

Math 125 Solutions to Midterm 1 (Theodore)

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points

(a) $\lim_{x \rightarrow 4} e^x \ln(x)$

Solution: For $x > 0$, $e^x \ln(x)$ is continuous, since it is the product of two continuous functions. Thus, the limit is what we get by evaluating the function at 4. Hence the limit is $e^4 \ln 4$.

3 points

(b) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{6x(x - 3)}$

Solution:

$$\lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{6x(x - 3)} = \lim_{x \rightarrow 3} \frac{(x + 3)}{6x} = \frac{3 + 3}{18} = \frac{1}{3}.$$

3 points

(c) $\lim_{x \searrow 2^+} \frac{x^2 - 9}{(x - 2)}$

Solution: Note for x close to 2, the numerator is close to -5 while the denominator tends towards zero. Thus, the function becomes unbounded at 2. Since we are looking at values of $x > 2$, the denominator is always positive. Hence, the limit is $-\infty$.

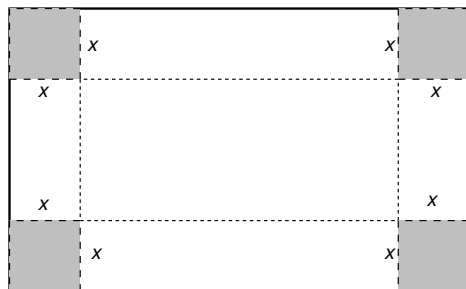
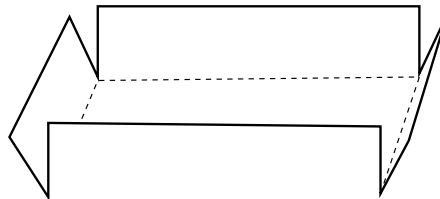
3 points

(d) $\lim_{x \rightarrow +\infty} \sqrt{16x^2 + x} - 4x$

Solution: This limit is of the form $\infty - \infty$, so we will have to combine the two terms to determine the limit. We can accomplish this by rationalizing the expression. Thus:

$$\begin{aligned} \lim_{x \rightarrow +\infty} \sqrt{16x^2 + x} - 4x &= \lim_{x \rightarrow +\infty} \left(\sqrt{16x^2 + x} - 4x \right) \frac{\sqrt{16x^2 + x} + 4x}{\sqrt{16x^2 + x} + 4x} \\ &= \lim_{x \rightarrow +\infty} \frac{16x^2 + x - 16x^2}{\sqrt{16x^2 + x} + 4x} \\ &= \lim_{x \rightarrow +\infty} \frac{x}{\sqrt{16x^2 + x} + 4x} \\ &= \lim_{x \rightarrow +\infty} \frac{1/x}{1/x} \cdot \frac{x}{\sqrt{16x^2 + x} + 4x} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{16 + \frac{1}{x}} + 4} = \frac{1}{\sqrt{16 + 4}} = \frac{1}{8} \end{aligned}$$

2. A box without a top is to be made from a rectangular piece of cardboard which is 14 inches by 20 inches by cutting out four equal squares of side length x inches from each corner, and then folding up the flaps to form the sides of the box (see figure).



6 points

- (a) Express the volume of the box V as a function of x .

Solution: The volume of the box is given by $V = (\text{length})(\text{width})(\text{height})$.

The height of the box will be x , since that is how long the flap we fold up is.

The width of the box is $20 - 2x$, because we started with a piece of cardboard 20" wide, and cut x " off either end.

Similarly, the length will be $14 - 2x$.

