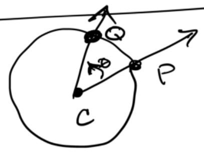
Transformations: Rotations

Monday, September 14, 2020 1:25 PM

LAST TIME: DEFINED A ROTATION ρ_Θ OR $\rho_{\Theta,C}$. CLOCKWISE ROTATION ABOUT POINT C WITH ANGLE Θ, $-360° \le \Theta \le 360°$

C ANY [FIXED] POINT IN PLANE



$\rho_{\Theta,C}(P) = Q$

F TRANSFORMATION $F(P) = Q$

Q IS THE IMAGE OF P (UNDER F)

S ANY SET IN DOM F, $F(S) = \{F(P) \mid P \in S\}$.

F MAPS S TO F(S), F(S) IMAGE OF S UNDER F.

INJECTIVE $\iff$ 1-1

SURJECTIVE $\iff$ ONTO

BIJECTIVE $\iff$ 1-1 & ONTO (BOTH SURJ INJ)

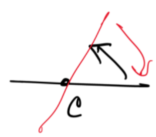
IF F & G ARE TWO TRANSFORMATIONS WITH $F \circ G = G \circ F = Id$

THEN G IS INVERSE OF F, $G = F^{-1}$

LEMMA: ANY FUNCTION F WITH A (GLOBAL) INVERSE $\implies$ F IS A BIJECTION

LEMMA: EVERY ROTATION $\rho_{\Theta,C}$ IS A BIJECTION

Pf: $(\rho_{\Theta,C})^{-1} = \rho_{-\Theta,C}$



DEF: A TRANSFORMATION F IS AN ISOMETRY IF FOR ALL P, Q

$$\mathrm{dist}(P,Q) = \mathrm{dist}(F(P),F(Q))$$

IE, ISOMETRIES PRESERVE DISTANCE AND LENGTHS OF SEGMENTS.

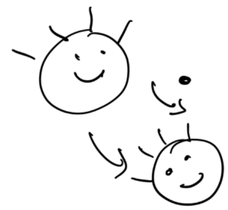
ISOMETRIES ARE INJECTIVE:

Pf:

SPOZE $P \ne Q$, $\mathrm{dist}(P,Q) \ne 0$

$\iff F(P) \ne F(Q)$ SINCE $\mathrm{dist}(F(P),F(Q)) \ne 0$

"OBVIOUS FACT" ROTATIONS PRESERVE GEOMETRIC FIGURES.



ISSUE: WE DEFINED LINES, POINTS & SO SEGMENTS ANGLES... IN TERMS OF PROPERTIES

MUST HAVE AXIOMS OR ASSUMPTIONS THAT TELL US HOW THEY WORK ON LINES, ETC.

ASSUMPTIONS ABOUT ROTATIONS

(ρ_1) ANY ROTATION MAPS LINES TO LINES, A SEGMENT TO A SEGMENT, A RAY TO A RAY, AN ANGLE TO AN ANGLE

(ρ_1) ANY ROTATION PRESERVES LENGTHS (ISOMETRY) AND ANGLE MEASURE

NOTE: ROTATIONS WITH SAME CENTER COMMUTE: $\rho_{\Theta,C} \circ \rho_{\sigma,C} = \rho_{\Theta+\sigma,C} = \rho_{\sigma,C} \circ \rho_{\Theta,C}$

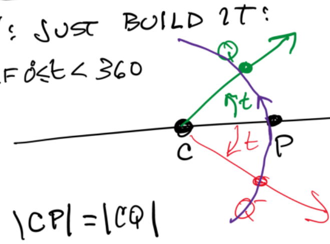
BUT DIFFERENT CENTERS DON'T COMMUTE: $A \ne B$

$$\rho_{\Theta,A} \circ \rho_{\sigma,B} \ne \rho_{\sigma,B} \circ \rho_{\Theta,A}$$

LEMMA: GIVEN A POINT C AND $-360 \le t \le 360$, THEN THERE IS A ROTATION BY t DEGREES AROUND C.

Pf/: JUST BUILD IT:

IF $0 \le t < 360$



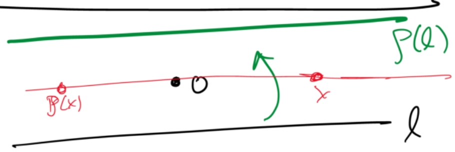
BY (L6) CAN FIND Q SO THAT $\measuredangle QCP = t^\circ$

(OR Q IN IN OTHER DIRECTION FOR $-360 < t < 0$)

$|CP| = |CQ|$

ASSIGN $\rho_t(P) = Q$

THM G1 LET ℓ BE A LINE, O A POINT NOT ON ℓ AND $\rho = \rho_{180,O}$. THEN $\rho(\ell) \parallel \ell$



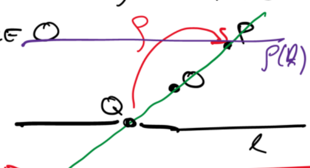
Pf/ CAN PROVE BY CONTRADICTION OR DIRECT PROOF.

LET P BE ANY POINT ON $\rho(\ell)$. WANT TO SHOW $P \notin \ell$.

