the value μ_2 , solution ϕ and period T are all unknown and a detailed analysis is required.

From this point, we can either have a **Feigenbaum cascade of period doubling**: ϕ_j with periods $2T, 2^2T, \dots, 2^j T$ or

④ For larger μ , $\mu_3 < \mu < \mu_4$ **invariant tori** can appear; i.e.

$$u(t) - \phi(t) \rightarrow 0$$

where ϕ is a **quasiperiodic solution** of (*)

$$\phi(t) = g(\omega_1 t, \dots, \omega_n t)$$

where g is periodic with period 2π in each variable and $\omega_i = 1/T_i$ are rationally independent real numbers.

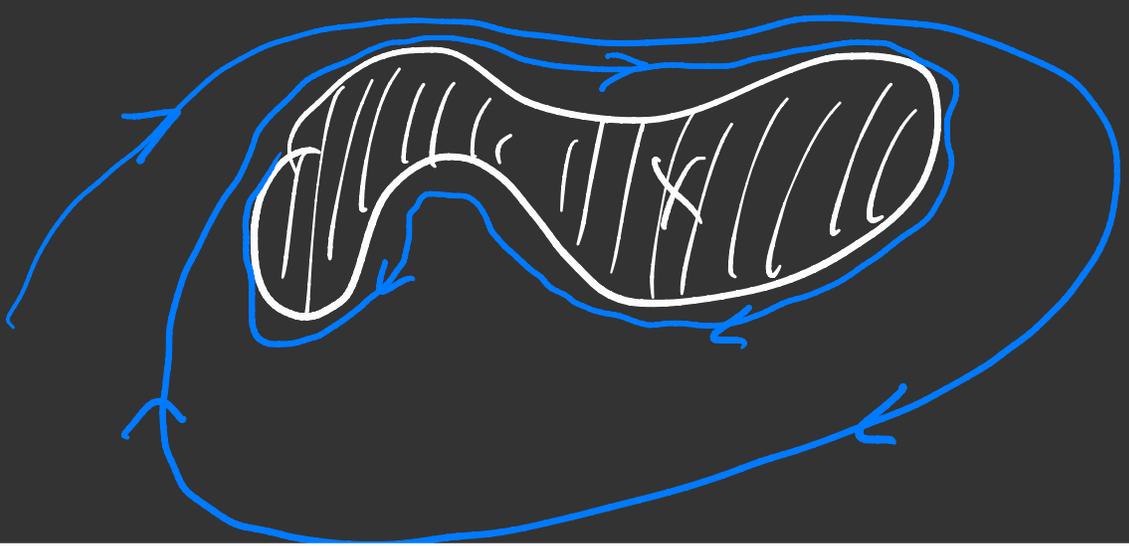
Flow looks chaotic, but further analysis shows system has discrete frequencies.

⑤ For $\mu > \mu_c$, the flow is chaotic, a stage where $u(t)$ looks completely random for all time. Fourier reveals a wide-band spectrum. One can show

$$u(t) \rightarrow X \text{ as } t \rightarrow \infty$$

in the sense

$$\text{dist}(u(t), X) \rightarrow 0 \text{ as } t \rightarrow \infty$$



Here, X is an invariant subspace of H ; i.e.

$$S_t(X) = X \quad \forall t \geq 0$$

where S_t is the semigroup of $u = F(u)$.

The set X may be extremely complicated (Fractal) with complex geometry (non-smooth).

The Kolmogorov Model

(5)

Analysis Seminar (1958) "Theory of dynamical systems and hydrodynamic stability".



Consider the motion of a two-dimensional fluid, periodic with period $2\pi L/\alpha$ in x and period $2\pi L$ in y :

Fix $\alpha > 0$, $L > 0$, $\gamma > 0$

On $\Pi_\alpha^2 := [0, 2\pi L/\alpha) \times [0, 2\pi L)$, consider

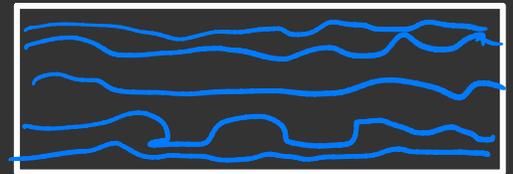
$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + \gamma f$$

$$\nabla \cdot u = 0$$

$$\int_{\Pi_\alpha^2} u \, dx \, dy = 0$$

$2\pi L/\alpha$

$2\pi L$



The forcing f is taken to be an eigenfunction of the Stokes operator: $A = -P\Delta$

$$A f_\lambda = \lambda f_\lambda$$

Can be constructed from eigenfunctions of Laplacian

$$f_\lambda = \nabla^\perp q_\lambda \quad \text{where} \quad -\Delta q_\lambda = \lambda q_\lambda, \quad \int q_\lambda = 0$$

example

$$\begin{pmatrix} 0 \\ \sin(\alpha n x) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sin(m y) \\ 0 \end{pmatrix} \quad n, m \in \mathbb{N}$$

