Solutions to Sample Final Exam Problems MAT 131, Fall 1998

As always, these solutions to the sample **may contain typos**. They are as correct as I could make them when I wrote them, but errors always creep in. However, the method of solution is certainly correct, so any errors will be only of a typographical or algebraic nature, which you should easily be able to pick out if you have already attempted the problem. And if you haven't attempted the problem, you shouldn't be reading these solutions anyway!

1. Write the equation of the linear function f with $f(0) = 1$ and $f(3) = 3$. Also write the equation of the exponential function g with $g(0) = 1$ and $g(3) = 3$.

Solution:
The linear function must be of the form $f(x) = ax + b$. If $f(0) = 1$, then $b = 1$. The fact that $f(3) = 3$ gives us $3a + 1 = 3$, so $a = 2/3$. Thus, the linear equation is $f(x) = 2x/3 + 1$.

An exponential function is of the form $g(x) = ke^{cx}$. Since $g(0) = 1$, we have $1 = ke^0$, so $k = 1$. Now using $g(3) = 3$ gives us $3 = 1 \cdot e^{cx}$. Taking the logarithm of both sides yields $\ln 3 = \ln(e^{3c}) = 3c$. Thus $c = \ln 3/3$, giving $e^{\frac{\ln 3}{3}x}$. Notice that this reduces to $g(x) = 3^{x/3}$.

2. Compute the derivatives with respect to x for each of the following:

(a) $x^{-1/2} + x + x^{1/2}$

(b) $x^2 \sin(x^2)$

(c) $\sin^2 x + \cos^2 x$

(c) $\arcsin(5x^2)$

(d) $\displaystyle\int_x^{\ln \pi} \cos(e^t)\, dt$

(e) $x^5 - \cos x \sin x$

(f) $\ln(\cos x)$

(g) $e^{\tan x} \cos x$

(h) $\dfrac{1+x^3}{1+x^2}$

Solution:

(a) $\dfrac{d}{dx}\left(x^{-1/2} + x + x^{1/2}\right) = -\dfrac{1}{2}x^{-3/2} + 1 + \dfrac{1}{2}x^{-1/2}$

(b) $\dfrac{d}{dx}\left(x^2 \sin(x^2)\right) = 2x\sin(x^2) - x^2\cos(x^2)\cdot(2x) = 2x\sin(x^2) - 2x^3\cos(x^2).$

(c) Since $\sin^2 x + \cos^2 x = 1$, the derivative is 0.

(d) $\dfrac{d}{dx}\left(\arcsin(5x^2)\right) = \dfrac{1}{\sqrt{1-(5x^2)^2}} \cdot 10x = \dfrac{10x}{\sqrt{1-25x^4}}$

(e) To do this, we use the Fundamental Theorem of Calculus.

$$\frac{d}{dx}\left(\int_x^{\ln \pi} \cos(e^t)\, dt\right) = \frac{d}{dx}\left(-\int_{\ln \pi}^{x} \cos(e^t)\, dt\right) = -(-\cos(e^x)) = \cos(e^x).$$

(f) $\dfrac{d}{dx}(\ln(\cos x)) = \dfrac{1}{\cos x}\cdot(-\sin x) = -\dfrac{\sin x}{\cos x} = -\tan x.$

(g) $\dfrac{d}{dx}\left(e^{\tan x}\cos x\right) = e^{\tan x}\dfrac{1}{\cos^2 x}\cos x + e^{\tan x}\cdot(-\sin x) = e^{\tan x}\left(\dfrac{1}{\cos x} - \sin x\right).$

(h) $\dfrac{d}{dx}\left(\dfrac{1+x^3}{1+x^2}\right) = \dfrac{3x^2(1+x^2)-(1+x^3)(2x)}{(1+x^2)^2} = \dfrac{x^4+3x^2-2x}{(1+x^2)^2}.$

3. Compute each of the following anti-derivatives (indefinite integrals):

(a) $\int x^5\ dx$

(b) $\displaystyle\int \frac{3}{2x}\ dx$

(c) $\int x\cos x^2\ dx$

(d) $\displaystyle\int \frac{x}{1+x^4}\ dx$

(e) $\int \tan x\ dx$

(f) $\int e^{2x+1}\ dx$

Solution:

(a) $\displaystyle\int x^5\ dx = \frac{x^6}{6} + C.$

(b) $\displaystyle\int \frac{3}{2x}\ dx = \frac{3}{2}\ln x + C.$

(c) To compute $\int x\cos x^2$, first we notice that because of the $\cos$, we must have something involving a $\sin$. Since $\frac{d}{dx}(\sin x^2) = 2x\cos x^2$, we are only off by a constant factor. Thus, the antiderivative must be $\frac{1}{2}\sin x^2 + C$.

