AN INTRODUCTION TO JULIA AND FATOU SETS

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ABSTRACT. We give an elementary introduction to the holomorphic dynamics of mappings on the Riemann sphere, with a focus on Julia and Fatou sets. Our main emphasis is on the dynamics of polynomials, especially quadratic polynomials.

1. INTRODUCTION

In this note, we briefly introduce some of the main elementary ideas in holomorphic dynamics. This is by no means a comprehensive coverage. We do not discuss the Mandelbrot set, despite its inherent relevance to the subject; the Mandelbrot set is covered in companion articles by Devaney [Dev13, Dev14].

For a more detailed and and comprehensive introduction to this subject, the reader is refered to the book [Mil06] by John Milnor, the survey article [Bla84] by Blanchard, the text by Devaney [Dev89], or any of several other introductory texts ([Bea91], [CG93], etc.)

Holomorphic dynamics is the study of the iterates of a holomorphic map f on a complex manifold. Classically, this manifold is one of the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$, or the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

In these notes, we shall primarily restrict our attention to the last case, where f is a rational map (in fact, most examples will be drawn from quadratic polynomials $f(z) = z^2 + c$, with z and $c \in \mathbb{C}$).

Based on the behavior of the point z under iteration of f, the Riemann sphere $\widehat{\mathbb{C}}$ is partitioned into two sets

- The **Fatou set** $\mathcal{F}_f$ (or merely $\mathcal{F}$), on which the dynamics are tame, and
- The **Julia set** $\mathcal{J}_f$ (or $\mathcal{J}$), where there is sensitive dependence on initial conditions and the dynamics are chaotic.

We shall define these sets more precisely in Section 5, but will use these informal definitions for the present to give some intuition.

Given the focus of this conference, we should point out that there is an inverse relationship between the approach of holomorphic dynamics and that of iterated function systems (IFS). More specifically, the Julia set of a rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ corresponds to the attractor for the IFS

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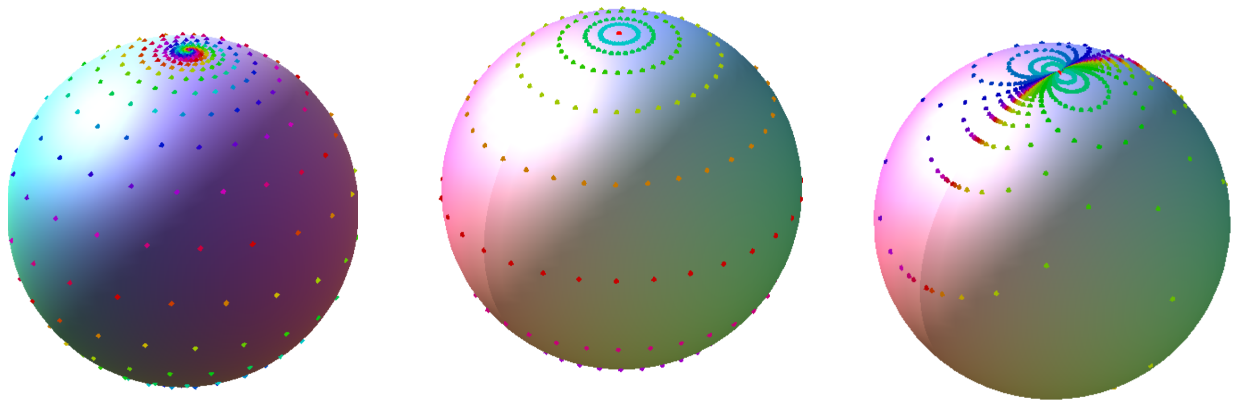


FIGURE 2.1. Left: under iteration of a loxodromic linear map, points spiral away from one fixedpoint (not visible) and towards the other. Center: an elliptic transformation corresponds to rotation of the Riemann sphere. Right: A parabolic transformation corresponds to $z \mapsto 1 + z$. Here points leave infinity from one side (the negative half-plane) and return on the other.

consisting of $\{g_1, g_2, \ldots, g_d\}$, where the g_i are branches of the inverse of f restricted to a suitable domain. For example, the Julia set of $z^2 - 1$ is the attractor of the iterated function system $\{w \mapsto \sqrt{w+1}, w \mapsto -\sqrt{w+1}\}$.

2. LINEAR MAPS

Before turning to the dynamics of rational maps, let us first discuss iteration of a single linear map. Much of the theory of holomorphic dynamics doesn't apply in this case, so it is always excluded. However, a brief overview of what happens will be helpful.

The case of iteration of a single linear map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is very simple. Such an f is a Möbius transformation of the form

$$f(z) = \frac{az+b}{cz+d} \qquad \text{with } ad - bc \neq 0,$$

which can be viewed as an element of $PSL(2, \mathbb{C})$, and has easily understood dynamics.

Except for the identity and the trivial map $z \mapsto 1/z$, these mappings have two fixed points, counted with multiplicity. If the fixed points are *distinct*, we may make a holomorphic change of coordinates moving one of them to infinity and the other to zero. In this case, the mapping becomes of the form $z \mapsto \lambda z$ with $\lambda \in \mathbb{C}$. See Figure 2.1.

- If $|\lambda| \neq 1$, the mapping $z \mapsto \lambda z$ is called **loxodromic** (or **hyperbolic**): under iteration, points move away from one of the fixed point (which is repelling) and toward the other (which is attracting).
- If $|\lambda| = 1$, $z \mapsto \lambda z$ has two **elliptic** fixed points: nonzero finite points orbit around the fixed points at zero and ∞ along an elliptic path. This corresponds to a rotation of the Riemann sphere by the angle $\operatorname{Arg} \lambda$.
- If the two fixed points of f *coincide*, the mapping is conjugate to $z \mapsto z + 1$. Here, the point at infinity is a **parabolic** fixed point: orbits leave from one side, and return on the other.

3. First examples

Now we turn to our main subject, beginning with some simple examples. First, we remark that in holomorphic dynamics, we only consider mappings f which have degree at least two. The notion of "degree" of a rational mapping of $\widehat{\mathbb{C}}$ is unambiguous (unlike in the case of higher-dimensional complex manifolds): the algebraic degree (i.e., the highest power z in the numerator and denominator of f) and the topological degree (i.e., the number of pre-images of a typical point) coincide.

The map $z \mapsto z^2$. We begin with an elementary analysis of the simplest rational map, $f(z) = z^2$.

If we write z in polar coordinates with $r = |z|, \theta = \operatorname{Arg} z$, then the mapping is $(r, \theta) \mapsto (r^2, 2\theta)$, and it is easy to make the following conclusions.