A. H. Колмогоров предложил рассмотреть решение системы

$$\left. \begin{aligned} \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \nu \Delta u + \gamma \sin y, \\ \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \nu \Delta v, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned} \right\}$$

(Arnold and Meshalkin, 1960)

Vorticity equation: $\omega = \nabla^\perp \cdot u = \partial_1 u_2 - \partial_2 u_1$

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega - \gamma \lambda q_\lambda \quad \leftarrow \Delta q_\lambda = \lambda q_\lambda$$

Representing $u = \nabla^\perp \psi$ by streamfunction ψ , $\omega = \Delta \psi$

$$\partial_t \Delta \psi + \mathcal{J}(\psi, \Delta \psi) = \nu \Delta^2 \psi - \gamma \lambda q_\lambda$$

$$\mathcal{J}(a, b) = \nabla^\perp a \cdot \nabla b$$

Note, this equation admits a laminar solution:

$$\bar{\psi} = \frac{1}{\nu \lambda} q_\lambda$$

Since, $\nu \Delta^2 \bar{\psi} = \lambda q_\lambda$, $\mathcal{J}(\bar{\psi}, \Delta \bar{\psi}) = \lambda \mathcal{J}(\bar{\psi}, \bar{\psi}) = 0$.

The velocity corresponding to this solution is

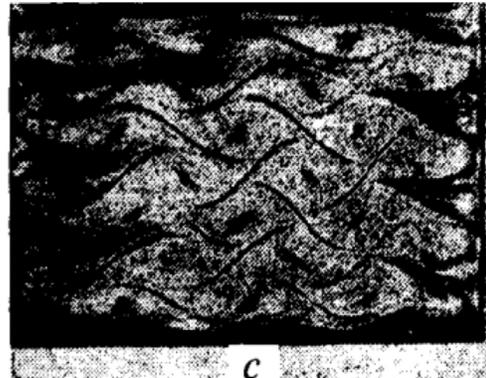
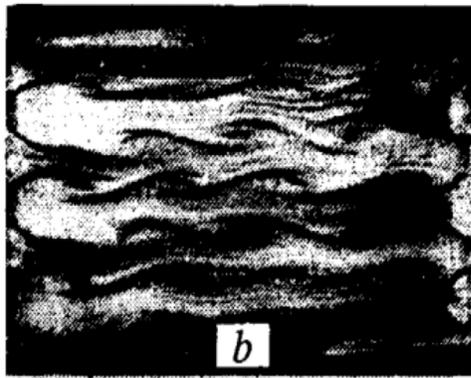
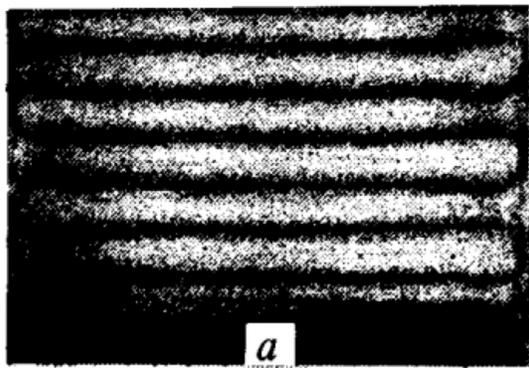
$$\bar{u} = \frac{1}{\nu \lambda} f_\lambda \stackrel{\text{top example}}{=} \frac{\gamma}{\nu} \begin{pmatrix} \sin y \\ 0 \end{pmatrix}$$

This laminar flow is the unique steady state when $\mu = \frac{\gamma}{\nu^2}$ is small, and is the global attractor for all data. μ is order parameter: instability, periodic in time, turbulence...