IF $P = O$ DONE SINCE O ISN'T ON ℓ

ASSUME $P \neq O$

CONSTRUCT $\overleftrightarrow{PO}$



SINCE $P \in \rho(\ell)$, THEN THERE IS A Q SO THAT $\rho(Q) = P$

$|PO| = |QO|$, P, O, Q ARE COLINEAR

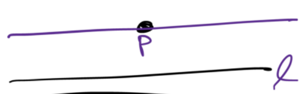
BUT $Q \in \ell$, $\overleftrightarrow{PO}$ INTERSECTS ℓ AT Q

SO $P \notin \ell$, SINCE OTHERWISE $\overleftrightarrow{QP} = \ell$ BUT P IS NOT O

SO $\rho(\ell) \parallel \ell$

RECALL PARALLEL POSTULATE (L2)

GIVEN ℓ AND POINT P NOT ON IT THEN THERE IS AT MOST ONE LINE THROUGH P PARALLEL TO ℓ.



COR TO G1 (PLAYFAIR VERSION OF // POSTULATE)

GIVEN ℓ, P NOT ON ℓ, THERE IS EXACTLY ONE LINE THRU P PARALLEL TO ℓ.

Pf: LET $Q \in \ell$

LET M BE MIDPOINT OF $\overline{PQ}$

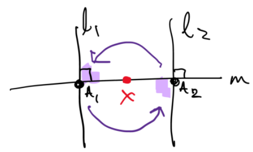
NOW $\rho = \rho_{180,M}$ LET ℓ_1 BE $\ell_1 = \rho(\ell)$

WE HAVE $P \in \ell_1$ SINCE $|MP| = |QM|$

BUT $\ell_1 \parallel \ell$, (L6) $\Rightarrow$ UNIQUE.

THM G2 LET m, ℓ_1, ℓ_2 BE LINES. SPOZE $\ell_1 \perp m$, $\ell_2 \perp m$. THEN $\ell_1 = \ell_2$ OR $\ell_1 \parallel \ell_2$.

IN HW, IF YOU TAKE A POINT ON A LINE, THERE IS A UNIQUE PERPENDICULAR.



TAKE $A_1 = \ell_1 \cap m$, $A_2 = \ell_2 \cap m$

SO ASSUME IF $A_1 = A_2$ DONE.

$A_1 \neq A_2$, LET X BE MIDPOINT OF $\overline{A_1 A_2}$

$\rho = \rho_{180,X}$. CERTAINLY $\rho(A_1) = A_2$, $\rho(A_2) = A_1$

$A_2 \in \ell_2$, $A_2 \in m$.

SINCE $\ell_1 \perp m$ AND (ρ_2) ROTATION PRESERVES ANGLES, SO $\rho(\ell_1) \perp m$ AT A_2

SO $\rho(\ell_1) = \ell_2$ SINCE BOTH ARE $\perp$ TO m AT A_2. APPLY HW#1 $\blacksquare$

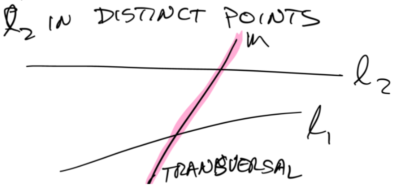
COR: RECTANGLES ARE PARALLELOGRAMS

RECTANGLE = QUADRILATERAL W/ 4 RIGHT ANGLES

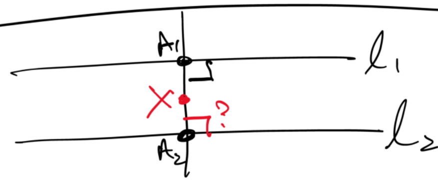
PARALLOGRAM = QUADRILATERAL W/ OPPOSITE SIDES $\parallel$

RED // BLACK //

DEF GIVEN ℓ_1, ℓ_2 LINES, A TRANSVERSAL OF ℓ_1, ℓ_2 IS A LINE THAT INTERSECTS ℓ_1 & ℓ_2 IN DISTINCT POINTS



THM G3: LET ℓ_1 AND ℓ_2 BE PARALLEL AND m BE A TRANSVERSAL. THEN $m \perp \ell_1 \iff m \perp \ell_2$



PARALLEL LINES HAVE MUTUAL PERPENDIC

Pf/ SAME AGAIN.

m TRANSVERSE TO ℓ_2 AND ℓ_1

$\ell_2 \cap m = A_2 \quad \ell_1 \cap m = A_1$

ASSUME $\ell_1 \perp m$ WANT TO SHOW $\ell_2 \perp m$.

LET X BE MIDPOINT OF $\overline{A_1 A_2}$

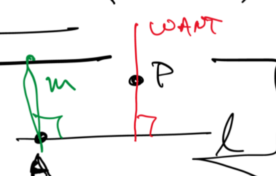
ROTATE $\rho_{180,X} = \rho$ BY (G1) $\rho(\ell_1) \parallel \ell_1$

(WANT TO SHOW $\rho(\ell_1) = \ell_2$)

BY // POSTULATE ℓ_2 IS UNIQUE // LINE THRU A_2 SO $\rho(\ell_1) = \ell_2$

SINCE ρ PRESERVES ANGLES (BY (ρ_2)) SO $m \perp \ell_2$

COR: GIVEN P NOT ON ℓ, THEN I CAN "DROP A PERPENDICULAR", IE THERE IS A UNIQUE LINE L THRU P SO THAT $L \perp \ell$



Pf/ CHOOSE $A \in \ell$. EITHER $\overleftrightarrow{AP} \perp \ell$ (SO DONE) OR NOT. IF NOT LET m BE UNIQUE LINE $\perp$ TO ℓ AT A.

$\exists!$ LINE L THRU P PARALLEL TO m

ℓ INTERSECTS BOTH L AND m SO TRANSVERSE SO APPLY THM TO GET $\perp$.

COR: RECTANGLES EXIST!