MAT513 Homework 10

Due Wednesday, April 20

Do *any five* of the problems below. If you also do a sixth problem, it will count as extra credit.

1. Suppose f is differentiable on an interval A. Prove that if $f'(x) \neq 0$ on A, then f must be one-to-one on A. Give an example that shows the converse does not always hold.

2. Let $f\colon [a,b] \to \mathbb{R}$ be a one-to-one function, and let $B = f([a,b])$. Then there is an inverse function $f^{-1}\colon B \to [a,b]$ given by $f^{-1}(y) = x$ where $f(x) = y$. You may assume† that if f is a continuous function, then so is f^{-1}.

Assume f is differentiable on $[a,b]$ with $f'(x) \neq 0$ for every $x \in [a,b]$. Show that f^{-1} is differentiable on B with $\left(f^{-1}\right)'(y) = 1/f'(x)$ where $y = f(x)$.

3. By analogy with the definition of uniform continuity, let's say that a function $f\colon A \to \mathbb{R}$ is **uniformly differentiable** on A if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \varepsilon \quad \text{whenever} \quad 0 < |x-y| < \delta \text{ with } x,y \in A.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$? What about $g(x) = x^3$?

(b) Show that if a function f is uniformly differentiable on an interval A, then the derivative of f must be continuous on A.

4. Let f be twice differentiable on an open interval containing the point c, and suppose that f'' is continuous at c.

(a) Show that $f'(c) = \lim_{h\to 0} \dfrac{f(c+h)-f(c-h)}{2h}$.

(b) Show that $f''(c) = \lim_{h\to 0} \dfrac{f(c+h)-2f(c)+f(c-h)}{h^2}$.

Hint: Write f'' in terms of f' via the Mean Value Theorem, then use the previous part.

5. Let $h : [0,3] \to \mathbb{R}$ be differentiable with $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Show there must be a point c with $h'(c) = 1/3$.

(b) Show there is another point b with $h'(b) = 1/4$.

6. In 2005, police in Scotland installed cameras at certain points along the A77 roadway to record license numbers and automatically calculate the average speed of individual cars between certain points along the road, then automatically issue speeding tickets to drivers whose average speed exceeded the limit. Some drivers objected that merely recording their positions at certain times was no proof that they were speeding at any given moment, especially since they slowed down when passing the cameras.

Write a paragraph or so responding to the claim, probably with some appeal to the mean value theorem.

† The proof is strightforward; essentially, just swap the roles of ε and δ to move the continuity of f to that of f^{-1}.