Exceptional (trivial case): when λ is smallest eigenvalue, "gravest mode" is unique global attractor for all $\mu > 0$!

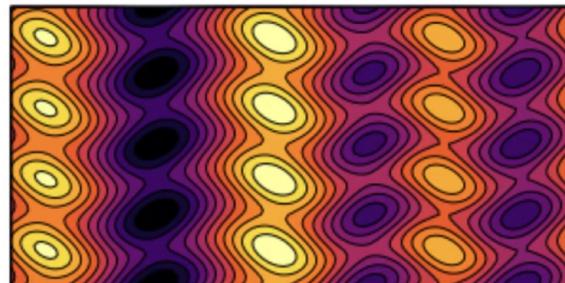
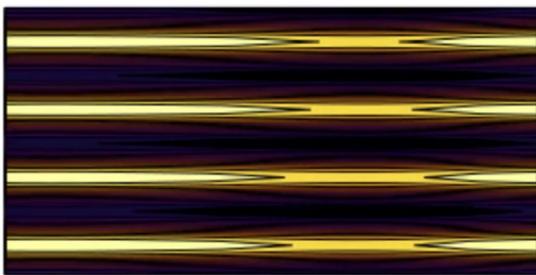
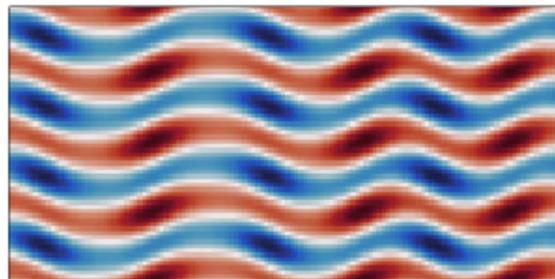
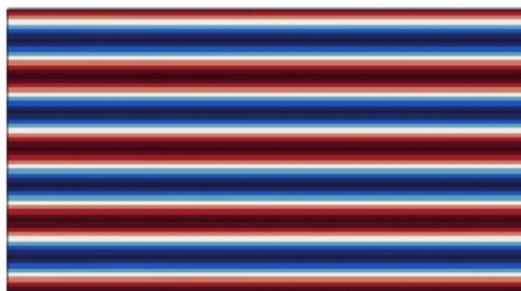
Real World

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Obukhov, "Kolmogorov flow and laboratory simulation of it" (1983)

Virtual Reality



simulations done on my personal computer
using publically available code written by

$$f = \begin{pmatrix} \cos(5y) \\ 0 \end{pmatrix}$$

Navid Constantinou

The flow in this 'Kolmogorov model' is usually supposed to be two-dimensional and the exterior force field has usually the form of the first harmonic. If the Reynolds number is small, the steady laminar flow emerges: its velocity profile has a sinusoidal form. This flow loses its stability as the Reynolds number grows. The Kolmogorov 1958 programme was to study rigorously the bifurcations of the attractors in this system and their dependence on the growing Reynolds number. This programme was partially fulfilled in the work of Mechalkin & Sinai (1961). As far as I know, the bifurcations in such systems have not been completely understood even now, even in the two-dimensional case and even numerically, using the Galerkin approximation for moderate Reynolds number (but see Nicolaenko & She 1990).

Upper bounds Kolmogorov viewed the sequence of bifurcation picture as one possible "route to turbulence" quasiperiodic flows with 2, then 3, 4 ...

(2-tori, 3-tori ... in phase space) appear subsequently.

Kolmogorov asked if an infinite sequence of such bifurcations could occur within a finite parameter range (as in logistic map picture)

NO: Ladyzhenskaya (1982), Il'yashenko (1983).