- There are three fixed points for f: 0, 1, and ∞.
- If $|z| < 1$, then $f^n(z) \to 0$; if $|z| > 1$, then $f^n(z) \to \infty$. Thus, 0 *and* ∞ *are attracting* fixed points (in fact, superattracting). A point p is called an **attracting fixed point** if for z in a neighborhood of p, we have $f^n(z) \to p$.
- If $|z| = 1$, then $|f^n(z)| = 1$ for all n. For this mapping, the unit circle is *forward-* and *backward-invariant*. That is, every point in a deleted neighborhood of 1 will eventually leave that neighborhood under f; so for this mapping 1 *is a repelling fixed point*.
- In fact, any small neigborhood of a point z on the unit circle contain points which tend to ∞, points which tend to 0, and points which remain on the unit circle. As we shall see, such behavior characterizes membership in the Julia set.

In the case of $f(z) = z^2$, $\mathcal{J}_f$ is the unit circle, and the Fatou set $\mathcal{F}_f$ has two components: the set of points which iterate to 0, and those which iterate to ∞. (See Figure 3.1.)

Even though behavior of f on the Julia set is merely that of angle doubling, there is a surprisingly rich collection of behaviors. For example, there are periodic points of all periods, the pre-images of any point are dense in the unit circle. There are also points whose forward orbit is dense in the circle.

We will refer to the set of points which iterate to ∞ as the **basin of** ∞ and denote it by $\operatorname{Bas}(\infty)$. More generally, for $p \in \widehat{\mathbb{C}}$ we shall use the notation

$$\operatorname{Bas}(p) = \left\{ z \in \widehat{\mathbb{C}} \mid f^n(p) \to p \right\}.$$

If p is a periodic point of of period m, we can extend this definition as

$$\operatorname{Bas}(p) = \bigcup_{0 \le j < m} \left\{ z \in \widehat{\mathbb{C}} \mid f^{n+j}(p) \to p \right\},$$

or, equivalently, as the union of the basins of each point in the orbit of p under the f^m.

Because ∞ plays a special role for polynomials maps $f(z)$, it is common and useful to define the **filled Julia set of** f as

$$\mathcal{K}_f = \widehat{\mathbb{C}} \setminus \operatorname{Bas}(\infty) = \{ z \in \mathbb{C} \mid f^n(z) \text{ is bounded} \}.$$

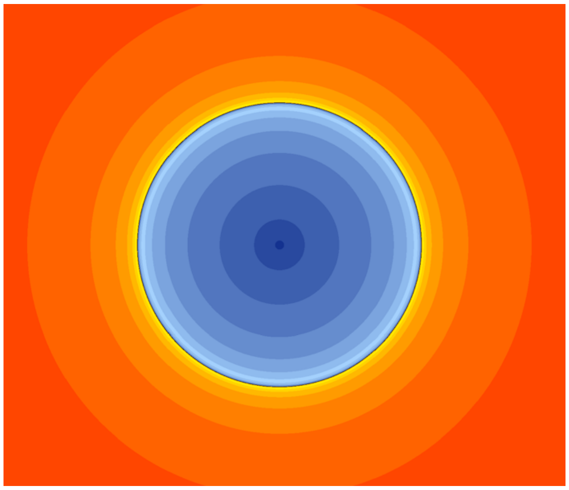


FIGURE 3.1. For $f(z) = z^2$, the Julia set J_f is the unit circle (black), and the Fatou set has two components: $\mathrm{Bas}(\infty)$ (red and orange) and $\mathrm{Bas}(0)$ (blue).

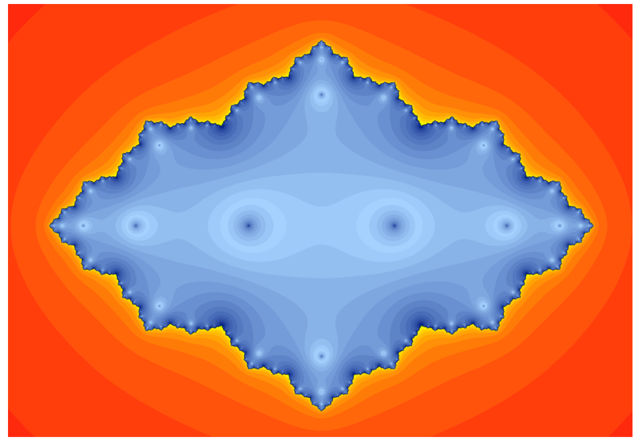


FIGURE 3.2. Julia and Fatou sets for $f(z) = z^2 - 1/2$. As in Figure 3.1, the Fatou set $\mathcal{F}_f$ consists of two components shown in reds and blues and J_f is in black, forming the boundary between the two components of $\mathcal{F}_f$.

It is worth noting that for $f(z) = z^2$, we can change coordinates to interchange the roles of ∞ and 0: $1/f(z) = f(1/z)$.

$$\begin{tikzcd} \mathrm{Bas}(\infty) \arrow[r, "f"] \arrow[d, "1/z"] & \mathrm{Bas}(\infty) \arrow[d, "1/z"] \\ \mathrm{Bas}(0) \arrow[r, "f"] & \mathrm{Bas}(0) \end{tikzcd}$$

That is, f acting on $\mathrm{Bas}(0)$ is *holomorphically conjugate* to f on $\mathrm{Bas}(\infty)$ via $z \mapsto 1/z$. Note also that $f'(0) = 0$, $f'(1) = 2$, and the derivative at ∞ is also 0 (in the $1/z$ coordinate chart).

THE MAP $z \mapsto z^2 + \epsilon$. Now let's change things a little, and consider $f(z) = z^2 + \epsilon$. How does this affect the dynamics?

- We still have three fixed points in $\widehat{\mathbb{C}}$:
  ∞ (with derivative 0), $\alpha = (1 - \sqrt{1 - 4\epsilon})/2$, and $\beta = (1 + \sqrt{1 - 4\epsilon})/2$.
- For $|z|$ large (i.e, near ∞), we still have $f^n(z) \to \infty$, and (as long as ϵ is not too big), for z near α, we have $f^n(z) \to \alpha$.
- Also as in the case of z^2, β is a repelling fixed point (since we have $|f'(\beta)| = |2\beta| > 1$). As such, it lies in the Julia set $\mathcal{J}_f$.
- Futhermore, (with a bit more effort) we can show that we have a conformal map ϕ so that

$$\begin{tikzcd} \mathrm{Bas}_f(\infty) \arrow[r, "f"] \arrow[d, "\phi"] & \mathrm{Bas}_f(\infty) \arrow[d, "\phi"] \\ \mathbb{D} \arrow[r, "z \mapsto z^2"] & \mathbb{D} \end{tikzcd}$$

However, things are a little more complicated near the fixed points α and β.

