Geometric (in)equalities

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In this notes, we review convex bodies and prove some fun geometrical inequalities used as Lemmas in the *Shape of the Earth* story.

1 Convex sets

Let's start with recalling a few of the basic properties of convex sets.

Definition: A closed subset $K \subset \mathbb{R}^n$ is said to be convex if the line segment connecting any points $p \in K$ and $q \in K$ also lies within K.

We use ∂K to denote the boundary of K. The boundary simply consists of those points for which every open neighborhood contains points of K and its complement. For us, the most important property of convex sets is that every point $p \in \partial K$ has a supporting hyperplane. A supporting hyperplane P at $p \in \partial K$ is a hyperplane with the property that all of K lies on one side of P.

Proposition: Let K be a compact convex subset of $\mathbb{R}^n$. At every $p \in \partial K$, there exists a supporting hyperplane P.

Proof. Here is a sketch of the argument. We consider a sequence of points $q_n \to p$ with $q_n \in K^c$, that is, we approach from points outside of the convex set K. We take the point closest to q_n in K, call it p_n, and we draw the line segment joining q_n and p_n. This uniquely defines a plane: it is the plane whose normal vector is colinear with this line segment and which passes through the midpoint of this line segment. Taking the limit, the points p_n will converge to p. Moreover, the unit normal vectors of the planes live in a compact set (the unit $n-1$ sphere), meaning that we can find a convergent subsequence. The limiting corresponding plane will be a supporting hyperplane. □

Exercise: Interpret the above argument in terms of linear functionals l on the vector space space $\mathbb{R}^n$ which satisfy $l(p) = 1$ and $l(x) \leq 1$ for all $x \in K$.

Given any set S, we can naturally associate a convex set to it by taking the intersection of all convex sets which contain it. We call this the *convex closure* of S.

Exercise: Let S be a closed set. Prove that the convex closure of S is equal to the union of all line segments joining points in S.

2 Transverse lines

Several problems boil down to properties of transverse sets, and somewhat surprisingly, specifically transverse lines. These considerations appear in several different circumstances. The following result is a simple manifestation of how transverse lines behave.

Theorem 1 (Transverse lines): Let L_1 and L_2 be straight lines in $\mathbb{R}^n$ which intersect in a point p, and let v_1 and v_2 be unit length vectors which generate the lines L_1 and L_2, respectively. Let θ denote the angle between L_1 and L_2, that is $\cos(\theta) = v_1 \cdot v_2$, and assume that $0 < \theta < \pi$. Finally, let $D > 0$ and let D_1 be the set of points $p_1 \in L_1$ with $d(p_1, L_2) \leq D$, and similarly, let D_2 be the set of points $p_2 \in L_2$ with $d(p_2, L_1) \leq D$ (we are using d to denote the distance function). Then, the diameter L of D_1 and D_2 has that $L \leq \frac{2}{\sin(\theta)} D$.

Before turning to the proof, let's think about what this is saying. We are basically trying to understand "how far" we have to travel along either L_1 or L_2 starting from p so that we get sufficiently far away from the other line (where sufficiently far is measured by D). The reason why this is reasonable is because the lines point in different directions. Thus, if we move along one of the lines, we start to move away from the other one. The "time" we have to travel at unit speed is dictated by how large D is and by how close to 0 or π the angle θ is (note that, as θ approaches one of these values, the lines start to become closer and closer to being the same line!). Note that the diameter of D_1 and D_2 should be the same by the symmetry of the setup, so the statement makes sense.

Proof. The lines L_1 and L_2 determine a two dimensional plane P. We can pick coordinates on this plane so that p is the origin and the line L_1 is the x axis and v_1 points in the direction of increasing x. Moreover, we can assume that $0 < \theta \leq \pi/2$, meaning that v_2 points into the upper right quadrant.