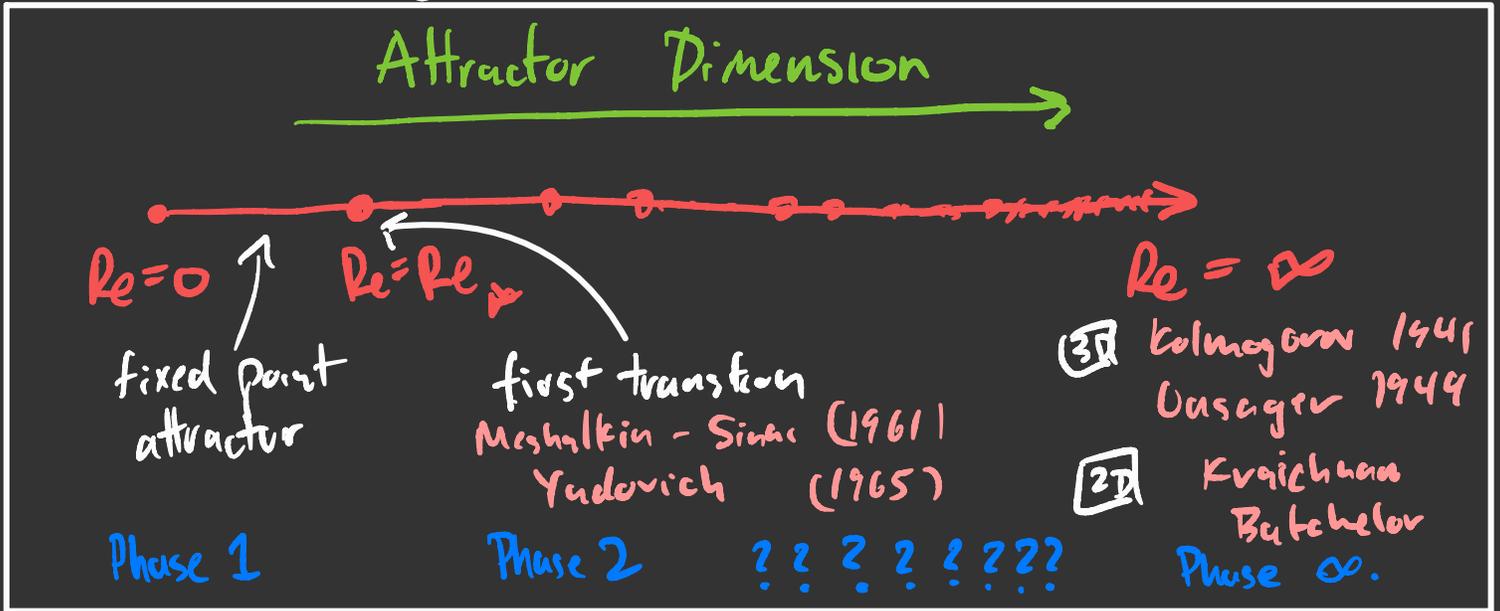
$$\dim_H(\text{Attractor}) \lesssim \frac{1}{\alpha} G^{1/2} \left(1 + \log \frac{G}{\alpha^{1/4}}\right)^{1/3}$$

where $G = \|f\|_{L^2} \frac{L^2}{\nu^2}$. Constantin - Foias - Temam (1988).

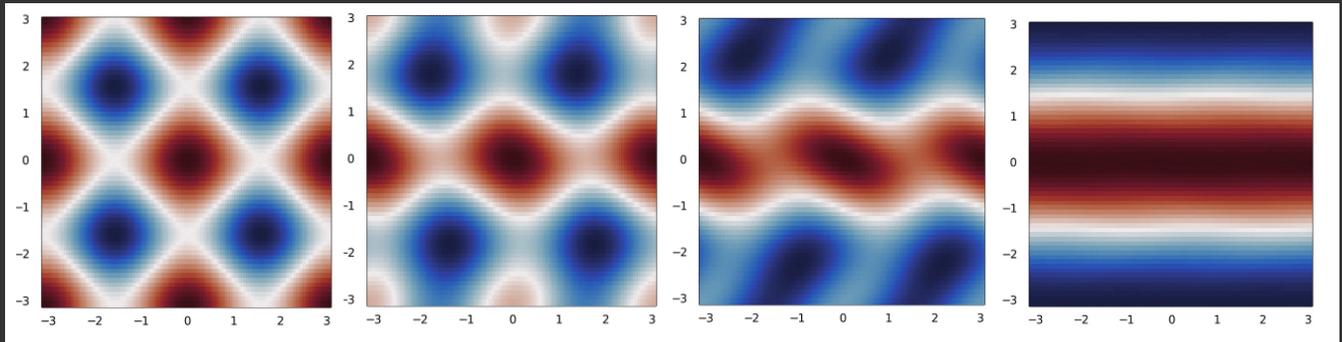
dependence expected sharp (agree's with Landau's D.O.F arg)

Does the minimum of the Hausdorff dimensions of the attractors grow with Reynolds number for the Navier-Stokes system on a compact two-dimensional manifold (for, say, the Kolmogorov model)?