Multiplying them all together gives

$$V(x) = x(20 - 2x)(14 - 2x)$$

3 points

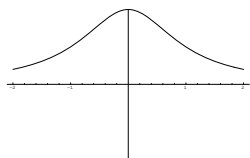
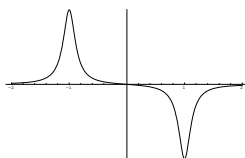
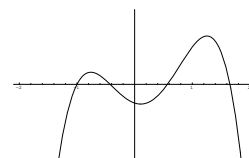
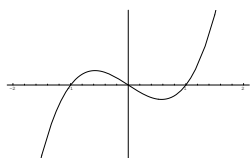
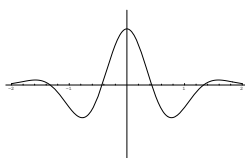
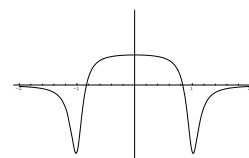
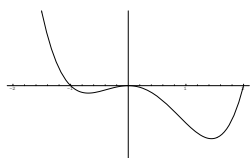
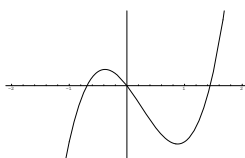
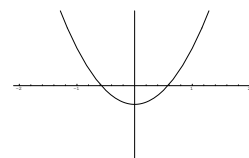
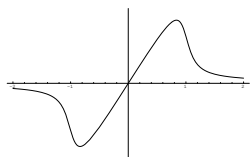
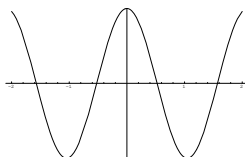
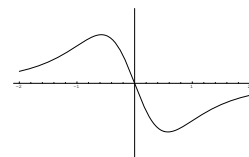
- (b) What is the domain of the function $V(x)$?

Solution: Remember, the domain is the values of x that are valid. We can't cut less than nothing off the piece of cardboard, so $x \geq 0$. Similarly, we can't cut more than half the smaller dimension, so $x \leq 7$. This means the domain is

$$0 \leq x \leq 7.$$

8 points

3. For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.

H**A:****B:****F****C:****D:****E****E:****F:****D****G:****H:**

8 points

4. What value of k is necessary so that the function

$$f(x) = \begin{cases} \frac{k}{x+3} & x < 1 \\ 2x^2 + x & x \geq 1 \end{cases}$$

is continuous for all positive values of x ? If there is no such value of k , say so. Justify your answer fully.

Solution: The function f is continuous at all positive values of x except possibly at $x = 1$, since $1/(x+3)$ is continuous for $-3 < x$ and $f(x)$ is a quadratic polynomial for $x \geq 1$.

Since $f(1) = 2 + 1 = 3 = \lim_{x \searrow 1^+} 2x^2 + x$, we need to find k so that $\lim_{x \nearrow 1^-} f(x) = 3$.

Observe that

$$\lim_{x \nearrow 1^-} f(x) = \lim_{x \nearrow 1^-} k/(1+3) = k/4,$$

so we need to take $k = 12$.

8 points

5. Write a limit that represents the slope of the line tangent to the graph of the function

$$f(x) = x \ln(2+x) \quad \text{at} \quad x = 4.$$

You **do not need to evaluate the limit**.

Solution: We just use the definition of the derivative at $x = 4$:

$$f'(4) = \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0} \frac{(4+h) \ln(6+h) - 4 \ln(6)}{h}.$$

There isn't a lot of simplification that can be done with what we have covered so far. If you prefer to use the version of the definition with $x \rightarrow 4$, you would have

$$f'(4) = \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \rightarrow 4} \frac{x \ln(2+x) - 4 \ln(6)}{x - 4}.$$

8 points

6. Sketch the graph of a function $f(x)$ which satisfies all of the following properties:

- $f(2) = -2$
- $\lim_{x \rightarrow 2} f(x) = 0$
- $\lim_{x \rightarrow 1} f(x) = 1$
- $\lim_{x \searrow 0^+} f(x) = +\infty$
- $\lim_{x \nearrow 0^-} f(x) = -\infty$
- $\lim_{x \rightarrow +\infty} f(x) = 0$
- $\lim_{x \rightarrow -\infty} f(x) = -\infty$
- $f'(-1) = 0$

Be sure to label all important values.

Solution: The graph below is one example, but there are several variations possible. The graph must pass through $(1, 1)$ and have a horizontal tangent at $x = -1$, and must have a removable discontinuity at $x = 2$ (with $f(2) = -2$) but limit on the point $(2, 0)$. In addition, it should have a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$ on the right side (ie, for $x > 0$).