Now, we begin by assuming that $\theta < \pi/2$. In this case, points on L_2 can be parameterized by $x = s$ and $y = \tan(\theta)s$ for any real number s. By symmetry, it suffices to consider $s > 0$. For every such s, the distance between such a point and the x axis is given by $\tan(\theta)s$. This means that the given point on L_2 will be closer than D to L_1 whenever $\tan(\theta)s \leq D$, meaning that $s \leq \cot(\theta)D$. The length of the part of L_2 between the origin and this point is given by $\frac{1}{\sin(\theta)}D$. We have to double this to account for the part of L_2 corresponding to negative s, meaning that we are left with $\frac{2}{\sin(\theta)}D$. This is the desired estimate on L. $\square$

3 Loomis–Whitney

We can ask a bunch of related questions where we want to understand geometric properties of a shape in terms of geometric properties of its projections. The Loomis–Whitney inequality provides an answer to the following question: suppose we have a subset of $\mathbb{R}^n$, and suppose we know the $n-1$ dimensional volume of each of its projections onto coordinate hyperplanes. What can we say about the n dimensional volume of the original set? The Loomis–Whitney inequality provides an upper bound for the volume of the set in terms of the $n-1$ dimensional volumes of the coordinate hyperplane plane projections.

This isn't a course about these kinds of geometric inequalities, so we won't study this in full generality. Let's just consider the two and three dimensional cases. The two dimensional case reads as follows:

Theorem 2 (Loomis–Whitney in $\mathbb{R}^2$): Let S be a measurable subset of $\mathbb{R}^2$, and let $\Pi_x(S)$ and $\Pi_y(S)$ denote the projections of S onto the coordinate hyperplanes orthogonal to x and y, respectively (so note that Π_x is a projection onto the y axis and Π_y is a projection onto the x axis). Then, we have that

$$|S| \leq |\Pi_x(S)||\Pi_y(S)|.$$

In this Theorem, note that $|S|$ denotes the two dimensional measure of S, whereas the analogous notation on the right-hand-side denotes the one dimensional measure.

Proof. The proof is straightforward: we first truncate the set S so it is bounded (the general case will follow by a limiting argument). Then, we realize that we can translate different parts of S around until $\Pi_x(S)$ and $\Pi_y(S)$ are continuous segments. Because translating pieces of S does not change the measure, we can assume that we in fact started with the such a set (note also that the lengths of the projections is not changed by this). The resulting set is clearly covered by a rectangle with side lengths $|\Pi_x(S)|$ and $|\Pi_y(S)|$, giving us the result. □

Let's now continue to the three dimensional case. This turns out to be much harder to go through rigorously, but the two dimensional result shows us that something ought to be possible.

Theorem 3 (Loomis–Whitney in $\mathbb{R}^3$): Let S be a measurable subset of $\mathbb{R}^3$, and let $\Pi_x(S)$, $\Pi_y(S)$, and $\Pi_z(S)$ denote the projections of S onto the coordinate hyperplanes orthogonal to x, y, and z, respectively. Then, we have that

$$|S| \leq |\Pi_x(S)|^{1/2}|\Pi_y(S)|^{1/2}|\Pi_z(S)|^{1/2}.$$

It is tempting to find a proof that translates pieces around and fits the shape into a slab of appropriate dimensions, but it is not so clear how to do this (for example, how would we get the side lengths to be the square roots of the areas, and if we don't get this, how would we get the inequality we want?). The following proof will be a bit harder to make intuitive (if anyone comes up with a proof in terms of translating shapes around, please let us know!).

