Solutions for Second Exam Review Questions MAT 131, Fall 1998

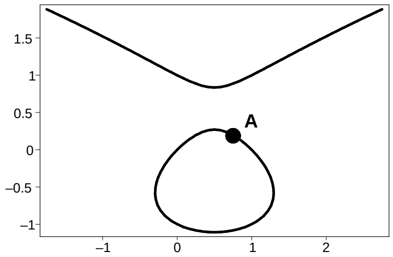
1. Compute the derivative with respect to x for each of the following expressions:

$$\pi x^8 - \sqrt{x} \qquad x^3 + \tan x \qquad x^2 \sin(x^2) \qquad \frac{x^3}{1+x^4} \qquad \sin(\cos(2x+1))$$

$$\ln \sqrt{x} \qquad (1+x^2)\arctan x \qquad \frac{\sin(3x)}{3x+1} \qquad e^{\cos x} \qquad x \ln x$$

Solution:

- $\dfrac{d}{dx}\left(\pi x^8 - \sqrt{x}\right) = 8\pi x^7 + \dfrac{1}{2\sqrt{x}}$
- $\dfrac{d}{dx}\left(x^3 + \tan x\right) = 3x^2 + \sec^2 x$
- $\dfrac{d}{dx}\left(x^2 \sin(x^2)\right) = 2x\sin(x^2) + 2x^3\cos(x^2)$
- $\dfrac{d}{dx}\left(\dfrac{x^3}{1+x^4}\right) = \dfrac{3x^2(1+x^4) - x^3 \cdot 4x^3}{(1+x^4)^2} = \dfrac{3x^2 - x^6}{(1+x^4)^2}$
- $\dfrac{d}{dx}\left(\sin(\cos(2x+1))\right) = \cos(\cos(2x+1))(-\sin(2x+1))(2) = -2\sin(2x+1)\cos(\cos(2x+1))$
- $\dfrac{d}{dx}\left(\ln\sqrt{x}\right) = \dfrac{d}{dx}\left(\dfrac{1}{2}\ln x\right) = \dfrac{1}{2x}$
- $\dfrac{d}{dx}\left((1+x^2)\arctan x\right) = 2x\arctan x + (1+x^2)\dfrac{1}{1+x^2} = 2x\arctan x + 1$
- $\dfrac{d}{dx}\left(\dfrac{\sin 3x}{3x+1}\right) = \dfrac{(3\sin 3x)(3x+1) - 3\sin 3x}{(3x+1)^2} = \dfrac{9x\sin 3x}{(3x+1)^2}$
- $\dfrac{d}{dx}\left(e^{\cos x}\right) = -e^{\cos x}\sin x$
- $\dfrac{d}{dx}\left(x\ln x\right) = \ln x + x\dfrac{1}{x} = \ln x + 1$



2. Consider the elliptic curve C which consists of the set of points for which

$$x^2 - x = y^3 - y$$

(see the graph at right).

a. Write the equation of the line tangent to C at the point $(1, 0)$.

Solution:
We use implicit differentiation to obtain $2x - 1 = 3y^2y' - y'$. Solving for y' gives $y' = \dfrac{2x-1}{3y^2-1}$. Plugging in at the point $(1, 0)$ says the slope of the relevant tangent line is $\frac{2-1}{-1} = -1$. Thus, the line tangent to C at $(1, 0)$ is $y = 0 - 1(x - 1)$— that is, $y = 1 - x$.

b. Use your answer to part **a** to estimate the y-coordinate of the point with x-coordinate $3/4$ marked A in the figure. Plug your estimate into the equation for C to determine how good it is.

Solution:
Plugging $x = \frac{3}{4}$ into the equation for the tangent line gives $y = \frac{1}{4}$. Trying the point $(\frac{3}{4}, \frac{1}{4})$ in the equation for C, we obtain

$$\frac{9}{16} - \frac{3}{4} \approx \frac{1}{64} - \frac{1}{4},$$

which is off from being true by $3/64$, or about 0.0469.

c. Write the equation of the parabola which best approximates C at the point $(1, 0)$.