(d) $\displaystyle\int \frac{x}{1+x^4}\ dx$ requires a bit of thought. Rewriting it as $\displaystyle\int \frac{x}{1+(x^2)^2}\ dx$ makes the answer more apparent, however. If we guess that the antiderivative is $\arctan x^2 + C$ and check by taking the derivative, we see it works perfectly.

(e) $\displaystyle\int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx$. If we hadn't just seen this derivative in problem 1(f), this would be harder. But we did, so the answer is obviously $-\ln(\cos x) + C$.

(f) To compute $\int e^{2x+1}\ dx$, we guess that something like e^{2x+1} would work. Taking the derivative of that shows that it is off by a factor of 2, so the correct antiderivative must be $\dfrac{1}{2}e^{2x+1} + C$

4. Evaluate each of the following definite integrals:

(a) $\int_0^1 x^5\, dx$ (c) $\int_0^{\sqrt{\pi}} x \sin x^2\, dx$ (e) $\int_{-1}^1 \sqrt{1-x^2}\, dx$

(b) $\int_0^{\frac{\pi}{4}} \sin x\, dx$ (d) $\int_{-1}^1 \frac{x^2}{1+x^3}\, dx$ (f) $\int_{-2}^2 |x|\, dx$

Solution:

(a) $\int_0^1 x^5\, dx = \left.\frac{x^6}{6}\right|_0^1 = \frac{1}{6} - 0 = \frac{1}{6}.$

(b) $\int_0^{\frac{\pi}{4}} \sin x\, dx = -\cos x|_0^{\frac{\pi}{4}} = -\cos(\pi/4) + \cos(0) = -\frac{\sqrt{2}}{2} + 1 = \frac{2-\sqrt{2}}{2}.$

(c) $\int_0^{\sqrt{\pi}} x \sin x^2\, dx = \left.-\frac{\cos x^2}{2}\right|_0^{\sqrt{\pi}} = -\cos\pi + \cos 0 = 1 + 1 = 2.$

(d) (Unfortunately, there was a typo on this problem, so its a bit uglier than expected. Oh well, we'll do it anyway.) To calculate $\int_{-1}^1 \frac{x^2}{1+x^3}\, dx$, first notice that the derivative of the bottom is $3x^2$, so $F(x) = \frac{1}{3}\ln(1+x^3)$ is an antiderivative of $\frac{x^2}{1+x^3}$. Thus, the value of the integral is $F(1) - F(-1) = \frac{\ln 2}{3} - \frac{\ln 0}{3} = \infty$. (Sometimes typos just happen, and we have to deal with them. Let's hope there aren't any on the real final.)

(e) The integral $\int_{-1}^1 \sqrt{1-x^2}\, dx$ looks really hard, unless you realize what it is the area of. This is the area of the upper half of a circle of radius 1, so its value must be $\pi/2$.

(f) To compute $\int_{-2}^2 |x|\, dx$, we can either realize this as the area of two triangles of base 2 and height 2 or we can write it as $\int_{-2}^0 (-x)\, dx + \int_0^2 (x)\, dx$. In either case, we get $2 + 2 = 4$ as the result.

5. What is the average value of $y = x^3$ over the interval $[0, 2]$?

Solution:

The average value is given by $\frac{1}{2}\int_0^2 x^3\, dx = \frac{1}{2}\left(\frac{2^4}{4}\right) = 2.$

6. Find the point on the graph of the curve $y = \sqrt{x}$ that is closest to the point $(5, 0)$. (Hint: if d is the distance from $(5, 0)$ to a point on the curve, then it is permissible (and easier!) to minimize d^2.)