In a neighborhood U of the attracting fixed point α, we have a conformal map ϕ so that

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"] & U \arrow[d, "\phi"] \\ \mathbb{C} \arrow[r, "z \mapsto f'(\alpha) z"] & \mathbb{C} \end{tikzcd}$$

This neighborhood U cannot include 0, since f is not one-to-one on any neighborood of 0. This means we cannot hope to extend the local conjugacy above to the whole of $\mathrm{Bas}(\alpha)$, unlike in the case of $\mathrm{Bas}(\infty)$.

One can show that in this case, the Julia set (which is the complement of $\mathrm{Bas}(\alpha) \cup \mathrm{Bas}(\infty)$) is a Jordan curve with Hausdorff dimension greater than one. See Figure 3.2.

4. Some history

The study of the iteration of complex analytic functions began more than a century and a quarter ago. In the 1870s, Ernst Schröder [Sch70, Sch71] investigated the convergence of iterative algorithms for solving equations, with a particular interest in the convergence of Newton's method, which corresponds to the iteration of the function $N_f(z) = z - f(z)/f'(z)$. He discovered that a root ρ of f corresponds to a super-attracting fixed point of N_f; this led him to generalize Newton's method to other numerical methods. In addition, Schröder showed that Newton's method for $f(z) = z^2 - 1$ converges globally in the right half-plane to the root 1, in the left half-plane to -1, and observed sensitive dependence to initial conditions along the imaginary axis. Later, Arthur Cayley independently (and via quite different methods) obtained similar results in [Cay79]. While both Cayley and Schröder had hopes of extending their understanding to higher degree polynomials, they were unable to do so. Figure 4.1 might give an idea as to the source of some of their difficulties.

Later, Gabriel Kœnigs [Kœ84] greatly extended Schröder's work to further generality, studying the **Schröder functional equation** $\phi(f(p)) = f'(p)\phi(p)$ in the neighborhood of a fixed point p. Kœnigs was able to show that the a mapping f was locally conjugate to multiplication by its derivative at the fixed point p, in the case where $|f'(p)|$ was not 1 or 0. The more complicated case of $f'(p)$ a root of unity was studied 1897 by Leau [Lea87], and the case of $f'(p) = 0$ was treated by Böttcher in 1904 [Böt04]. (Böttcher was one of the first, if not the first, researcher concerned with developing a general, global theory of iteration of rational maps.) The very difficult cases where $|f'(p)| = 1$ with $f'(p)$ not a root of unity were not understood until the work of Cremer in 1927 [Cre27] and Siegel in 1942 [Sie42]. We will summarize these results in Section 8.

In the time immediately after World War I, Pierre Fatou [Fat17, Fat19] and Gaston Julia [Jul18] laid down the foundations of complex dynamics, looking at the theory of iterated rational functions from a global point of view. Both of the had recently encountered Montel's theory of normal families [Mon12, Mon27] and realized its importance to the theory. Both Fatou and Julia independently proved that the domain of normality must either be empty, or have one, two, or infinitely many components. Each showed that Julia sets are typically fractal, and clearly were able to visualize and understand the very complicated structure of Julia sets, studying and explaining complicated behavior.

The huge amount of interest in the field of iteration of functions of one complex variable, led by the work of Fatou and Julia, continued until the 1930s when it inexplicably faded into obscurity. Although a few important mathematicians worked in this field during that time, it wasn't until the

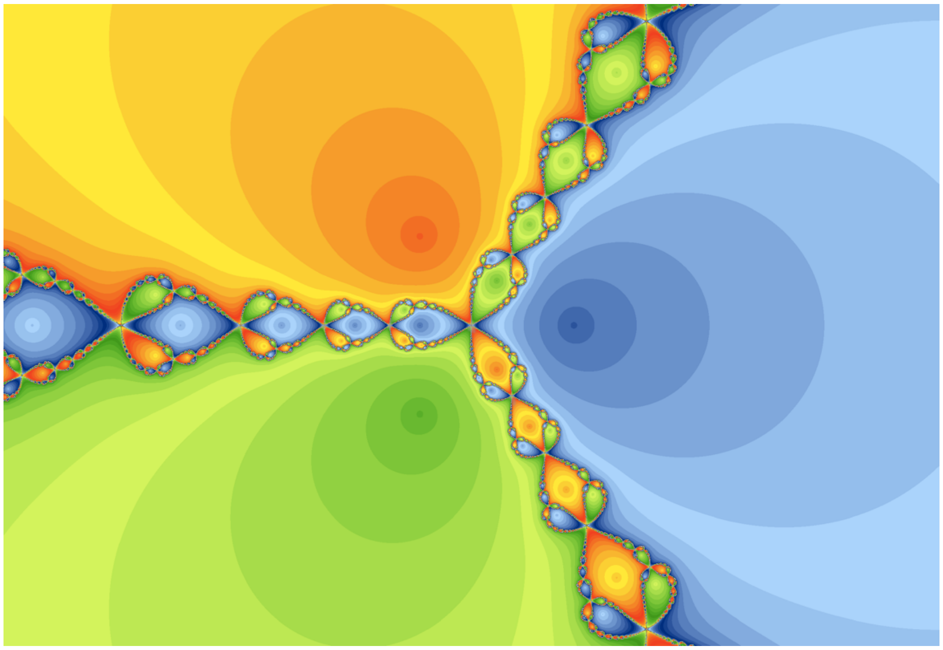


FIGURE 4.1. The dynamics of Newton's method for the polynomial $z^3 - 1$. The basins of each of the three roots are colored in shades of blue, green, and orange; the Julia set is the boundary between the colors.

1980s that it revived, probably due to the advent of accessible computers which enabled others to visualize the extraordinary beauty and complexity that Julia and Fatou obviously understood.

5. NORMAL FAMILIES

The key tool that enabled Fatou and Julia's breakthrough was their realization of the relevance of Montel's work on normal families to the theory of iteration.

Definition 5.1 (Normal family). *Let U be an open subset of $\widehat{\mathbb{C}}$ and $\mathfrak{F} = \{f_i \mid i \in I\}$ be a family of meromorphic functions indexed by I and defined on U with values in $\widehat{\mathbb{C}}$. The family $\mathfrak{F}$ is a **normal family** if every sequence f_n contains a subsequence f_{n_j} which converges uniformly on compact subsets of U.*

Arzela's theorem gives us an equvialent, and often more useful, condition for checking normality. If X is a metric space with metric d, a family of functions $\{f_i : X \to X\}$ is **equicontinuous** if for every $\epsilon > 0$, there exists $\delta >$ so that $d(x, y) < \epsilon$ implies $d(f_i(x), f_i(y)) < \delta$ for all i.