Here, we try to build up the picture:



PHASE: I



Theorem: There exists $\mu^* = \frac{\delta}{\nu^2}$ such that for all $\mu < \mu^*$, the laminar solution is the unique stationary solution and attracts all orbits as $t \rightarrow \infty$. If $\lambda = \lambda_1$, true for all μ .

PHASE II:

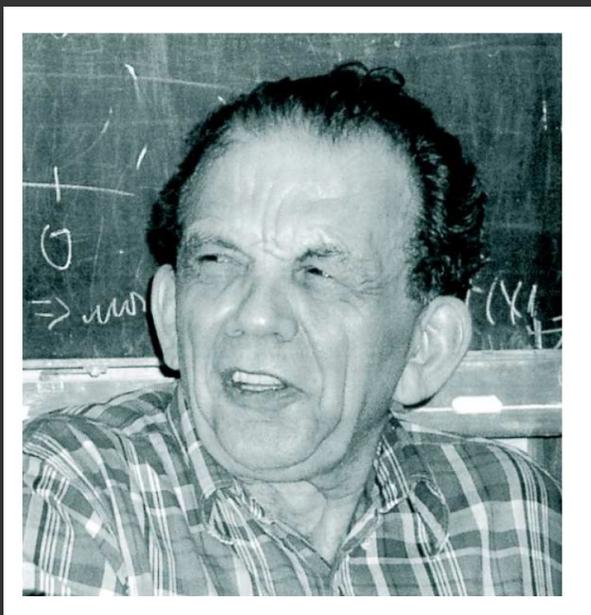
(10)

THEOREM: (Yudovich 1965). Let $\alpha \in (0, 1)$. Consider Navier-Stokes forced by $\gamma \cos y$ with $\gamma > 0$ on Π_α^2 . Then, there exists a **critical Grashof number** $\mu_*(\alpha)$ which is a monotonically increasing function of α satisfying

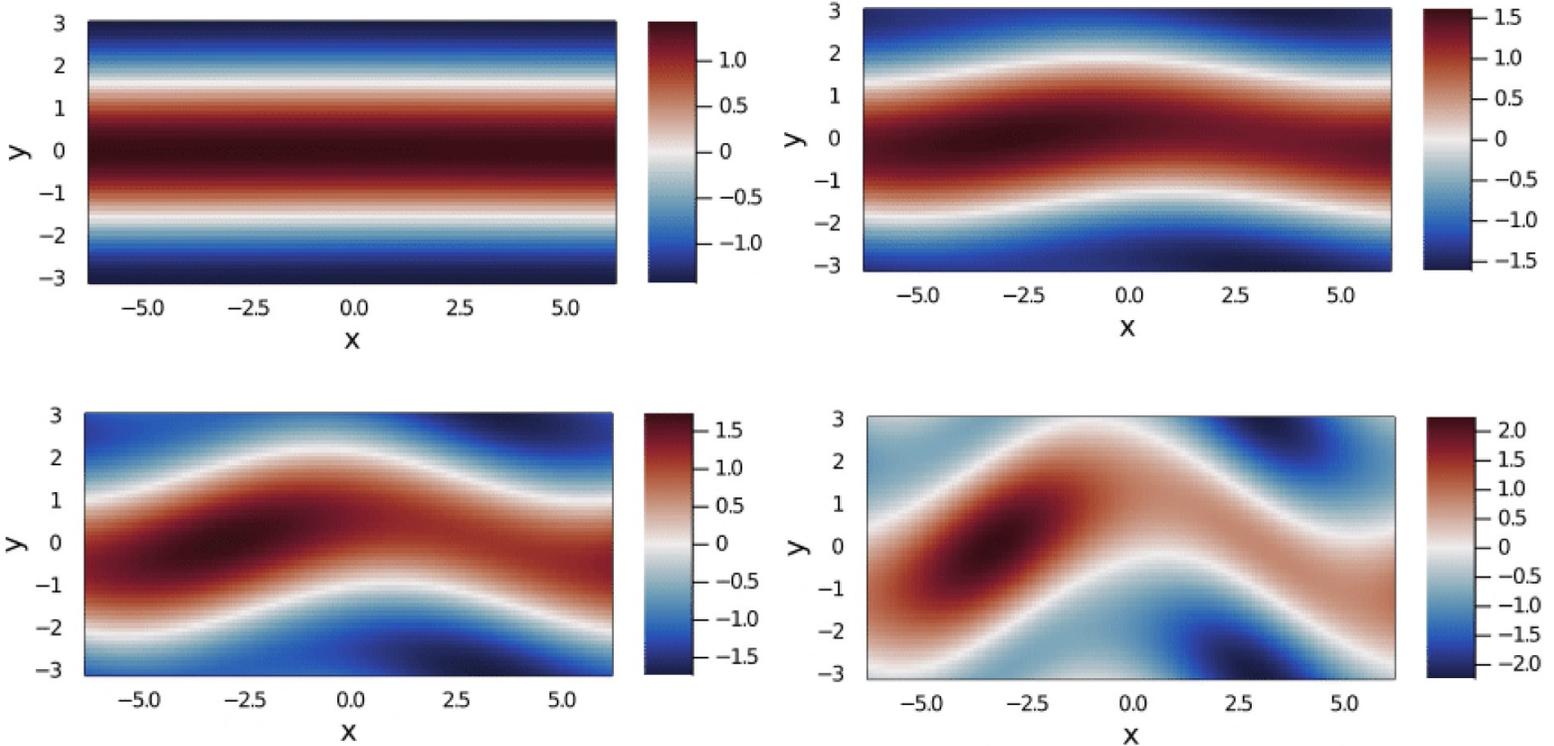
$$\lim_{\alpha \rightarrow 0} \mu_*(\alpha) \rightarrow \sqrt{2}, \quad \lim_{\alpha \rightarrow 1} \mu_*(\alpha) \rightarrow \infty.$$