Solution:
Let $D(x)$ be the square of the distance from $(5, 0)$ to a point $(x, \sqrt{x})$ on the graph of $y = \sqrt{x}$. Then

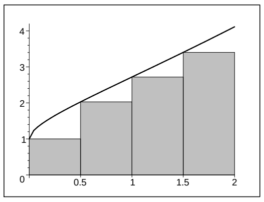
$$D(x) = (x-5)^2 + (\sqrt{x} - 0)^2 = (x-5)^2 + x = x^2 - 11x + 25$$

To find the closest point, we find the minimum of $D(x)$, which must occur at either $x = 0$ or a critical point of D. $D'(x) = 2x - 11$, so the only critical point is $x = 11/2$. This is clearly a minimum, since $D(x)$ is a parabola, opening upwards. The closest point occurs at $\left(\frac{11}{2}, \sqrt{\frac{11}{2}}\right)$.

7. Give the left-hand sum, right-hand sum, and trapezoid approximation for the integral $\int_0^2 e^{\sqrt{x}}\, dx$, using $n = 4$ rectangles. What should n be to ensure that the right-hand sum is accurate to within 0.001? (Hint: compare the expressions for the left-hand and right-hand sums for arbitrary n— what does this tell you about the exact value of the integral?)

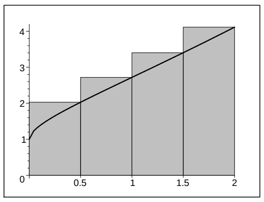
Solution:
If we are using 4 rectangles, our points are $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, $x_3 = 3/2$, and $x_4 = 2$. The left-hand sum is given by

$$L_4 = \frac{1}{2}\sum_{i=0}^{3} e^{\sqrt{x_i}} = \frac{1}{2}\left(e^0 + e^{\sqrt{1/2}} + e^1 + e^{\sqrt{3/2}}\right) \approx 4.574847253.$$



The right-hand sum is

$$R_4 = \frac{1}{2}\sum_{i=1}^{4} e^{\sqrt{x_i}} = \frac{1}{2}\left(e^{\sqrt{1/2}} + e^1 + e^{\sqrt{3/2}} + e^2\right) \approx 6.131472442,$$



and the trapezoid rule is the average of the two, that is, the trapezoid rule gives

$$T_4 = \frac{1}{4}\left(1 + 2e^{\sqrt{1/2}} + 2e^1 + 2e^{\sqrt{3/2}} + e^2\right) \approx 5.353159847.$$

To determine what n should be to ensure that the left-sum has an error of no more than 0.001, first notice that since $e^{\sqrt{x}}$ is an increasing function, we always have

$$L_n < \int_0^2 e^{\sqrt{x}}\, dx < R_n.$$

This means that the error in either one can be at most $R_n - L_n$, and is probably even less than that. So we must find n so that

$$0.001 > R_n - L_n = \frac{2}{n}\sum_{i=1}^{n} e^{\sqrt{x_i}} - \frac{2}{n}\sum_{i=0}^{n-1} e^{\sqrt{x_i}} = \frac{2}{n}\left(e^{\sqrt{2}} - e^0\right),$$

because all the middle terms cancel out. This means that if we take $n > \frac{2e^{\sqrt{2}}-2}{0.001} \approx 6227$, we will be guaranteed to have an answer that is within 0.001 of the value of the integral. (In fact, using 3250 rectangles gives a left-sum that is within 0.00096 of the correct answer.)

8. Write the equation of the line tangent to the curve $y = 3x^2 + 2x + 1$ at the point $(1, 6)$.

Solution:
The derivative of this function is $y' = 6x + 2$, so at the desired point, the slope is $6 + 2 = 8$. The tangent line (also known as the first Taylor polynomial) has the equation

$$y = 6 + 8(x - 1) \qquad \text{or} \qquad y = 8x - 2.$$

9. Write the equation of the line tangent to the curve $y^3 - 2xy + x^3 = 0$ at the point $(1, 1)$.

Solution:
To find the slope at $(1, 1)$, we use implicit differentiation to obtain $3y^2y' - 2(y + xy') + 3x^2 = 0$. At the point $(1, 1)$, this becomes $3y' - 2 - 2y' + 3 = 0$, or $y' = -1$. Using this, we get a tangent line of

$$y = 1 - (x - 1), \qquad \text{or} \qquad y = 2 - x.$$

10. From physics, we know that the illumination at a point x which is provided by a light source at L is proportional to the intensity of the light at L divided by the square of the distance between x and L. Suppose that two lights L_1 and L_2 are placed 20 meters apart, and that the intensity of L_2 is 8 times the intensity of L_1. Where is the point on the line between L_1 and L_2 where the illumination is at a minimum?