Theorem 5.2 (Arzela). *A family of meromorphic functions $\left\{f_i : U \to \widehat{\mathbb{C}}\right\}$ is normal if and only if it is equicontinuous on every compact subset of U.*

Corollary 5.3. *If a family of holomorphic functions $\{f_i : U \to \mathbb{C}\}$ is is locally uniformly bounded, then it is a normal family.*

The concept of normal families enables us to define the Julia set $\mathcal{J}_f$ and the Fatou set $\mathcal{F}_f$.

Definition 5.4. *A point z is in the **Fatou set** for f if there is a neighborhood U of z such that the family of iterates $\{f^n|_U\}$ is normal. The complement of the Fatou set is called the **Julia set**.*

If p is a periodic point of period n, the **multiplier** λ of the periodic orbit is $\lambda = (f^n)'(p)$; by the chain rule, this is the product of the derivatives of f along the periodic orbit.

Definition 5.5. *A periodic orbit p with multiplier λ is*

- ***superattracting*** *if $\lambda = 0$,*
- ***attracting*** *if $0 < |\lambda| < 1$,*
- ***neutral*** *if $|\lambda| = 1$, and*
- ***repelling*** *if $|\lambda| > 1$.*

The next result follows easily from the definitions and Arzela's theorem.

Proposition 5.6. *If p is an attracting or superattracting periodic point, then $\mathrm{Bas}(p) \subset \mathcal{F}$. If p is a repelling periodic point, then $p \in \mathcal{J}$.*

The Fatou set is sometimes called "the domain of normality" or "the domain of equicontinuity."

Corollary 5.7. *The Fatou set $\mathcal{F}$ is an open set which is completely invariant. That is, if $z \in \mathcal{F}$, then $f(z) \in \mathcal{F}$ and $f^{-1}(z) \subset \mathcal{F}$.*

The Julia set $\mathcal{J}$ is a completely invariant and compact set in $\widehat{\mathbb{C}}$.

Proof. The fact that $\mathcal{F}$ is open follows immediately from the definition. We have the invariance of $\mathcal{F}_f$ since f is a continuous, open mapping. Since J is the complement of a completely invariant, open set, its invariance and compactness follows. □

A few examples of $\mathcal{J}$ and $\mathcal{F}$ for polynomials with attracting periodic points are shown in Figure 5.8.

While the previous examples have all had attracting periodic orbits, for many polynomials, all finite periodic orbits are repelling. As examples, consider $f(z) = z^2 - 2$ or $f(z) = z^2 + i$.

Since infinity is always an attracting fixed point for a polynomial, we have $\mathrm{Bas}(\infty) \subset \mathcal{F}$. But if all orbits in $\mathbb{C}$ are repelling, there can be no other components of $\mathcal{F}$. That is, the filled Julia set is equal to the Julia set, i.e. $\mathcal{K} = \mathcal{J}$. In such a situation, if $\mathcal{J}$ is connected it will be a **dendrite**. See Figure 5.9.

In the case of a polynomial where the Julia set is not connected, it consists of infinitely many components. We will examine this case shortly.

6. THE LOCAL THEORY

As mentioned in Section 4, the behavior in a neighborhood of a fixed point or periodic point was, in most cases, well-known to Fatou, Julia, and their predecessors. In this section, we briefly address those situations.

BÖTTCHER COORDINATES. In the case where $f(z)$ is a polynomial, we have $\mathrm{Bas}(\infty) \subset \mathcal{F}$. Furthermore, since polynomials have no poles, $\mathrm{Bas}(\infty)$ is a completely invariant subset of the Fatou set.

