

MAT 125

Solutions to Midterm 1 (Alvin)

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points

(a) $\lim_{x \rightarrow 2} e^x \ln(x)$

Solution: For $x > 0$, $e^x \ln(x)$ is continuous, since it is the product of two continuous functions. Thus, the limit is what we get by evaluating the function at 2. Hence the limit is $e^2 \ln 2$.

3 points

(b) $\lim_{x \rightarrow 1} \frac{x^2 - 1}{7x(x - 1)}$

Solution:

$$\lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{7x(x - 1)} = \lim_{x \rightarrow 1} \frac{(x + 1)}{7x} = \frac{1 + 1}{7} = \frac{2}{7}.$$

3 points

(c) $\lim_{x \searrow 1^+} \frac{x^2 - 9}{(x - 1)}$

Solution: Note for x close to 1, the numerator is close to -8 while the denominator tends towards zero. Thus, the function becomes unbounded at 1. Since we are looking at values of $x > 1$, the denominator is always positive. Hence, the limit is $-\infty$.

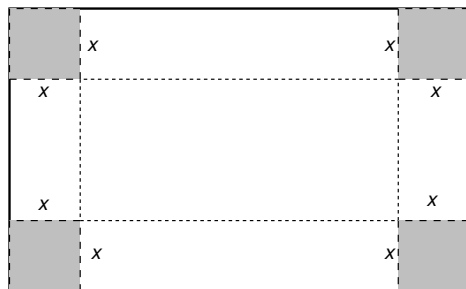
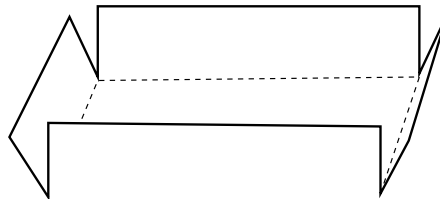
3 points

(d) $\lim_{x \rightarrow +\infty} \sqrt{4x^2 + x} - 2x$

Solution: This limit is of the form $\infty - \infty$, so we will have to combine the two terms to determine the limit. We can accomplish this by rationalizing the expression. Thus:

$$\begin{aligned} \lim_{x \rightarrow +\infty} \sqrt{4x^2 + x} - 2x &= \lim_{x \rightarrow +\infty} \left(\sqrt{4x^2 + x} - 2x \right) \frac{\sqrt{4x^2 + x} + 2x}{\sqrt{4x^2 + x} + 2x} \\ &= \lim_{x \rightarrow +\infty} \frac{4x^2 + x - 4x^2}{\sqrt{4x^2 + x} + 2x} \\ &= \lim_{x \rightarrow +\infty} \frac{x}{\sqrt{4x^2 + x} + 2x} \\ &= \lim_{x \rightarrow +\infty} \frac{1/x}{1/x + 2} \cdot \frac{x}{\sqrt{4x^2 + x} + 2x} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{4 + \frac{1}{x}} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4} \end{aligned}$$

2. A box without a top is to be made from a rectangular piece of cardboard which is 16 inches by 20 inches by cutting out four equal squares of side length x inches from each corner, and then folding up the flaps to form the sides of the box (see figure).



6 points

- (a) Express the volume of the box V as a function of x .

Solution: The volume of the box is given by $V = (\text{length})(\text{width})(\text{height})$.

The height of the box will be x , since that is how long the flap we fold up is.

The width of the box is $20 - 2x$, because we started with a piece of cardboard 20" wide, and cut x " off either end.

Similarly, the length will be $16 - 2x$.