Proof. We shall write the volume of S as an integral over various slicings. By an abuse of notation, we shall use S to denote the characteristic function of itself, that is, $S(x, y, z) = 1$ for $(x, y, z) \in S$ and $S(x, y, z) = 0$ otherwise. We let S_x, S_y, and S_z denote the characteristic functions restricted to translates of coordinate hyperplanes. Thus, $S_x(y, z) = 1$ if $(x, y, z) \in S$ and $S_x(y, z) = 0$ otherwise (and similarly for S_y and S_z). Now, we can write that

$$|S| = \int_{-\infty}^{\infty} |S_z| dz.$$

Now, we shall "interpolate" between two bounds. The first is that $|S_z| \leq |\Pi_z(S)|$. The second comes from noticing that $|S_z|$ is now the two dimensional area of a set living in (a copy of) $\mathbb{R}^2$, so we can use the two dimensional Loomis–Whitney inequality we have already established. This allows us to write that

$|S_z| \leq |\Pi_y(S_z)||\Pi_x(S_z)|$, where we note that the projections here are onto the x and y axes, respectively, of the planes of constant z. We can then relate $|\Pi_y(S_z)|$ and $|\Pi_x(S_z)|$ to $|\Pi_y(S)|$ and $|\Pi_x(S)|$, respectively, by noting that $\bigcup_z \Pi_y(S_z) = \Pi_y(S)$, and similarly that $\bigcup_z \Pi_x(S_z) = \Pi_x(S)$. Then, we write

$$|\bigcup_z \Pi_x(S_z)| = \int_{-\infty}^{\infty} |\Pi_x(S_z)| dz \leq |\Pi_x(S)|,$$

and similarly

$$|\bigcup_z \Pi_y(S_z)| = \int_{-\infty}^{\infty} |\Pi_y(S_z)| dz \leq |\Pi_y(S)|.$$

Now, in order to interpolate between these bounds, we can write

$$\begin{aligned} |S| = \int_{-\infty}^{\infty} |S_z| dz &\leq |\Pi_z(S)|^{1/2} \int_{-\infty}^{\infty} |\Pi_y(S_z)|^{1/2} |\Pi_x(S_z)|^{1/2} dz \\ &\leq |\Pi_z(S)|^{1/2} |\Pi_x(S)|^{1/2} |\Pi_y(S)|^{1/2}, \end{aligned} \tag{1}$$

where we have used the Cauchy–Schwarz inequality. This is the desired result. □

Exercise: The *Cauchy–Schwarz* inequality on $\mathbb{R}^n$ reads $\int fg dx \leq \left(\int f^2 dx\right)^{1/2} \left(\int g^2 dx\right)^{1/2}$, where dx denotes the usual Lebesgue (or Riemann) integration. Prove the Cauchy–Schwarz inequality by considering the quadratic form $\int (f - \lambda g)^2 dx$ and minimizing this in the real parameter λ.

The above proof has to "interpolate" between different possibilities. Indeed, one can consider different situations where one of $|\Pi_x(S)|$, $|\Pi_y(S)|$, or $|\Pi_z(S)|$ is much larger than the others, and we have to account for this possiblity. The bound ends up essentially taking the geometric mean of these quantities. The extremizers in these kinds of situations are often very symmetric. Extremizers refer to situations which saturate a given inequality, meaning the inequality in fact becomes equality.

Exercise: Show that the unit cube is an extremizer of the Loomis–Whitney inequality. In fact, show that any cube with faces parallel to the coordinate planes is an extremizer.

4 Averaged projected areas

Here we prove the Cauchy-Crofton formula:

Theorem 4 (Cauchy-Crofton formula): Let $B \subset \mathbb{R}^d$ be a convex body bounded by surface S. Then

$$\mathrm{Vol}_{d-1}(S) = \frac{\sigma_{d-1}}{\omega_{d-1}} \fint_{\mathbb{S}^{d-1}} \mathrm{Vol}_{d-1}(B|u^\perp)\mathrm{d}u \tag{2}$$

where $\omega_d = \mathrm{Vol}_d(B_1(0))$ and $\sigma_d := \mathrm{Vol}_d(\mathbb{S}^d)$, and $\mathrm{Vol}_{d-1}(B|u^\perp)$ is the area of the projection of B onto the plane orthogonal to $u \in \mathbb{S}^{d-1}$.