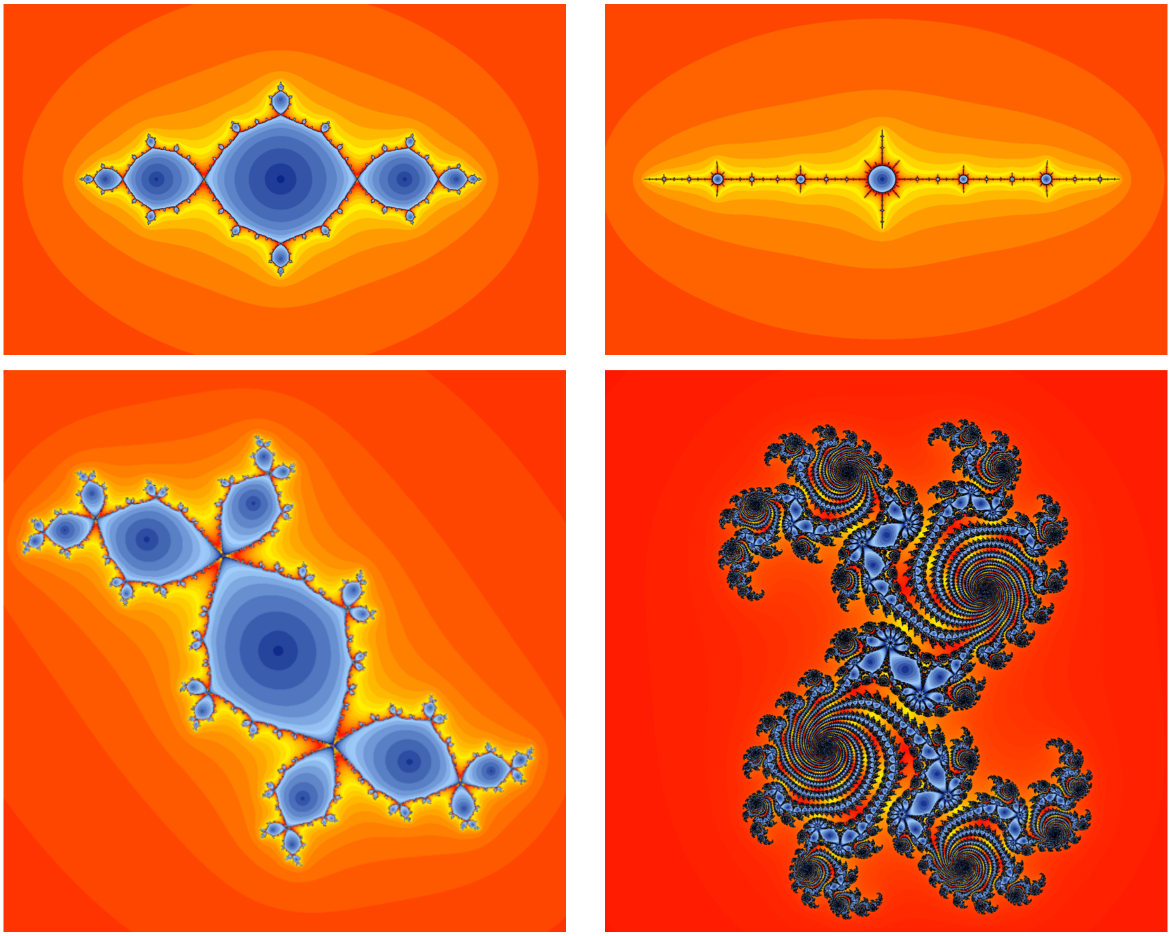


FIGURE 5.8. Pictured are J_f (in black) and $\mathcal{F}_f$ (in reds and blues) for several examples.
(top left) "The basilica" corresponds to $f(z) = z^2 - 1$, and has an attracting period 2 orbit.
(top right) "The airplane", which has an attracting period 3 orbit.
(bottom left)"Douady's rabbit", also has an attracting period 3 orbit.
(bottom right) A map with an attracting period 72 orbit.

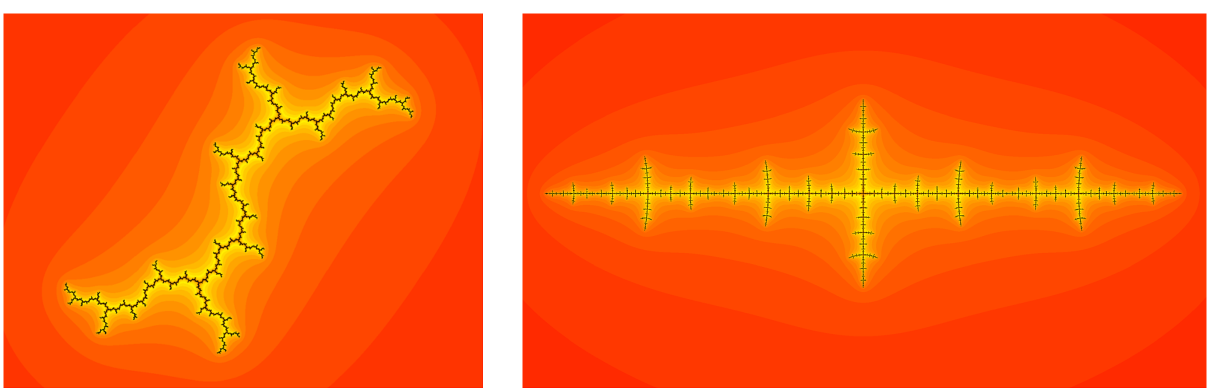


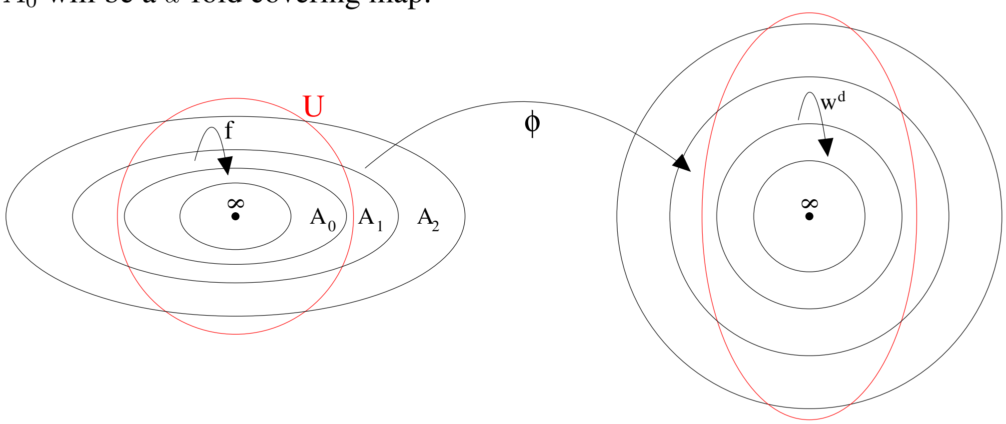
FIGURE 5.9. The Julia sets for $z^2 - i$ and the limit of period doubling ($z^2 - 1.4011552\ldots$), which are connected but have no attracting periodic orbits.

For $|z|$ large, $f(z)$ is conjugate to $w \mapsto w^d$, where d is the degree of f. Let U be a neighborhood of ∞; there is a holomorphic map ϕ so that

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"] & U \arrow[d, "\phi"] \\ \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \arrow[r, "w \mapsto w^d"] & \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}, \end{tikzcd}$$

where d is the degree of the polynomial f.[1]

We now explain how to extend this conjugacy to a larger neighborhood. Choose r large so that $A_0 = \phi^{-1}\left(\overline{\mathbb{D}}_{r^d} \setminus \mathbb{D}_r\right)$ is contained in U. Here $\mathbb{D}_r$ denotes the disk of radius r centered at the origin. Now define $A_1 = f^{-1}(A_0)$. As long as there is no critical point of f in A_0, the map $f\colon A_1 \to A_0$ will be a d-fold covering map.



We can extend the conjugacy ϕ to $U \cup A_1$ by setting $\phi(z) = \phi(f(z))^{1/d^k}$, where we take care to choose the appropriate branches of the inverses. There is no problem doing this, since both maps are covering maps. This process can be continued inductively as long as A_k does not contain a critical point of f. This gives us a set of coordinates on $\{\infty\} \cup \bigcup_{k=0}^{\infty} A_k$; these are usually called **Böttcher coordinates**.

The preimages of curves of constant radius under ϕ are called **equipotentials**; the preimage of the radial line $te^{2\pi i\theta}$ ($t > r_0$) is called the **external ray of angle** θ (where θ is measured in *turns*); we denote it by $\mathcal{R}_\theta$, and use $\mathcal{R}_\theta(t)$ as a parameterization, with $\phi(\mathcal{R}_\theta(t)) = te^{2\pi i\theta}$.

If the orbits of all critical points for the polynomial f are bounded, these coordinates will extend to all of $\mathrm{Bas}(\infty)$. Even in the case where a finite critical value lies in $\mathrm{Bas}(\infty)$, equipontentials and all but a countable number of external rays can be defined on the whole of $\mathrm{Bas}(\infty)$, as we shall see shortly.

If $\lim_{t\to 1} \mathcal{R}_\theta(t)$ exists, we say that the ray $\mathcal{R}_\theta$ **lands** at a point z in the Julia set. If all rays land, then the Julia set must be locally connected (since then we have ϕ^{-1} extending as a continuous mapping from $\overline{\mathbb{D}}$ to $\overline{\mathrm{Bas}(\infty)}$).