Solution:
Let x represent the position of a point on the line from L_1 to L_2, with L_1 being at $x = 0$ and L_2 being at $x = 20$. We need to write an expression for the illumination at x, and find the value of x which minimizes it. The illumination provided by L_1 at x is given by $1/x^2$, and that provided by L_2 is $8/(20 - x)^2$, so the total illumination at x is

$$I(x) = \frac{1}{x^2} + \frac{8}{(20 - x)^2}$$

Taking the derivative, we get

$$I'(x) = \frac{-2}{x^3} + \frac{16}{(20 - x)^3} = \frac{-18x^3 + 120x^2 - 2400x + 16000}{x^3(x - 20)^3} = \frac{-(6x - 40)(3x^2 + 400)}{x^3(x - 20)^3}.$$

This is only zero when $6x = 40$, that is, when $x = 20/3$. This is the only critical point, and it is clearly a minimum because $I(0)$ and $I(20)$ are both infinite. So the dimmest point is $1/3$ of the way from L_1 to L_2.

11. Find the maximimum and minimum values of the function $f(x) = x^3 + 3x^2 - 42x - 22$ on the interval $-5 \le x \le 2$.

Solution:
First we locate any critical points of the function by solving $f'(x) = 0$:

$$f'(x) = 3x^2 + 6x - 42 = 0 \quad \Longleftrightarrow \quad x = -1 \pm \sqrt{15}$$

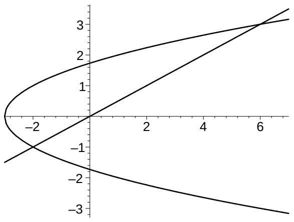
Notice that $-1 + \sqrt{15} \approx 2.873$, which is not in our domain. So now we should check the values of the function at the relevant critical point ($-1 - \sqrt{15} \approx -4.873$) and at the endpoints of our interval.

$$f(-5) = 138 \qquad f(-1-\sqrt{15}) \approx 138.1895 \qquad f(2) = -86$$

and so the maximum of f on $[-5, 2]$ occurs at $x = -1 - \sqrt{15}$, and the minimum at $x = 2$.

12. Calculate the area of the region bounded by the graphs of $y = x/2$ and $x = y^2 - 3$.

Solution:



The two curves cross at the points $(-2, -1)$ and $(6, 3)$. This is much easier to do if we integrate with respect to y, because then top of the rectangles is always given by $y = x/2$ (that is, $x = 2y$), and the bottom is the curve $x = y^2 - 3$. So the area is given by the integral

$$\int_{-1}^{3} 2y - (y^2 - 3)\, dy = y^2 - y^3/3 + 3y\Big|_{-1}^{3} = 9 + 1\frac{2}{3} = 10\frac{2}{3}.$$

Notice that if you integrate with respect to x, things are harder— the area is given by

$$\int_{-3}^{-2} 2\sqrt{x+3}\, dx + \int_{-2}^{6} \sqrt{x+3} - \frac{x}{2}\, dx$$

Do you see why?

13. Write the equation of the parabola which best approximates $y = \sin(x)$ at $x = \frac{\pi}{2}$ (that is, the second Taylor polynomial). Use your polynomial to find approximations of the nonzero solutions to $\sin(x) = x/3$. (Hint: graph the relevant functions for $-2\pi \le x \le 2\pi$ to make sure your answers make sense. The fact that $\sin(-x) = -\sin(x)$ is helpful.)

Solution:
Since $y' = \cos x$ and $y'' = -\sin x$, the second Taylor polynomial at $x = \pi/2$ is

$$P_2(x) = \sin(\pi/2) + \cos(\pi/2)(x - \pi/2) - \sin(\pi/2)(x - \pi/2)^2 = 1 - (x - \pi/2)^2 = 1 - \frac{\pi^2}{4} + \pi x - x^2.$$

Using the quadratic formula to solve where $P_2(x) = x/3$ gives $\dfrac{3\pi - 1 \pm \sqrt{37 - 6\pi}}{6}$, or about 0.694 and 2.114. The solution we want is clearly 2.114, and because of the symmetry, the other non-zero solution is near -2.114.

14. Use Newton's method to determine the nonzero solutions of $\sin(x) - x/3 = 0$ to within 0.000005. You may either use your answer to the previous problem as x_0, or use $x_0 = 2$.