and is such that the following hold:

- (1) If $\mu < \mu_*$, the **laminar state** is unique steady solution in $H^2(\Pi_\alpha^2)$ and is **attractor** for all data.
- (2) For $\mu > \mu_*$, the laminar state becomes **unstable** (Meshalkin - Sibirskii (1961))
- (3) For $|\mu - \mu_*| \ll 1$, there exists a **two-dimensional manifold of stationary states** which bifurcate from the laminar state and are **linearly stable**.



Yudovich's Bifurcation, and its fate



Elements of the proof:

Step 1: express nonlinear problem and its linearization as eigenvalue problems on the Hilbert space H^2

$$\psi = \nu \mu (\phi - \cos y) \quad \text{Stationary.}$$

Then the perturbation ϕ satisfies nonlinear equation

$$\Delta^2 \phi = \mu \sin y \partial_x (\phi + \Delta \phi) + \mu J(\phi, \Delta \phi)$$

Its linearization reads

$$\Delta^2 \phi = \mu \sin y \partial_x (\phi + \Delta \phi)$$

These read $N[\phi] = \frac{1}{\mu} \phi$ in H_2 , $(f, g)_{H^2} = \int_{\mathbb{T}^2} \Delta f \Delta g$.

Review of Bifurcation Theory

$$G[u] = \lambda u \quad \text{eqn in Banach space } X$$

(a) $G[0] = 0$

(b) G is compact

(c) G is differentiable at $u=0$ with compact $A = G'[0]$.

Consider $S = \{(\lambda, u) \in \mathbb{R} \times X : u \neq 0, \lambda u = G[u]\}$

DEF: A **bifurcation point** λ_* is such that any neighborhood of $(\lambda_*, 0)$ there is a point $(\lambda, u) \in S$

Bifurcation points must be eigenvalues of A by implicit function theorem. Reverse not true but

THM (Krasnoselskii) Suppose (a) - (c) hold and that λ_* is an eigenvalue of A with **odd multiplicity**. Then λ_* is a **bifurcation point** for $F[u] = \lambda u - G[u]$

Global refinement by **Rabinowitz**.

• To obtain information about multiplicity of eigenvalue, we study also **adjoint problem** and find eigenfunc with same λ and not orthogonal to original eigenfunction $\Rightarrow \lambda$ simple

• (Simplicity in subspace). Consider $\frac{d^2}{dx^2} h = -\lambda h$
eigenvalues: $\lambda = k^2$ eigenfunctions $\sin(kx)$, $\cos(kx)$
geo = alg mult = 2 odd even
simple in even and odd subspaces.

