Polygons, separation, convexity

Monday, August 31, 2020 6:01 PM

LAST TIME: POINTS, LINES, PLANES

(L1) THROUGH ANY 2 POINTS THERE IS A UNIQUE LINE

(L2) PARALLEL POSTULATE
GIVEN A LINE ℓ AND POINT P NOT ON ℓ, THERE IS AT MOST ONE LINE m WITH $m \parallel \ell$

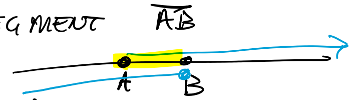
DEFINE RAY: $\overrightarrow{AB}$ = RAY FROM A IN DIRECTION OF B

DEF: MAKE $\overleftrightarrow{AB}$ INTO A #LINE WITH #(A) = 0 AND #(B) = 1.
THEN $\overrightarrow{AB} = \{C \mid A < C < B$ OR $C > B$ AND A AND B$\}$ — CORRECT BUT COMPLICATED

$= \{C \mid C \geq A\}$ ← BETTER

BOTH OK

DEF: SEGMENT $\overline{AB}$



AS $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$

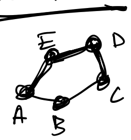
OR $\{C \mid \#C$ IS BETWEEN #A & #B, INCLUSIVELY$\}$

DEF C IS BETWEEN A & B ON $\overleftrightarrow{AB}$ IS $C \in \overline{AB}$, BUT $C \neq A$, $C \neq B$

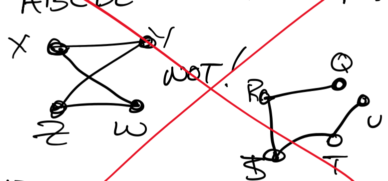
WRITE $A * C * B$

$A*C*B \iff B*C*A$

POLYGONS



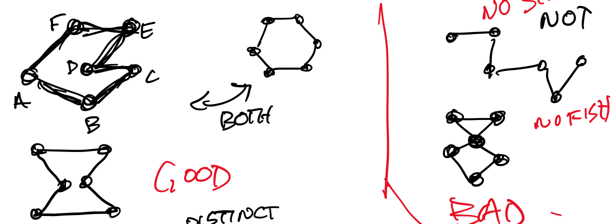
ABCDE IS A POLYGON.



HOW CAN WE DEFINE POLYGON CAREFULLY?

EXAMPLE HEXAGON

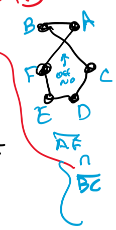
SIX POINTS A B C D E F



① NEED SIX DISTINCT POINTS A, B, C, D, E, F.

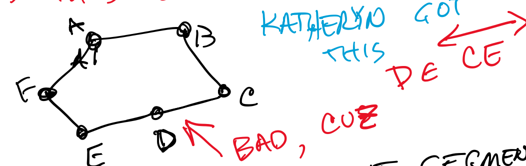
② THESE SIX SEGMENTS $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FA}$

③ SEGMENTS ONLY INTERSECT AT SHARED ENDPOINTS.

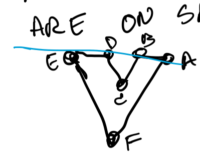


IS THIS ENOUGH? NOPE

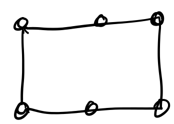
KATHERYN GOT THIS



④ NO TWO CONSECUTIVE SEGMENT ARE ON SAME LINE.



IS OK. $\overline{AB}$ AND $\overline{DE}$ ARE ON $\overleftrightarrow{AE}$ IS OK



NOT A HEXAGON

$\{A, B, C, D, E, F\}$ ARE VERTICES (EACH IS VERTEX)

$\overline{AB}$, $\overline{CD}$, ETC ARE EDGES

EASY TO GENERALIZE TO GET

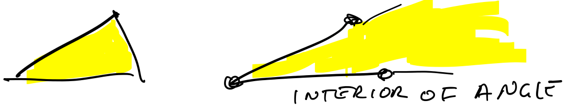
- TRIANGLE (3)
- QUADRILATERAL (4)
- PENTAGON (5)
- HEXAGON (6)
- ⋮
- n-GON (n SIDES)

TO DESCRIBE A 247-GON

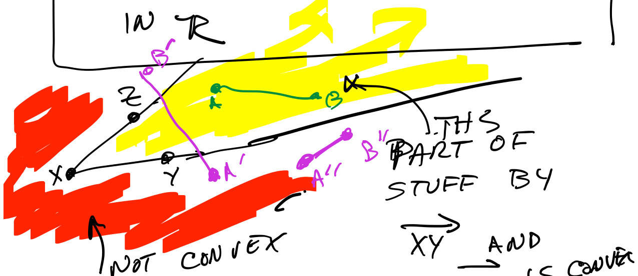
$P_1, P_2, P_3, \ldots, P_{247}$

YOU CAN FIGURE OUT THE DEF.

STILL LEFT OUT IDEA OF INTERIOR OF TRIANGLE OR ..