In the Theorem and below, we denoted the average integral by

$$\fint_{\mathbb{S}^{d-1}} = \frac{1}{\mathrm{Vol}_{d-1}(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}}.$$

Before going to the proof, we note some important special cases:

Remark: Theorem 4, applied to *two-dimensional convex bodies*, gives

$$\text{Average Width}(B) = \frac{1}{\pi}\text{Perimeter}(S). \tag{3}$$

This follows since $\omega_1 = 2$ (which makes sense, as the uniform measure on the sphere degenerates to the counting measure on two points) and $\sigma_1 = 2\pi$. Note that, as a consequence of this formula, we established *Barbier's theorem* that, for a body of constant width, its perimeter is exactly π times its width.

Applied to *three-dimensional convex bodes*, $\sigma_2/\omega_2 = 4\pi/\pi = 4$, so

$$\text{Average Shadow Area}(B) = \frac{1}{4}\text{Surface Area}(S). \tag{4}$$

For *higher dimensional convex bodies*, we recall $\omega_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$ and note

$$\frac{\sigma_{d-1}}{\omega_{d-1}} = \frac{d\omega_d}{\omega_{d-1}} = \sqrt{\pi}d\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2}+1)} \approx \sqrt{2\pi d} + O(\tfrac{1}{\sqrt{d}}). \tag{5}$$

Thus, for increasing dimensions, we see

$$\frac{\text{Average Shadow Volume}(B)}{\mathrm{Vol}_{d-1}(S)} \sim \frac{1}{\sqrt{2\pi d}}, \tag{6}$$

indicating bodies with surface area growing slower than $\sqrt{d}$ (inexplicably, the diagonal of the unit cube in $\mathbb{R}^d$) don't block much light in high dimensions.

Proof. We require the following

Lemma 1: Let B be a convex body in $\mathbb{R}^d$ with bounding surface S and unit outer normal field $\hat{n}$. For any $u \in \mathbb{S}^{d-1}$, the following identity holds

$$\int_S |\hat{n}(x) \cdot u| \mathrm{d}S(x) = 2\mathrm{Vol}_{d-1}(B|u^\perp) \tag{7}$$

Proof # 1. To prove this assertion, we note that – due to convexity – we can represent the set of points on the surface S such that $u \cdot \hat{n}(x) \neq 0$ as the union of two graphs over over the plane perpendicular to u. Without loss of generality, we can assume that u is the unit vector pointing along the z axis, so that we are projecting onto the (x, y) plane. The key observation is that the area element $dS(x)$ of the surface is related to the area element of the plane $dxdy$ exactly via $u \cdot \hat{n}(x)$. To see this, we compare the two area elements as follows. Let p be a point in S for which $u \cdot n(p) \neq 0$, and take two unit vectors tangent to S. Denote these vectors by e_1 and e_2, and note that we can actually take e_1 to be orthogonal to u. Because of this, note that $\mathbb{Q}_u e_1 = e_1$, where $\mathbb{Q}_u v = v - (u \cdot v)u$ for any vector v is the orthogonal projection. Moreover, we can write that $\mathbb{Q}_u e_2 \cdot e_1 = 0$. Indeed, we can write that $\mathbb{Q}_u e_2 \cdot e_1 = e_2 \cdot e_1 - (u \cdot e_2)(u \cdot e_1) = 0$, as desired.