It should be apparent that the previous construction can be immediately adapted to the situation where we have a fixed (or periodic) point p with $f'(p) = 0$. In this situation, the mapping will

[1] More precisely, we set $\phi(z) = \left(\lim_{k\to\infty} f^k(z)\right)^{1/d^k}$.

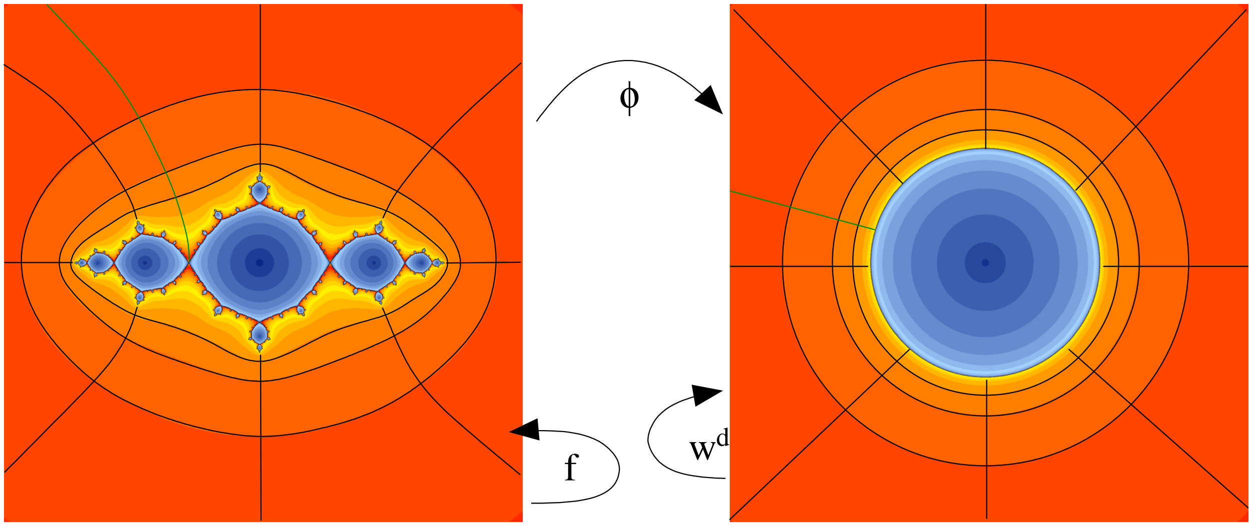


FIGURE 6.1. Böttcher coordinates on $\mathrm{Bas}(\infty)$, with equipotential lines and external rays shown.

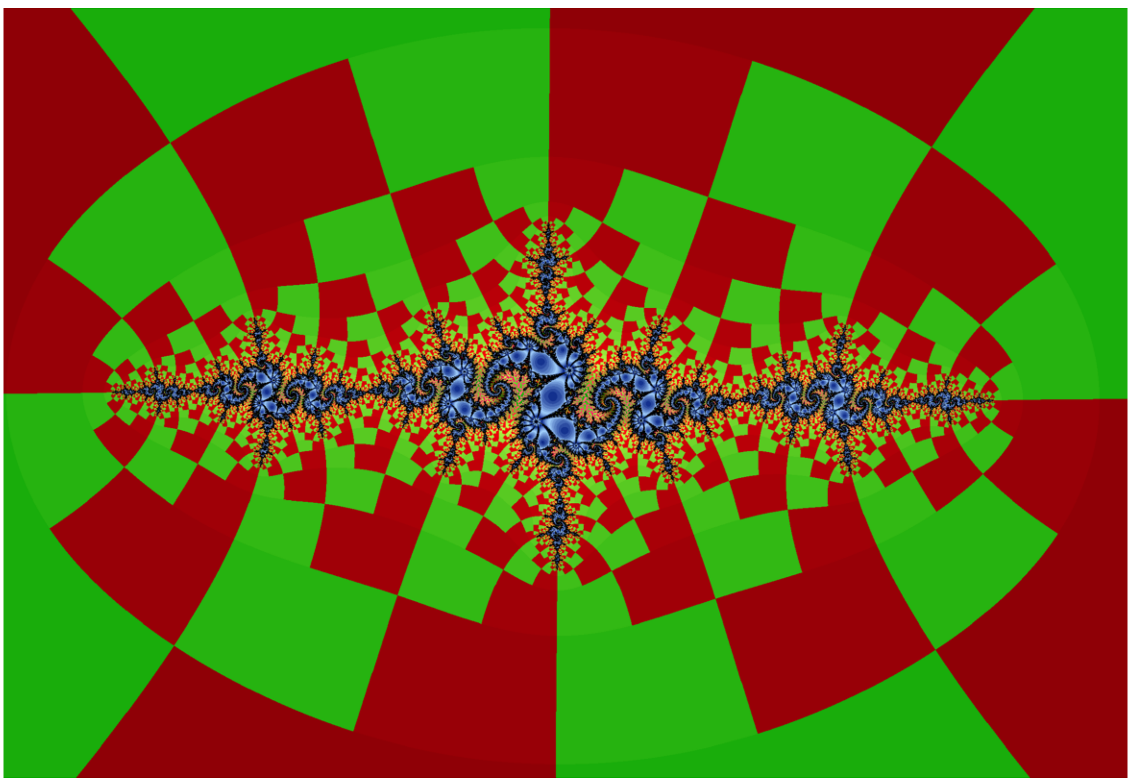


FIGURE 6.2. On $\mathrm{Bas}(\infty)$, preimages of the annulus A_0 (as described earlier) are shown, where points z for which $f^n(z) \in A_0$ has positive imaginary part are colored in shades of red, and those where $\mathrm{Im}(f^n(z)) < 0$ are colored green. This coloring enables one to read off equipotentials and external angles of the form $p/2^n$.

be conjugate to w^k, where k is the smallest integer such that the kth derivative $f^{(k)}(p)$ is nonzero ($k \geq 2$). In this case, the corresponding ray $\mathcal{R}$ is called an **internal ray**.

For all points $z \in \mathrm{Bas}(\infty)$, the **escape rate** ($|\phi(z)|$) is well-defined, since we have

$$|\phi(z)| = \left(\lim_{k\to\infty} |f^k(z)|\right)^{1/d^k},$$

and the dth root is unambiguous for non-negative numbers.

This enables us to define the **Green's function** for the filled Julia set $\mathcal{K}_f$ (sometimes also called the **canonical potential function**) as

$$G_f(z) = \begin{cases} \log|\phi(z)| & z \notin \mathcal{K}_f \\ 0 & z \in \mathcal{K}_f \end{cases}.$$

One sees easily that G_f is continuous everywhere and harmonic, and that $G_f(f(z)) = dG_f(z)$.