Solution:
We need to determine y'', so we take the derivative of y' from part **a**. Again, we use implicit differentiation, this time together with the quotient rule.

$$y'' = \frac{2(3y^2 - 1) - (2x - 1)(6y)(y')}{(3y^2 - 1)^2}$$

Thus, $y''(1, 0) = \frac{-2}{1} = -2$. This means our desired parabola is $1 - x - (x - 1)^2$.

d. Use your answer to part **c** to improve your answer from part **b**. How close does this new estimate come to being right?

Solution:
Here we obtain the estimate $y \approx 1 - \dfrac{3}{4} - \left(1 - \dfrac{3}{4}\right)^2 = \dfrac{3}{16}$. Plugging $(\frac{1}{4}, \frac{3}{16})$ into the equation for C gives

$$\frac{9}{16} - \frac{3}{4} \approx \frac{27}{4096} - \frac{3}{16},$$

off by $\frac{27}{4096}$, or about .00659, a dramatic improvement.

3. A mold culture is growing on the world's largest slice of bread. The culture starts in the center of the bread, and remains approximately circular.

a. The size of the culture grows at a rate proportional to the square of its diameter. Write a differential equation which expresses this relationship.

Solution:
Let's let $y(t)$ be the diameter of the culture. This changes at a rate proportional to (that is, a constant times) its square, so we have

$$y' = ky^2$$

b. Verify that $y(t) = \dfrac{1}{C - kt}$ satisfies the differential equation for any choice of k and C.

Solution:
We just need to check that this particular $y(t)$ satisfies the equation in **a**. Taking the derivative, we have

$$\frac{d}{dt}\left((C-kt)^{-1}\right) = -(C-kt)^{-2}(-k) = \frac{k}{(C-kt)^2}.$$

This is exactly $k(y(t))^2$, so the equation holds.

c. If the diameter of the culture was 1 mm at 8 A.M. and 2 mm at noon, what is the size of the culture at 2 P.M.? What about at 3 P.M.? Does anything surprising happen at 4 P.M.?

Solution:
Let's let $t = 0$ correspond to 8 A.M., so we have

$$y(0) = \frac{1}{C} = 1$$

That is, $C = 1$.

Since the diameter is 2 at noon, four hours later, we have

$$y(4) = \frac{1}{1-4k} = 2.$$

Solving for k gives $1 - 4k = 1/2$, or $k = 1/8$, so our particular solution is

$$y(t) = \frac{1}{1 - t/8}.$$

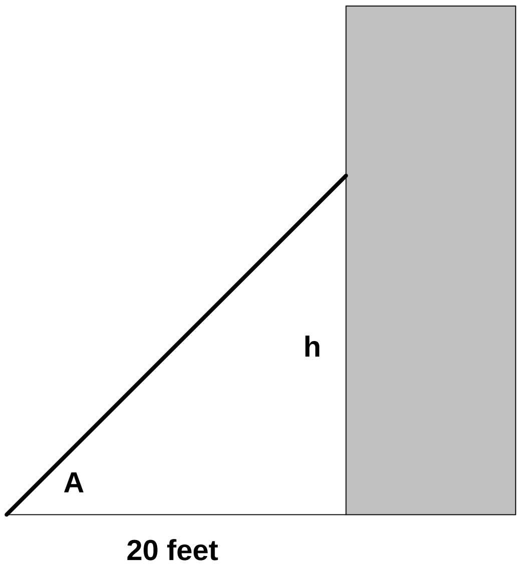
At 2 P.M., the size of the culture is given by $y(6) = 1/(1 - 6/8) = 4$, so it is 4 mm across.

At 3 P.M., the culture has a diameter of $1/(1 - 7/8)$, doubling to $8mm$ in one hour.

At 4 P.M., the universe comes to an end, because the size of the mold is now infinite.

4. A spotlight is aimed at a building whose base is 20 feet away. If the light is raised so that its angle increases at a constant rate of 5 degrees per second, how fast is the image rising when the light makes a 45 degree angle with the ground?