Now, note that e_1 and e_2 define a unit square tangent to S, meaning they correspond to a unit area element. In order to compare with the area element on the plane, we must understand the area of the projection of this square. The above considerations tell us that the projection of the square is a rectangle with one side length equal to 1 and the other side length given by $|\mathbb{Q}_u e_2|$, so we must now simply compute this. We can form a right triangle in the plane spanned by e_2 and u where the hypotenuse has length 1 and points in the same direction as e_2 (note that the vertical direction corresponds to u). Now, the length of the base of this triangle is exactly equal to $|\mathbb{Q}_u e_2|$, which we can compute as being $\cos(\theta)$, where θ is the angle between the hypotenuse and the base. Now, if we rotate e_2 by $\pi/2$, we get the normal vector $n(p)$, and if we rotate a unit vector pointing along the base of the triangle in the same direction, we get u. This tells us that $|\cos(\theta)| = |u \cdot n(p)|$. From this, it follows that $|u \cdot n(p)| dS(p) = |\cos(\theta)| dS(p) = dxdy$, from which the desired result follows upon integrating (note that integrating the area element of the plane $dxdy$ over the shadow certainly produces the area of that shadow). □

Proof # 2. We now offer another proof of Lemma 1, in dimension 3 and the zero set $\Gamma_u := \{\hat{n}\cdot u = 0\}$ is a smooth closed curve (a condition which is generic and holds on approximations). Let $S_u^{\pm} := \{x \in S \ \ \pm\hat{n}\cdot u > 0\}$. Note, denoting $v_u(x) = \frac{1}{2}u \times x$, we have $\nabla \times v_u(x) = u$[1] and, by Stokes' theorem,

$$\begin{aligned}
\int_S |\hat{n}(x)\cdot u|\mathrm{d}S(x) &= \int_{S_u^+} \hat{n}(x)\cdot u\mathrm{d}S(x) - \int_{S_u^-} \hat{n}(x)\cdot u\mathrm{d}S(x) \\
&= \int_{S_u^+} \hat{n}\cdot\nabla\times v_u\mathrm{d}S(x) - \int_{S_u^-} \hat{n}\cdot\nabla\times v_u\mathrm{d}S(x) \\
&= 2\int_{\Gamma_u} v_u\cdot(u\times\hat{n}(x))\mathrm{d}s = \int_{\Gamma_u}(u\times x)\cdot(u\times\hat{n}(x))\mathrm{d}s \\
&= \int_{\Gamma_u} x\cdot\hat{n}(x)\ \mathrm{d}s = \int_{\mathbb{P}_{u^\perp}\Gamma_u} x\cdot\hat{n}(x)\ \mathrm{d}s \\
&= 2\mathrm{Area}(B|u^\perp)
\end{aligned}$$

where we used that $\hat{n}$ is orthogonal to u to write the integral over Γ_u as the integral over the curve projected onto the plane perpendicular to u, and the last line follows by an application of the divergence theorem in two dimensions. □

Let us now proceed to use this fact to prove the Cauchy-Crofton formula. The key is to create a symmetric expression between the convex body B and the unit sphere $\mathbb{S}^{d-1}$. If we integrate the above expression in u for all u in the unit sphere, we get

$$\int_{\mathbb{S}^{d-1}}\int_S |\hat{n}(x)\cdot u|dS(x)du = \int_{\mathbb{S}^{d-1}} 2\mathrm{Vol}_{d-1}(B|u^\perp)du \tag{8}$$

Note that the expression on the right is exactly (twice) the average projected area of the body in all possible directions. Thus, it suffices to show that the expression on the left is proportional to the surface area of S. To see that this is the case, we apply Fubini's theorem to interchange the order of integration. This gives us that

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}}\int_S |\hat{n}(x)\cdot u|dS(x)du &= \int_S\int_{\mathbb{S}^{d-1}} |\hat{n}(x)\cdot u|dudS(x) \\
&= 2\int_S \mathrm{Vol}_{d-1}(B_1|u^\perp)dS(x) = 2\omega_{d-1}\mathrm{Vol}_{d-1}(S).
\end{aligned}$$

Here we used Lemma 1 again in the second line, and then that the projection of the unit ball onto any hyperplane is the $d-1$ ball. Combining with (8), we find $\mathrm{Vol}_{d-1}(S) = \frac{1}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}}\mathrm{Vol}_{d-1}(B|u^\perp)du$, as desired. □

[1] $\nabla\times(a\times b) = a\nabla\cdot b - b\nabla\cdot a + b\cdot\nabla a - a\cdot\nabla b$

5 Isoperimetric inequality

We now give a quick calculus proof of the classical isoperimetric inequality.