Level curves $G_f(z) = \text{constant} > 0$ are **equipotentials for** f; observe that f sends one equipotential to another equipotential. While we won't go into details here, one can use G_f to define a measure on the Julia set which corresponds to the harmonic measure on K_f.

Observe that if there is a critical point which lies in $\mathrm{Bas}(\infty)$, we cannot extend the conjugacy ϕ without ambiguity to the entire basin of infinity, since there will be at least two rays which land at the critical point. (Of course, we should not expect to be able to extend the conjugacy, since w^d has no non-zero, finite critical points. Instead, the conjugacy will instead correspond to a Blaschke product.)

However, the mapping ϕ can be analytically continued to the whole of $\mathrm{Bas}(\infty)$ to obtain a conformal isomorphism between $\mathrm{Bas}(\infty)$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Consider $f(z) = z^2 + c$ with $c < -2$. In this case, $f^n(0) \to \infty$, so $0 \in \mathrm{Bas}(\infty)$. The equipotential $G_f(z) = \log|c|$ will be an (analytic) circle, but its preimage $G_f(z) = \log|c|/2$ will be will branch at the origin, making a figure-8 shape. Note that the two preimages of the ray passing through c must cross at the origin (since it is a critical point), although all other rays extend without ambiguity to the interior of the figure-8. Of course, there will be two more rays which branch at the pre-image of 0, and so on. See Figure 6.3.

Observe the close similiarity to the classical construction of a Cantor set: the Julia set must lie in the interior of the nested figure-8 shapes. If these figure-8s contract to points, the result will be homeomorphic to a Cantor set.

This is indeed what happens if *all* the critical points of f lie in the basin of ∞. For quadratic polynomials, since there is only one critical point, we have the following.

Lemma 6.5 (Dichotomy for quadratic polynomials)**.** *Let $f(z)$ be a quadratic polynomial with critical value c.*

- *If the orbit of c is unbounded, then $\mathcal{J}_f$ is homeomorphic to a Cantor set.*
- *If the orbit of c is bounded, then $\mathcal{J}_f$ is connected.*

Using this dichotomy, one can define the **Mandelbrot set** as

$$\mathcal{M} = \left\{ c \in \mathbb{C} \mid \text{the orbit of } 0 \text{ under } z \mapsto z^2 + c \text{ is bounded} \right\}.$$

Of course, for a polynomial of degree three or higher there can be an intermediate case where the Julia set consists of infinitely many components but is not a Cantor set; see Figure 6.4.

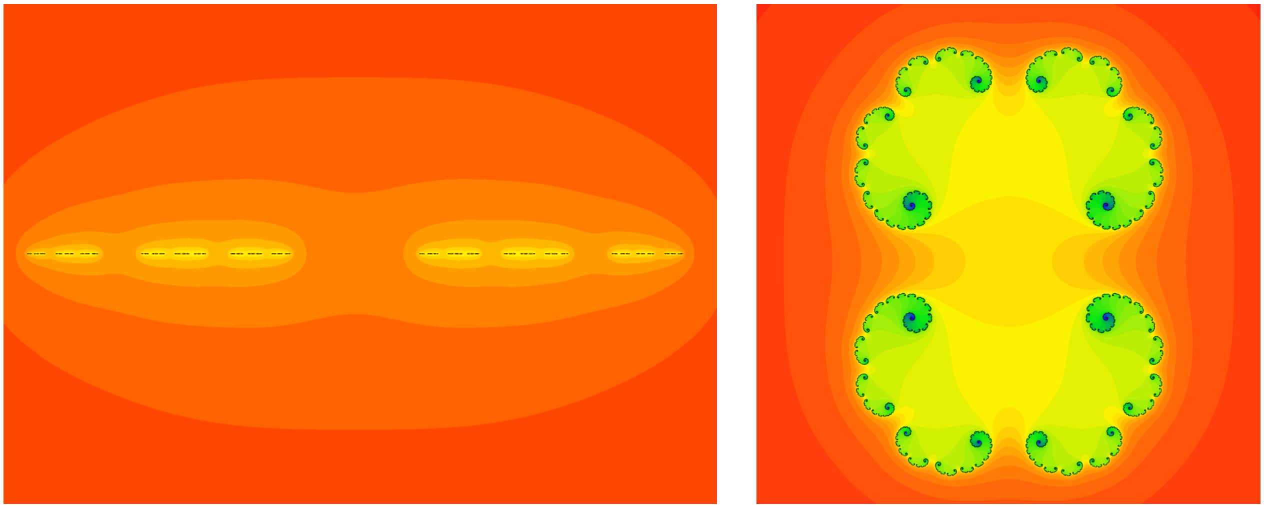


FIGURE 6.3. Two quadratic Julia sets homeomorphic to the Cantor set.

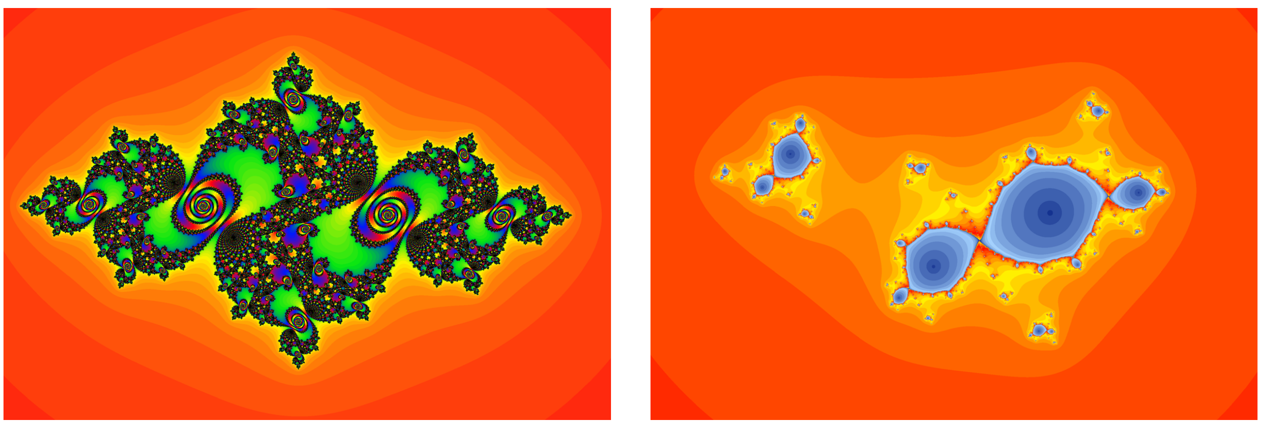


FIGURE 6.4. Another quadratic Cantor Julia set, and a cubic with only one critical point which escapes.