Multiplying them all together gives

$$V(x) = x(20 - 2x)(16 - 2x)$$

3 points

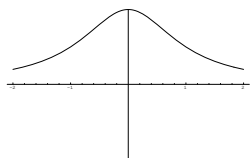
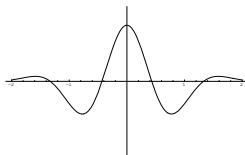
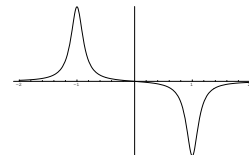
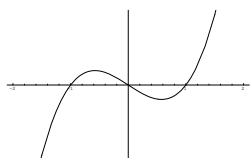
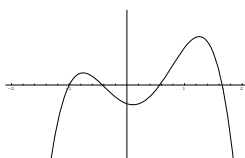
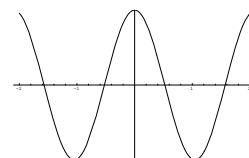
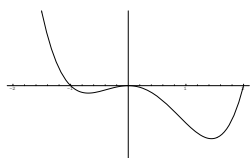
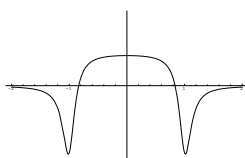
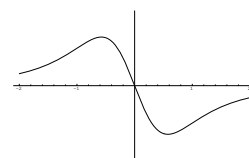
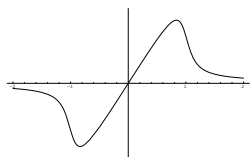
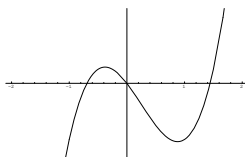
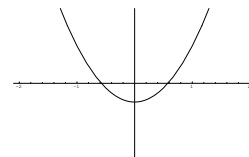
- (b) What is the domain of the function $V(x)$?

Solution: Remember, the domain is the values of x that are valid. We can't cut less than nothing off the piece of cardboard, so $x \geq 0$. Similarly, we can't cut more than half the smaller dimension, so $x \leq 8$. This means the domain is

$$0 \leq x \leq 8.$$

8 points

3. For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.

F**A:****B:****H****C:****D:****G****E:****F:****E****G:****H:**

8 points

4. What value of k is necessary so that the function

$$f(x) = \begin{cases} \frac{k}{x+2} & x < 1 \\ 3x^2 + x & x \geq 1 \end{cases}$$

is continuous for all positive values of x ? If there is no such value of k , say so. Justify your answer fully.

Solution: The function f is continuous at all positive values of x except possibly at $x = 1$, since $1/(x+2)$ is continuous for $-2 < x$ and $f(x)$ is a quadratic polynomial for $x \geq 1$.

Since $f(1) = 3 + 1 = 4 = \lim_{x \searrow 1} 3x^2 + x$, we need to find k so that $\lim_{x \nearrow 1} f(x) = 4$.

Observe that

$$\lim_{x \nearrow 1} f(x) = \lim_{x \nearrow 1} k/(1+2) = k/3,$$

so we need to take $k = 12$.

8 points

5. Write a limit that represents the slope of the line tangent to the graph of the function

$$f(x) = x \ln(2+x) \quad \text{at} \quad x = 1.$$

You **do not need to evaluate the limit**.

Solution: We just use the definition of the derivative at $x = 1$:

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{(1+h) \ln(3+h) - \ln(3)}{h}.$$

There isn't a lot of simplification that can be done with what we have covered so far. If you prefer to use the version of the definition with $x \rightarrow 1$, you would have

$$f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x \ln(2+x) - \ln(3)}{x - 1}.$$

8 points

6. Sketch the graph of a function $f(x)$ which satisfies all of the following properties:

- $f(2) = -1$
- $\lim_{x \rightarrow 2} f(x) = 0$
- $\lim_{x \rightarrow 1} f(x) = 1$
- $\lim_{x \searrow 0^+} f(x) = +\infty$
- $\lim_{x \nearrow 0^-} f(x) = -\infty$
- $\lim_{x \rightarrow +\infty} f(x) = 0$
- $\lim_{x \rightarrow -\infty} f(x) = -\infty$
- $f'(-1) = 0$

Be sure to label all important values.

Solution: The graph below is one example, but there are several variations possible. The graph must pass through $(1, 1)$ and have a horizontal tangent at $x = -1$, and must have a removable discontinuity at $x = 2$ (with $f(2) = -1$) but limit on the point $(2, 0)$. In addition, it should have a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$ on the right side (ie, for $x > 0$).