Solution:
In this case, the Newton iteration is

$$x_{n+1} = x_n - \frac{\sin x_n - x_n/3}{\cos x_n - 1/3}$$

Starting with $x_0 = 2.114$, we get $x_1 = 2.292061527$, $x_2 = 2.278928744$, $x_3 = 2.278862662$, and $x_4 = 2.278862660$. So the desired result is ± 2.27886266.

15. Compute the following limits. Distinguish between $+\infty$, $-\infty$, and "does not exist".

Solution:

(a) $\lim_{x\to+\infty} \tan x$ does not exist, since the tangent is periodic with period π.

(b) $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6$

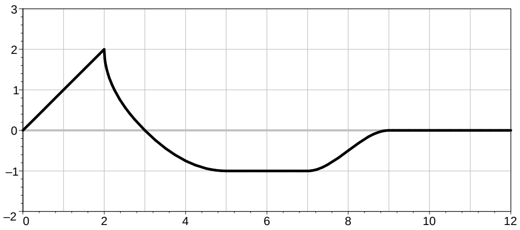
(c) $\lim_{x\to 3} \frac{x-3}{x^2-9} \lim_{x\to 3} \frac{1}{(x+3)} = \frac{1}{6}$

(d) $\lim_{x\to 3} \frac{x^2+9}{x-3}$: As $x \to 3$, the numerator tends towards 18, and the denominator tends towards 0. However, since $x > 3$ means the denominator is positive and $x < 3$ means it is negative, we have $\lim_{x\to 3+} \frac{x^2+9}{x-3} = +\infty$ and $\lim_{x\to 3-} \frac{x^2+9}{x-3} = -\infty$. Hence the two-sided limit does not exist.

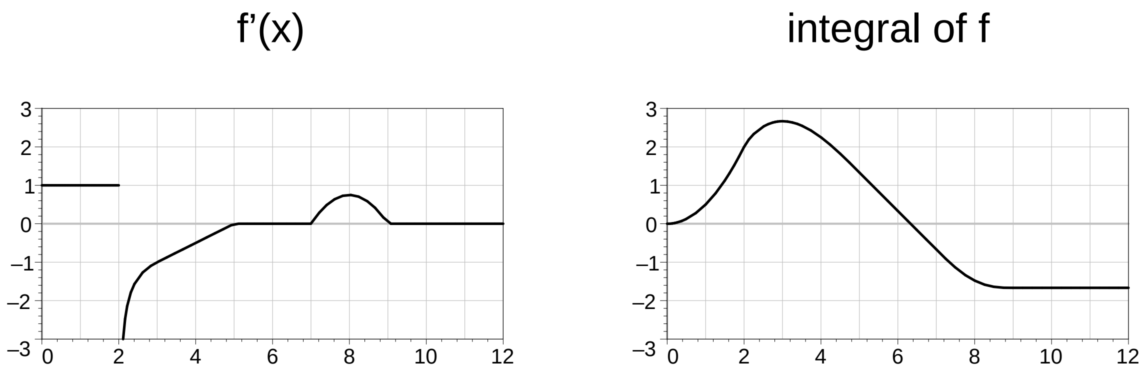
(e) $\lim_{x\to+\infty} x^{27}e^{-x} = 0$, since e^{-x} tends to zero faster than any polynomial grows.

(f) $\lim_{x\to 3+} \frac{x^2+9}{x-3} = +\infty$ (see part (d)).

16. The figure below is the graph of a function $f(x)$. Use it to sketch the graph of $f'(x)$ and the graph of $\int_0^x f(t)\,dt$.



Solution:



17. For each differential equation on the left, indicate which function on the right is a solution.

(a) $y' = 5y$	(1) $y = x^2$
(b) $y' = y(y-1)$	(2) $y = \ln x$
(c) $x^2y'' + 2xy' = 1$	(3) $y = (1+e^{-x})^{-1}$
(d) $yy'' = xy'$	(4) $y = e^{5x}$

Solution:

(a) The solution to $y' = 5y$ is (4): $y = e^{5x}$, because for this function, $y' = 5e^{5x}$, which is $5y$.

(b) (3) is the solution to $y' = y(y-1)$: $\dfrac{d}{dx}\left(\dfrac{1}{1+e^{-x}}\right) = \dfrac{e^{-x}}{(1+e^{-x})^2}$. On the other side, we have $y(y-1) = \dfrac{1}{1-e^{-x}}\left(\dfrac{1}{1-e^{-x}} - 1\right) = \dfrac{1}{1-e^{-x}}\left(\dfrac{e^{-x}}{1-e^{-x}}\right)$, so they are the equal.