Solution:



It is helpful to draw a figure. Let's call the angle the spotlight makes with the ground A, so we have

$$\frac{dA}{dt} = 5\frac{deg}{sec} = \frac{\pi}{36}\frac{radians}{sec}.$$

Let's also call the distance from the ground to where the spotlight hits the building h, so what we want to know is $\frac{dh}{dt}$ when $A = 45$ degrees, or $\frac{\pi}{4}$ radians.
It is probably safe to assume that the building is at least approximately perpendicular to the ground, so

$$\tan A = \frac{h}{20}.$$

Differentiating this with respect to t gives

$$\sec^2 A\,\frac{dA}{dt} = \frac{1}{20}\frac{dh}{dt}, \qquad \text{or} \qquad \frac{20}{cos^2 A}\frac{dA}{dt} = \frac{dh}{dt}.$$

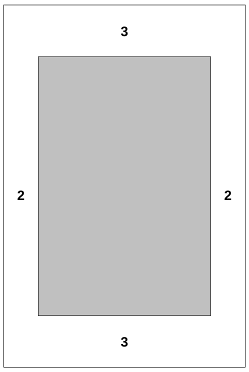
Plugging in what we know gives the result:

$$\frac{dh}{dt} = \frac{20}{(1/\sqrt{2})^2}\cdot\frac{\pi}{36}, \qquad \text{so} \qquad \frac{dh}{dt} = \frac{10\pi}{9},$$

or about $3.49\frac{ft}{sec}$.

5. A poster is to be made which requires 150 in^2 for the printed part, and is to have a 3" margin at the top and bottom, and a 2" margin on the sides. What should the dimensions be in order to minimize the total area of the poster?

Solution:



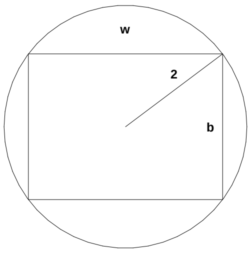
If we let w represent the width of the printed message and h be its height, then $hw = 150$, or $h = 150/w$. The dimensions of the poster are $4 + w$ wide by $6 + h$ tall, so we want to minimize

$$A(w) = (4 + w)(6 + \frac{150}{w}) = 24 + 6w + \frac{600}{w} + 150.$$

Thus, $A'(w) = 6 - \frac{600}{w^2}$, and $A'(w) = 0$ when $w = 10$. So the dimensions of the optimal poster are 14×21, including the margins. The area for the message is 10×15.

6. The stiffness of a beam is directly proportional to the product of its width and the cube of its breadth. What are the dimensions of the stiffest beam that can be cut from a cylindrical log with a radius of 2'?

Solution:



We want to maximize the product wb^3, where $\left(\frac{w}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = 4$, that is, $b = \sqrt{16 - w^2}$. We also must have $0 < w < 4$.
So, $S(w) = w(16 - w^2)^{3/2}$, and

$$S'(w) = (16 - w^2)^{\frac{3}{2}} - 3w^2\sqrt{16 - w^2} = (16 - 4w^2)\sqrt{16 - w^2}.$$

S' is zero when $w = 4$ and $w = 2$. For $w = 4$ we have $b = 0$, which gives a strength of 0, as does $w = 0$. When $w = 2$, we have $b = 2\sqrt{3}$, so the optimal dimensions are $2 \times 2\sqrt{3}$.

7. At what x value does the maximum of $\ln(x)/x$ occur? What is the maximum value of the function?

Solution:
Write $f(x) = \dfrac{\ln x}{x}$. Then $f'(x) = \dfrac{1 - \ln x}{x^2}$. Thus, $f'(x) = 0$ if $\ln x = 1$, and $f'(x)$ does not exist if $x = 0$. Thus $x = e$ is a relative maximum, and it is a global maximum because of the shape of the graph. The maximum value is $1/e$.

8. Compute $\frac{dy}{dx}$ for the curve $\sin(x) + \cos(y) = 1$. What is the slope of the tangent line at the point $(\pi/6, \pi/3)$?

Solution:
Using implicit differentiation,

$$\cos x - y' \sin y = 0, \qquad \text{so} \qquad y' = \frac{\cos x}{\sin y}.$$

At $(\pi/6, \pi/3)$, the slope is 1.