• we restrict to even and odd subspaces to construct a simple eigenvalue

• We apply these theorems in the setting of

$x = \mathbb{H}_e^2, \mathbb{H}_o^2, G = N, A = L.$

• To construct eigenfunction, note any such is a linear combination of

$$\psi_n(x,y) = e^{i\alpha_n x} \sum_{m \in \mathbb{Z}} c_m(\alpha_n) e^{im y}$$

Then $c_n = d_n / (k^2 + m^2 - 1)$ with $k = \alpha_n$ &

(*) $d_m d_m + d_{m-1} - d_{m+1} = 0$

with
$$d_m = \frac{2(k^2 + m^2)^2}{\mu k (k^2 + m^2 - 1)}$$

One derives characteristic eqn:

(**) $F(\mu) := \frac{1}{[a_1, a_2, \dots]} = -\frac{d_0}{2}$

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

converges by Van Vleck convergence theorem and lower bound on a_n .

μ is a real root of (*) iff d_n solves (**) $d_n \neq 0$

LEMMA: Let $k = \alpha n \in (0, 1)$. Then $(**)$ admits a positive root $\mu_* = \mu_*(k)$ and it has only this one root. If $k > 1$, there is no root.

Lemma: let $\alpha \in (0, 1)$. The problems $L[\phi] = \frac{1}{\mu} \phi$ and $L^*[\psi] = \frac{1}{\mu} \psi$ have exactly $\lfloor \frac{1}{\alpha} \rfloor$ positive (as many negative) eigenvalues $1/\mu$. Each of multiplicity two.

THEOREM: There exist exactly $\lfloor \frac{1}{\alpha} \rfloor$ positive numbers $0 < \mu_*^{(1)} < \mu_*^{(2)} < \dots < \mu_*^{(n)} < \infty$ which are points of bifurcation for N_α . Each corresponds to a branch of eigenfunctions $-\infty > -\mu_*^{(m)} > \dots > -\mu_*^{(2)} > -\mu_*^{(1)} > 0$. are also points of bifurcation. The spectrum also contains the intervals: $(\mu_*^{(1)}, \mu_*^{(2)})$, $(\mu_*^{(3)}, \mu_*^{(4)})$, \dots , $(\mu_*^{(m-1)}, \mu_*^{(m)})$. If m is odd, includes $(\mu_*^{(m)}, \infty)$. Also reflections.

PHASE ∞ : (aspects of it)

Kolmogorov's conjecture more precisely

1992-11. Consider the Navier–Stokes equation (say, on the 2-torus) with external force proportional to the viscosity (Kolmogorov's model). Is it true that, as the viscosity tends to 0 (i. e., the Reynolds number grows), there appear attractors of dimensions increasing with the Reynolds number (and containing no smaller attractors)?

Is it true that, moreover, the minimum dimension of all attractors unboundedly increases with the Reynolds number?

A. N. Kolmogorov suggested (in 1958) that the answer to the first question was affirmative, but he doubted that so was the answer to the second because of the experiments on delaying loss of stability.

v. I. Arnold'd
1991,
1999,
2004.

\mathcal{R} There are two Kolmogorov's 1958 conjectures on the behavior of the dimension of minimal attractors (i. e., the attractor which does not contain a smaller attractor) when its Reynolds number R tends to the infinity:

a) weak

$$\max_{\text{min Attr}} \dim \text{min Attr} \rightarrow \infty \quad \text{for } \nu \rightarrow 0;$$

b) strong

$$\min_{\text{min Attr}} \dim \text{min Attr} \rightarrow \infty \quad \text{for } \nu \rightarrow 0;$$

(see [1], Ch. I) where ν is the (kinematic) viscosity of a current.

THEOREM: (Babin-Vishik) ¹⁹⁸³ For $d \in (0, 1)$ and $\mu > \mu_* = \mu_*(d)$
 $2 \lfloor \frac{1}{\alpha} \rfloor \leq \dim_H (\text{Attractor}) \leq \frac{1}{\alpha} (1 + \alpha^{1/2} \mu).$

↑
dimension of unstable manifold of laminar steady state.



of boxes of side length 2π in $2\pi/\alpha$ by 2π rectangle.

