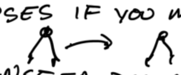
Constructions

~~RULER~~ STRAIGHTEDGE & COMPASS

Wednesday, December 2, 2020 5:58 PM

IDEA STARTED $\leq$ 500 BC W/ ANCIENT GREEKS.
CODIFIED BY EUCLID (c. 300 BC) { "CLASSICAL CONSTRUCTIONS" / "CONSTRUCTIVE GEOMETRY" }

- STRAIGHTEDGE (RULE): ARBITRARILY LONG
  NOT TO MEASURE, JUST TO DRAW STRAIGHT LINE THROUGH 2 POINTS OR TO EXTEND A SEGMENT
- COMPASS CAN DRAW A CIRCLE KNOWING THE CENTER & RADIUS.
  (FORMALLY, IT COLLAPSES IF YOU MOVE IT



  CAN'T USE TO TRANSFER DISTANCE.
  DOESN'T MATTER, CAN SHOW THAT CAN STILL TRANSFER DISTANCE USING COLLAPSING KIND)

IN FACT: EVERY POINT YOU CAN MAKE BY RULER & COMPASS CAN JUST BE DONE WITH A COMPASS (MOHR-MASCHERONI THM) 1672 1797

OLDER: RULER & A RUSTY COMPASS,
IE STRAIGHTEDGE & ONE CIRCLE.
IS ENOUGH. IDEA ~ 900 AD BUT PROVEN
PONCELET-STEINER THM (1833)

- EVERY CONSTRUCTION IS EXACT. (NO MEASURING)
- EACH CONSTRUCTION TERMINATES IN FINITE # OF STEPS (NO LIMITING PROCESS)

SEEMS LIKE JUST A "GAME".
THE POINT IS CONSTRUCTION CAN BE PROVEN TO GIVE EXACT SOLUTION - EXACTLY CORRECT.

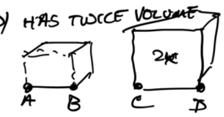
CONSTRUCTIONS CORRESPOND EXACT TO EUCLID'S 1ST 3 POSTULATES:

RULER {
I. A STRAIGHT LINE CAN BE DRAWN JOINING ANY TWO POINTS
II. ANY LINE SEGMENT CAN BE EXTENDED INDEFINITELY. } (L1)

COMPASS { III. GIVEN ANY STRAIGHT SEGMENT, A CIRCLE CAN BE DRAWN WITH ITS LENGTH AS RADIUS & ONE END AS CENTER.

AT THE TIME OF EUCLID, SOME UNSOLVED QUESTIONS:

- TRISECTING ANY ANGLE
  GIVEN AN ANGLE OF $X°$, MAKE ONE OF $X/3°$
- DOUBLING THE CUBE
  GIVEN $\overline{AB}$, CONSTRUCT $\overline{CD}$ SO THAT CUBE WITH SIDE LENGTH $|CD|$ HAS TWICE VOLUME OF THE ONE WITH $|AB|$



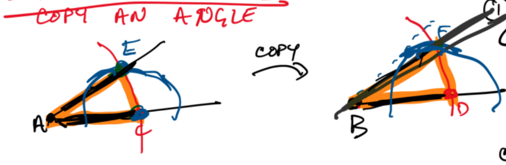
- SQUARING THE CIRCLE
  GIVEN A CIRCLE OF RADIUS r, CONSTRUCT A SQUARE OF THE SAME AREA

- CONSTRUCTING REGULAR POLYGONS OF n-SIDES
  KNOW HOW TO DO $n = 3, 4, 5, 6, \overset{?}{7}, 8, 9, 10, 11$
  GAUSS (1796)

THESE PROBLEMS / CONSTRUCTIONS LED TO
- CONIC SECTIONS (MENAECHMUS ~ 350 BC) ELLIPSES, PARABOLAS, HYPERBOLAE
- COORDINATES (DESCARTES 1600s)
- MODERN ALGEBRA, NUMBER THEORY, ANALYSIS.

BASIC CONSTRUCTIONS

~~COPY AN ANGLE~~

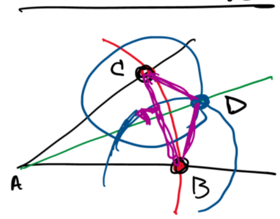


1. PICK A POINT C ON ONE SIDE OF $\angle A$. DRAW A CIRCLE OF RADIUS $|AC|$ CENTER A
2. REPRODUCE CIRCLE W/ CENTER AT B SO THAT $|AC| = |BD|$)
3. DRAW A CIRCLE WITH CENTER C, RADIUS IS DIST FROM C TO INTERSECTION OF SIDE W/ FIRST CIRCLE AT E.
4. BRING THAT DISTANCE, TO CIRCLE AT D WITH RADIUS $|CE|$.
5. $\angle FBD = \angle EAC$

WHY DOES $\angle FBD = \angle EAC$?

SHOWN: $\triangle CAE \cong \triangle DBF$ BY SSS: $|AC| = |BD|$, $|AE| = |BF|$, $|CE| = |DF|$

BISECT AN ANGLE



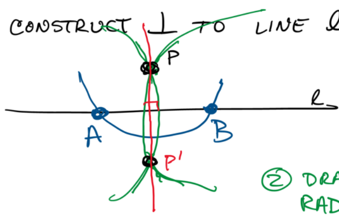
1. DRAW A CIRCLE WITH CENTER A [NOW $|AB| = |BC|$]
2. DRAW TWO CIRCLES OF SAME RADIUS (BIG ENOUGH TO INTERSECT) CENTERS AT B & C.

