MATH 125 Solutions to Midterm 1 (Simon)

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points (a) $\lim_{x\to 3} e^x \ln(x)$

Solution: For $x > 0$, $e^x \ln(x)$ is continuous, since it is the product of two continuous functions. Thus, the limit is what we get by evaluating the function at 3. Hence the limit is $e^3 \ln 3$.

3 points (b) $\lim_{x\to 4} \dfrac{x^2-16}{5x(x-4)}$

Solution:

$$\lim_{x\to 4} \frac{(x-4)(x+4)}{5x(x-4)} = \lim_{x\to 4} \frac{(x+4)}{5x} = \frac{4+4}{10} = \frac{4}{5}.$$

3 points (c) $\lim_{x\searrow 4^+} \dfrac{x^2-9}{(x-4)}$

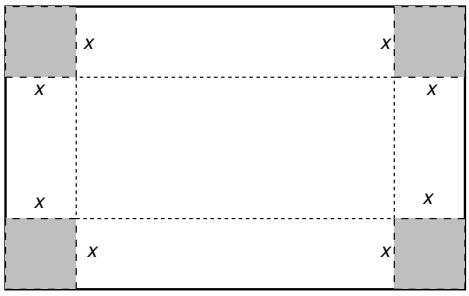
Solution: Note for x close to 4, the numerator is close to 7 while the denominator tends towards zero. Thus, the function becomes unbounded at 4. Since we are looking at values of $x > 4$, the denominator is always positive. Hence, the limit is $+\infty$.

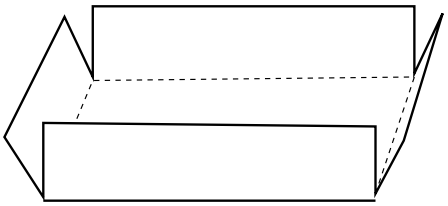
3 points (d) $\lim_{x\to+\infty} \sqrt{9x^2+x} - 3x$

Solution: This limit is of the form $\infty - \infty$, so we will have to combine the two terms to determine the limit. We can accomplish this by rationalizing the expression. Thus:

$$\begin{aligned}
\lim_{x\to+\infty} \sqrt{9x^2+x} - 3x &= \lim_{x\to+\infty} \left(\sqrt{9x^2+x} - 3x\right) \frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to+\infty} \frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to+\infty} \frac{x}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to+\infty} \frac{1/x}{1/x} \cdot \frac{x}{\sqrt{9x^2+x}+3x} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}
\end{aligned}$$

2. A box without a top is to be made from a rectangular piece of cardboard which is 12 inches by 20 inches by cutting out four equal squares of side length x inches from each corner, and then folding up the flaps to form the sides of the box (see figure).





6 points (a) Express the volume of the box V as a function of x.

Solution: The volume of the box is given by $V = (length)(width)(height)$.

The height of the box will be x, since that is how long the flap we fold up is.

The width of the box is $20 - 2x$, because we started with a piece of cardboard 20" wide, and cut x" off either end.

Similarly, the length will be $12 - 2x$.

Multiplying them all together gives

$$V(x) = x(20 - 2x)(12 - 2x)$$

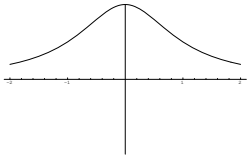
3 points (b) What is the domain of the function $V(x)$?

Solution: Remember, the domain is the values of x that are valid. We can't cut less than nothing off the piece of cardboard, so $x \geq 0$. Similarly, we can't cut more than half the smaller dimension, so $x \leq 6$. This means the domain is

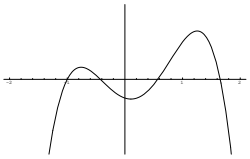
$$0 \leq x \leq 6.$$

8 points 3. For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.

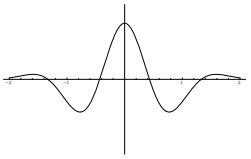
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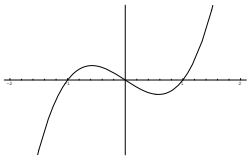
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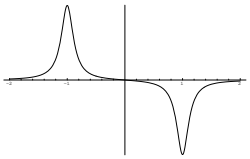
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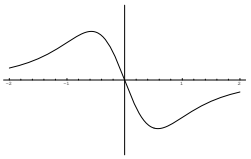
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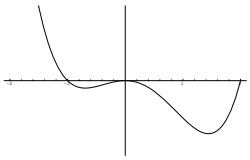
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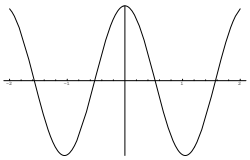
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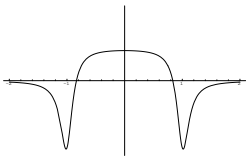
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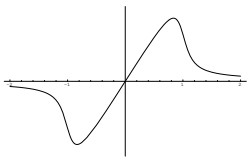
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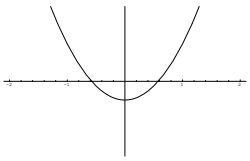
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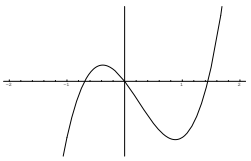
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8 points 4. What value of k is necessary so that the function

$$f(x) = \begin{cases} \dfrac{k}{x+4} & x < 1 \\ 5x^2 + x & x \geq 1 \end{cases}$$

is continuous for all positive values of x? If there is no such value of k, say so. Justify your answer fully.

Solution: The function f is continuous at all positive values of x except possibly at $x = 1$, since $1/(x+4)$ is continuous for $-4 < x$ and $f(x)$ is a quadratic polynomial for $x \geq 1$.

Since $f(1) = 5 + 1 = 6 = \lim_{x \searrow 1^+} 5x^2 + x$, we need to find k so that $\lim_{x \nearrow 1^-} f(x) = 6$.

Observe that

$$\lim_{x \nearrow 1^-} f(x) = \lim_{x \nearrow 1^-} k/(1+4) = k/5,$$

so we need to take $k = 30$.

8 points 5. Write a limit that represents the slope of the line tangent to the graph of the function

$$f(x) = x \ln(2 + x) \qquad \text{at} \qquad x = 3\,.$$

You **do not need to evaluate the limit.**

Solution: We just use the definition of the derivative at $x = 3$:

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)\ln(5+h) - 3\ln(5)}{h}.$$

There isn't a lot of simplification that can be done with what we have covered so far. If you prefer to use the version of the definition with $x \to 3$, you would have

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{x\ln(2+x) - 3\ln(5)}{x - 3}.$$

8 points 6. Sketch the graph of a function $f(x)$ which satisfies all of the following properties:

- $f(2) = 2$
- $\lim_{x \to 2} f(x) = 0$
- $\lim_{x \to 1} f(x) = 1$
- $\lim_{x \searrow 0^+} f(x) = +\infty$
- $\lim_{x \nearrow 0^-} f(x) = -\infty$
- $\lim_{x \to +\infty} f(x) = 0$
- $\lim_{x \to -\infty} f(x) = -\infty$
- $f'(-1) = 0$

Be sure to label all important values.

Solution: The graph below is one example, but there are several variations possible. The graph must pass through $(1, 1)$ and have a horizontal tangent at $x = -1$, and must have a removable discontinuity at $x = 2$ (with $f(2) = 2$) but limit on the point $(2, 0)$. In addition, it should have a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$ on the right side (ie, for $x > 0$).