Theorem 5 (Isoperimetric inequality): Let $B \subset \mathbb{R}^d$ be a simply connected compact body bounded by a continuously differentiable surface S. Then

$$\frac{\mathrm{Vol}_{d-1}(S)}{\sigma_{d-1}} \geq \left(\frac{\mathrm{Vol}_d(B)}{\omega_d}\right)^{\frac{d-1}{d}}, \tag{9}$$

where $\omega_d = \mathrm{Vol}_d(B_1(0))$ and $\sigma_d := \mathrm{Vol}_d(\mathbb{S}^d)$. Moreover, for a fixed surface area, the unique body that maximizes volume is the sphere.

Proof. Say, wlog, that S encloses 0. Compute the volume via Gauss's theorem

$$\mathrm{Vol}_d(B) = \frac{1}{d}\int_B \mathrm{div} x \; \mathrm{d}x = \frac{1}{d}\int_S x \cdot \hat{n}(x) \; \mathrm{d}S \tag{10}$$

where $\hat{n}$ is the unit outer normal to S. Clearly,

$$\mathrm{Vol}_d(B) \leq \frac{1}{d}\int_S |x| \; \mathrm{d}S \tag{11}$$

To finish...

To see that the sphere is the only extremize of the inequality, note that the string of inequalities results in equality if and only if S is a sphere. First note (11) holds with equality if and only if $x \parallel \hat{n}(x)$ for all $x \in S$. Next we claim if $x \parallel \hat{n}(x)$ for all $x \in S$, then $|x|$ must be constant on S, and thus S must be a sphere. To see this, let $f(x) := \frac{1}{2}|x|^2$ and consider any C^1 path $y(t) \in S$. Since, $\dot{y}(t) \in T_{y(t)}S$, we have that for any $0 = \nabla f(y(t)) \cdot \dot{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(y(t))$. Thus, $|x||_S$ is constant, so S is a sphere. □

Remark (Behavior in large dimensions): The behavior of volumes and areas in large ambient dimensions is somewhat counterintuitive. For example, consider the relative volume in a δ thickness shell of a radius R sphere in large dimension

$$\frac{\mathrm{Vol}(B_{R+\delta})}{\mathrm{Vol}(B_R)} = \left(1 - \frac{\delta}{R}\right)^d \tag{12}$$

If you have a water with radius of 1 meter, and skin of 1 cm thickness in dimension 1000, then after you peel the skin less than one one-thousandth of

the volume remains. Thus, the overwhelming part of the volume of a body in high dimensions is concentrated in a small neighborhood of its boundary. This phenomenon is called *concentration of measure*.

Consider a multidimensional cube with side length R. Peel the cube, like an orange, by removing the $\delta/2$-neighborhood of its boundary. What remains is a cube with side length $R-\delta$, whose relative volume is $\frac{\text{Cube}_{R-\delta}}{\text{Cube}_R} = \left(1-\frac{\delta}{R}\right)^d$. Let us now analyze the isoperimetric inequality: we see that the surface area dominates the volume of a body in large dimensions

$$\frac{\text{Vol}_{d-1}(S)}{\text{Vol}_d(B)^{\frac{d-1}{d}}} \geq d\omega_d^{\frac{1}{d}} \to \infty \quad \text{as} \quad d \to \infty. \tag{13}$$

Recall also that, for increasing dimensions, the Cauchy-Crofton formula tells

$$\frac{\text{Vol}_{d-1}(S)}{\text{Average Shadow Volume}(B)} \to \infty \quad \text{as} \quad d \to \infty. \tag{14}$$

which is somehow a manifestation of the same fact.

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