DEF A SUBSET $\mathcal{R}$ OF THE PLANE IS CONVEX IF GIVEN ANY TWO POINTS $A, B \in \mathcal{R}$ THE SEGMENT $\overline{AB}$ IS ENTIRELY IN $\mathcal{R}$



THIS PART OF STUFF BY $\overrightarrow{XY}$ AND $\overrightarrow{XZ}$ IS CONVEX

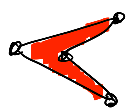
THE PLANE IS CONVEX

LINES ARE CONVEX (EASY Pf)

RAY IS CONVEX

TRIANGLE? YES.

QUADRILATERAL? NO



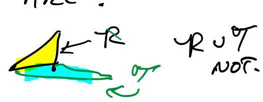
LEMMA GIVEN $\mathcal{S}$ CONVEX, $\mathcal{R}$ CONVEX, THEN $\mathcal{S} \cap \mathcal{R}$ IS CONVEX.

Pf/? SPOZE $\mathcal{S}, \mathcal{R}$ ARE CONVEX BUT $\mathcal{S} \cap \mathcal{R}$ NOT.

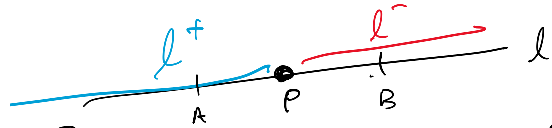
SO $\exists\, X, Y \in \mathcal{S} \cap \mathcal{R}$ WITH $\overline{XY}$ NOT IN $(\mathcal{S} \cap \mathcal{R})$

BUT $\overline{XY} \subseteq \mathcal{S}$ AND $\overline{XY} \subseteq \mathcal{R}$ $\Rightarrow\Leftarrow$

IS $\mathcal{R} \cup \mathcal{T}$ CONVEX IF $\mathcal{R}$ & $\mathcal{T}$ ARE? NOPE



(L3) LINE SEPARATION.
A POINT P ON A LINE ℓ SEPARATES ℓ INTO TWO NON-EMPTY SUBSETS ℓ^+ & ℓ^- (HALF LINES OF ℓ WRT. P)
ℓ^+ & ℓ^- SATISFY
(i) $\ell^+ \cup \{P\} \cup \ell^- = \ell$ AND ARE PAIRWISE DISJOINT
(ii) IF A & B ARE IN DIFFERENT HALF-LINES, THEN $\overline{AB}$ CONTAINS P



DEF IF $P \notin \overline{AB}$, THEN A & B ARE ON SAME SIDE OF P

IF $P \in \overline{AB}$ THE A & B ARE ON OPPOSITE SIDES OF P

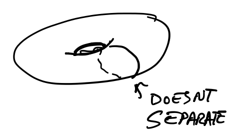
AXIOM RULES OUT P ℓ

(THIS IS ALREADY THERE FROM BETWEENNESS, BUT LETS US LEAVE OUT $\neq$ LINE)

THESE HALF-LINES WITH P FORM RAYS

PLANE SEPARATION:

NEED AN AXIOM TO ENSURE LINE SEPARATES PLANE.

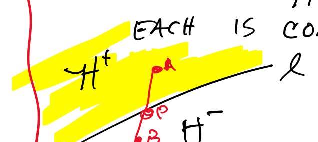


(L4) ANY LINE ℓ SEPARATES THE PLANE INTO TWO NON-EMPTY SETS $\mathcal{H}^+$ & $\mathcal{H}^-$ CALLED HALF-PLANES OF ℓ

THESE SATISFY

- $\mathcal{H}^+, \mathcal{H}^-, \ell$ ARE DISJOINT; $\mathcal{H}^+ \cup \ell \cup \mathcal{H}^-$ IS THE PLANE

  EACH IS CONVEX



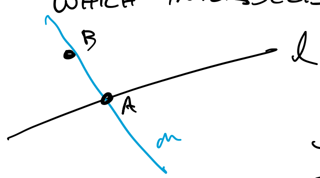
- IF A & B ARE IN DIFFERENT HALF PLANES, THEN $\overline{AB}$ INTERSECTS ℓ

A & B ARE IN SAME HALF PLANE
$\iff \overline{AB} \cap \ell = \emptyset$

$\mathcal{H}^+ \cup \ell$ IS CALLED A
CLOSED HALF PLANE

LEMMA WITH A TEDIOUS PROOF

LET ℓ BE A LINE $B \in \mathcal{H}^+$ OF ℓ
SPOSE m IS A LINE CONTAINING B
WHICH INTERSECTS ℓ AT A



THEN THE HALF-LINE OF A ON m IS $\mathcal{H}^+ \cap \ell$ AND $\overrightarrow{AB}$ IS THE PART OF m IN THE CLOSED HALF PLANE OF ℓ CONTAINING B.

DEF SAME SIDE OF ℓ IN OBVIOUS WAY

LEMMA LET ℓ & m BE TWO DISTINCT LINES P_1, P_2, P_3 SATISFYING $P_1 * P_2 * P_3$.
TAKE 3 MUTUALLY // LINES PASS THRU EACH OF P_1, P_2, P_3 INTERSECTING m AT Q_1, Q_2, Q_3
THEN $Q_1 * Q_2 * Q_3$