Consider only steady states $\tilde{u} = \frac{v}{\delta} u$, $\mu = \frac{\gamma}{v\lambda}$

steady NS $\left\{ \begin{aligned} \mu (\tilde{u} \cdot \nabla \tilde{u} - \nabla \tilde{p}) &= \nabla \tilde{p} + \Delta \tilde{u} + f_\lambda \\ \nabla \cdot \tilde{u} &= 0 \\ \int_{\mathbb{T}_\alpha^2} \tilde{u} \, dx dy &= 0 \end{aligned} \right.$

THEOREM: Let $\{u^\mu\}_{\mu > 0}$ be a family of $H^2(\mathbb{T}_\alpha^2)$ steady solutions of Navier-Stokes. Then $\frac{v}{\delta} u^\mu \rightarrow u^E$ strong in H^1 as $\mu \rightarrow \infty$ where $u^E \in H^2$ solves

$$\begin{aligned} u^E \cdot \nabla u^E &= -\nabla p^E \\ \nabla \cdot u^E &= 0 \\ \int u^E \, dx dy &= 0 \end{aligned}$$

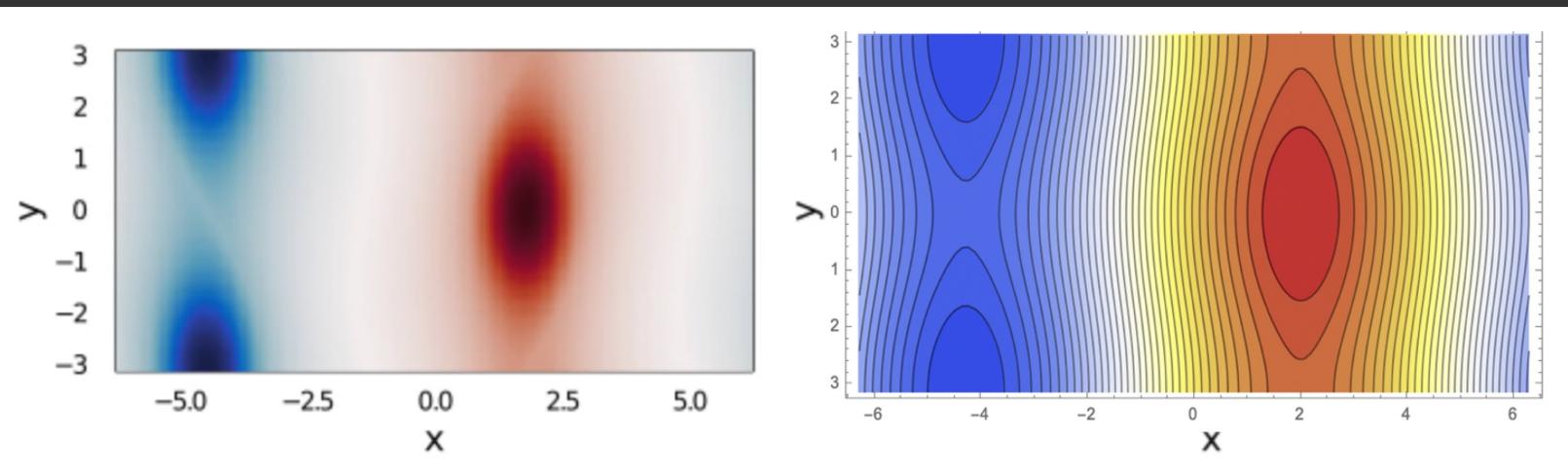
with the additional constraint

$$\lambda_1 \|\nabla u^E\|_{L^2}^2 \leq \|\Delta u^E\|_{L^2}^2 \leq \lambda \|\nabla u^E\|_{L^2}^2$$

Remark: • to normalize, Kolmogorov suggests $\gamma = v$
 • nonempty: applies to laminar state (saturates)

$$\bar{u} = \frac{\gamma}{v\lambda} f_\lambda$$

Numerical observation: By following on branch
of the bifurcation diagram. (17)



seems to converge to elliptical vortex pattern.

On right: assuming solution is 2 modes and saturates lower bound. Not solution except when $\alpha \rightarrow 1$.

Conjecture: \exists a sequence of steady NS solutions with non-trivial but shrinking basin of attraction as $\mu \rightarrow \infty$. Thus strong Kolmogorov could be false.

A.N. Kolmogorov always emphasized that preservation of stability of a steady flow, even for the infinitely growing Reynolds number, would not contradict hydrodynamical experiments, under the assumption that the basin of the corresponding attractor shrinks fast enough.

Arnold - Khesin (1999)