(c) The answer is (2): For $y = \ln x$, we have $y' = 1/x$ and $y'' = -1/x^2$. Plugging into the expression $x^2y'' + 2xy'$, we get $x^2\frac{-1}{x^2} + 2x\frac{1}{x} = -1 + 2 = 1$, so it satisfies the equation.

(d) Of course the only choice left is (1), but let's check anyway. Here we have $y = x^2$, so $y' = 2x$ and $y'' = 2$. Thus, $yy'' = 2x^2$, and $xy' = 2x^2$, so the equation is satisfied.

18. A coffee filter has the shape of an inverted cone. Water drains from it at a constant rate of 10 cm^3/min. When the depth is 8 cm, the water level drops at a rate of 2 cm/min. What is the ratio of the height of the cone to its radius? You may find it useful to recall that the volume of a cone of radius r and height h is $\pi r^2 h/3$.

Solution:
Let $V(t)$ denote the volume of water in the cone at time t, and $h(t)$ be the depth of the water, with $r(t)$ being the radius at height h. Since the water is in a cone, the ratio r/h is always constant; let us denote this constant by c, and it is $1/c$ we need to determine.

We are told that water drains out at a rate of $10\ cm^3/min$— this says that $V'(t) = -10$. Further, we know that when $h(t) = 8$, $h'(t) = -2$. Finally, the volume of water at any given time is $V(t) = \pi(r(t))^2 h(t)/3$.

Note that if we just differentiate the expression for the volume, we will get something involving both $r(t)$ and $r'(t)$, but we don't know anything about either of those. However, if we use that $r(t)/h(t) = c$, we can rewrite the expression for $V(t)$ not to involve $r(t)$ at all:

$$V(t) = \frac{\pi}{3}(ch(t))^2 h(t) = \frac{\pi}{3}c^2(h(t))^3.$$

Differentiating and remembering to use the chain rule, we get $V'(t) = \pi c^2 (h(t))^2 h'(t)$, and plugging in gives $10 = c^2\pi \cdot 64 \cdot 2$. Solving for c yields

$$c = \sqrt{\frac{10}{128\pi}} \qquad \text{so} \qquad 1/c = \sqrt{\frac{128\pi}{10}} \approx 6.3413.$$

19. Let $\mathcal{C}$ be the parametric curve given by

$$x = t - \sin 2\pi t \qquad y = \sqrt{t} + \cos \pi t$$

What is the slope of $\mathcal{C}$ at the point $(4, 2)$, when $t = 4$?

Solution:
We want to compute $\frac{dy}{dx}$ at $(4, 2)$— remember that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. In our case,

$$\frac{dx}{dt} = 1 - 2\pi \cos 2\pi t \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{t}} - \pi \sin \pi t$$

or, when $t = 4$, we have

$$\frac{dx}{dt} = 1 - 2\pi \cos 8\pi = 1 - 2\pi \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{4}} - \pi \sin 4\pi = \frac{1}{4} \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{4 - 8\pi} \approx -0.473$$

20. An angle θ is known to vary periodically with time, in such a way that its rate of change is proportional to the product of itself and the cosine of the time t. Write a differential equation which expresses this relationship. Show that $\theta = c\, e^{k \sin t}$ is a solution to the differential equation. If you know that $\theta(0) = 2$ and $\theta(\pi/2) = 3$, what is the equation for $\theta(t)$?

Solution:
This relationship can be written as

$$\frac{d\theta}{dt} = k\theta \cos t.$$

The function $\theta(t) = c\, e^{k \sin t}$ is a solution, since $\theta'(t) = c\, e^{k \sin t} k \cos t = \theta(t) \cdot k \cos t$.

If we know $\theta(0) = 2$, then we have $2 = ce^0 = c$. Using $\theta(\pi/2) = 3$, we have $3 = 2e^{k \sin(\pi/2)} = 2e^k$. So $e^k = \frac{3}{2}$. Taking logarithms, we get $k = \ln \frac{3}{2}$. Thus

$$\theta(t) = 2e^{\ln \frac{3}{2} \sin t} \qquad \text{or} \qquad \theta(t) = 2\left(\frac{3}{2}\right)^{\sin t}.$$