POINT OF INTERSECTION D SATISFIES $|CD| = |BD|$.

SO $\overrightarrow{AD}$ IS BISECTOR.

APPLYING FACT THAT $AD \perp CB$ SINCE BISECTS CHORD.

CONSTRUCT $\perp$ TO LINE ℓ THRU P.



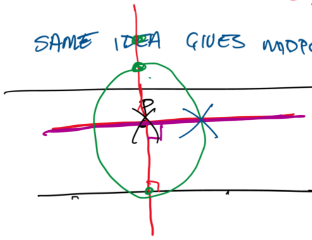
DRAW A CIRCLE OF ANY RADIUS CENTER P

GET $|AP| = |BP|$ ON ℓ

(2) DRAW CIRCLE AT B W/ RAD $|BP|$ & AT A, SAME RADIUS

(3) CONNECT INTERSECTION $\overline{PP'} \perp \overline{AB}$

SAME IDEA GIVES MIDPOINT $\overline{AB}$

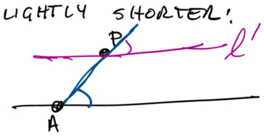


CONSTRUCT ℓ' WHICH IS // TO ℓ CONTAINING P.

(1) CONSTRUCT $\perp$ TO ℓ THRU P m

(2) CONSTRUCT $\perp$ TO m AT P. $= \ell'$

SLIGHTLY SHORTER:



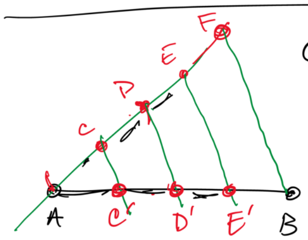
(1) PICK A ON ℓ

(2) DRAW $\overrightarrow{AP}$

(3) COPY THE ANGLE $\overrightarrow{AP}$ MAKES WITH ℓ TO GET ℓ'

(CORRESP. ANGLES)

SUBDIVIDE SEGMENT INTO n EQUAL PIECES.



(1) DRAW SOME LINE WITH A POINT C ON IT

(2) COPY $|AC|$ n TIMES TO GET D, E, ... F

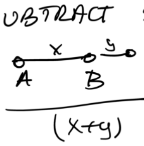
(3) DRAW $\overleftrightarrow{FB}$ & PARALLELS TO FB THRU E, D, C, ...

WHY DOES IT WORK?

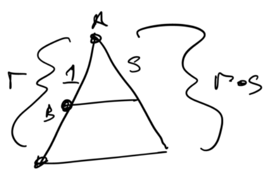
USE FTS* SIMILAR TRIANGLES ...

CAN DO ARITHMETIC WITH CONSTRUCTIONS.

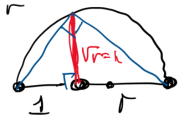
- ADD & SUBTRACT SEGMENTS



- MULTIPLY & DIVIDE USE SIMILAR $\triangle$'s



- CAN TAKE SQUARE ROOTS. GIVEN r



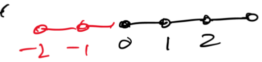
- MAKE SEGMENT OF LENGTH r & OF 1
- CONSTRUCT CIRCLE W/ DIAM $= r+1$

HEIGHT IS $\sqrt{r}$ BY SIMILAR TRIANGLES

$$\frac{1}{h} = \frac{h}{r} \Rightarrow h^2 = r \Rightarrow h = \sqrt{r}$$

CAN MAKE NUMBERS:

$\mathbb{Z}$



$\mathbb{Q}$ BY DIVIDING, MULTIPLYING

CAN TAKE SQUARE ROOTS

$\sqrt{\mathbb{Q}}$ $\left(\text{GET \#'s LIKE } \frac{\sqrt[4]{5} + \sqrt{2} - 1}{1\sqrt[8]{7} - 8\sqrt{3}} \right)$

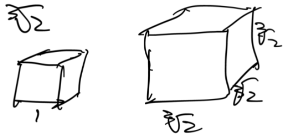
BUT NO $\sqrt[3]{2}$, $\sqrt[5]{8}$, ...

NO TRANSCENDENTALS LIKE π

THM: $\boxed{\text{A NUMBER IS CONSTRUCTIBLE} \iff x \in \sqrt{\mathbb{Q}}}$

WHY IS DOUBLING THE CUBE IMPOSSIBLE?

NEED TO MAKE $\sqrt[3]{2}$



- SQUARING CIRCLE $\Rightarrow$ MAKE π BY CONSTRUCTION (ACTUALLY $\sqrt{\pi}$)

- TRISECTING ANGLE MUST SOLVE A CUBIC.

THM: (GAUSS) LET P BE PRIME

- A REGULAR P-GON IS CONSTRUCTIBLE
  $\iff$ P IS A FERMAT PRIME $2^{2^k}+1$ $k \in \mathbb{N}$
- A REGULAR n-GON IS CONSTRUCTIBLE
  $\iff$ $n = 2^k P_1 P_2 P_3 \cdots P_m$ P_i: DISTINCT FERMAT

ORIGAMI: MAKES ALL CONSTRUCTIBLE
+ CAN TAKE $\sqrt[3]{\ }$

ORIGAMI $\Rightarrow$ CAN DOUBLE CUBE
CAN TRISECT ANGLE
– NO SQUARE CIRCLE –