7. MONTEL'S THEOREM AND ITS CONSEQUENCES

One powerful tool to understanding the Julia set $\mathcal{J}$ is Montel's theorem.

Theorem 7.1 (Montel)**.** *Let $\mathfrak{F}$ be a family of meromorphic functions defined on a domain U. If there exist three points $a, b, c \in \widehat{\mathbb{C}}$ so that*

$$\{a, b, c\} \cap \Big(\bigcup_{f \in \mathfrak{F}} f(U) \Big) = \emptyset,$$

then $\mathfrak{F}$ is a normal family.

Corollary 7.2. *Let $z \in \mathcal{J}_f$. Then for any neighborhood U of z, the set $\mathcal{E}_U = \widehat{\mathbb{C}} \setminus \bigcup_{n>0} f^n(U)$ contains at most two points.*

The points in $\mathcal{E}_U$ are called **exceptional points**.

Note that for a polynomial $p(z)$, ∞ is an exceptional point. We have $p^{-1}(\infty) = \{\infty\}$. Furthermore, since ∞ is a superattracting point, $\mathrm{Bas}(\infty) \subset \mathcal{F}$, and the Julia set $\mathcal{J}$ lies in a bounded region of $\mathbb{C}$. Since p has no poles, $p(\mathbb{C}) = \mathbb{C}$; thus ∞ is an exceptional point.

Theorem 7.3. *Suppose $z \in \mathcal{J}_f$, and let $E_z = \bigcup E_U$, where U ranges over all neighborhoods of z. Then*

- *If $\mathcal{E}_z$ contains exactly one point, then f is conjugate to a polynomial.*
- *If $\mathcal{E}_z$ contains two points, then either f is conjugate to z^d or $1/z^d$, where d is the degree of f.*

In both cases, E_z does not depend on the choice of $z \in \mathcal{J}$, and $\mathcal{E}_z \subset \mathcal{F}$.

The proof follows from conjugating f by a Möbius transformation which moves $\mathcal{E}_z$ to $\{\infty\}$ (in the first case) or $\{0, \infty\}$ in the second. An easy calculation finishes the proof.

Corollary 7.4. *If $\mathcal{J}_f$ has nonempty interior, then $\mathcal{J}_f$ is the entire Riemann sphere $\widehat{\mathbb{C}}$.*

Proof. Suppose there is a domain $U \subset \mathcal{J}$. Since $\mathcal{J}$ is forward invariant,

$$\bigcup_{n>0} f^n(U) \subset \mathcal{J} \quad \text{and} \quad \bigcup_{n>0} f^n(U) = \widehat{\mathbb{C}} \smallsetminus \mathcal{E}_u.$$

But $\mathcal{J}$ is closed and $\mathcal{E}_u$ contains at most two points, so $J = \widehat{\mathbb{C}}$. □

The map

$$L(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$$

is known as a **Lattès' example**. $L(z)$ corresponds to multiplication by 2 on the torus $\mathbb{T}^2$ via the Weierstrass $\wp$ function. Furthermore, the Julia set of L is all of $\widehat{\mathbb{C}}$. For further details, see [Mil06], for example. Other examples of rational maps where $\mathcal{J}$ is the entire Riemann sphere can be found in [Ree84].

Corollary 7.5. *If z is any point in the Julia set of f, then preimages of z are dense in $\mathcal{J}_f$:*

$$\mathcal{J}_f = \overline{\bigcup_{n\geq 0} f^{-n}(z)}.$$

Proof. Observe that for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, the preimages of w accumulate on the Julia set:

$$\mathcal{J}_f \subset \overline{\bigcup f^{-n}(w)}.$$

This follows since for any $z \in \mathcal{J}$ and any neighborhood U of z, we have $w \in f^n(U)$ for some n.

Since $\mathcal{J}$ is a completely invariant set and $\mathcal{J}$ is closed, we also have

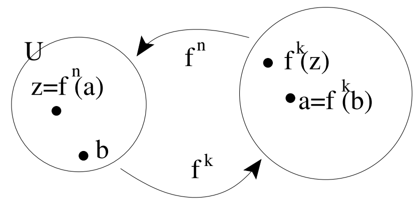
$$\overline{\bigcup f^{-n}(z)} \subset \mathcal{J} \quad \text{for any } z \in \mathcal{J}.$$

□

Theorem 7.6. *The Julia set is a perfect set. That is, it has no isolated points.*

Proof. Let $z \in \mathcal{J}$. We can find another point $a \in \mathcal{J}$ so that $f^n(a) = z$ for some n, but a is not a forward image of z. (If z is not periodic, any inverse of z will do. If z is periodic of period m, consider $g = f^m$, and choose a so that $g(a) = z$ but $a \neq z$.)

Now let U be a neighborhood of z. Since $a \in \mathcal{J}$, a is not an exceptional point, so there is a $k > 0$ so that $f^k(U)$ contains a.



Let b be another point of U for which $f^k(b) = a$. Then $b \neq z$ since z is not in the forward orbit of a. Since $\mathcal{J}$ is completely invariant, a and b are in $\mathcal{J}$. Thus, every point z of $\mathcal{J}$ is an accumulation point of other points of $\mathcal{J}$. $\square$

We now show that the Julia set is the closure of the repelling periodic points.

Lemma 7.7. $\mathcal{J} \subset \overline{\{\textit{periodic points}\}}$.

Proof. We give the proof for polynomials $p\colon \mathbb{C} \to \mathbb{C}$; a similar idea works for rational functions.

Let J_0 be the Julia set with any critical values of p removed, and let $w \in J_0$. Since J_0 contains no critical values, there is a neighborhood U of w and a local inverse $S\colon U \to \mathbb{C} \setminus U$. Let

$$g_n(z) = \frac{p^n(z) - z}{S(z) - z}, \quad n > 0.$$

Observe that g_n is bounded and takes the values 0 and 1 only on periodic orbits z. Consequently, the family $\{g_n\}$ is normal if and only if $\{p_n\}$ is normal[2].

But since $w \in \mathcal{J}$, the family $\{p_n\}$ and hence $\{g_n\}$ cannot be normal on any neighborhood of w. But unless there is a sequence of periodic orbits accumulating to w, the family $\{g_n\}$ omits the values 0, 1, and ∞ on a neighborhood of w, contradicting Montel's theorem. $\square$

To prove the inclusion the other way, we show that there are only finitely many non-repelling orbits. Then the previous lemma, coupled with the fact that $\mathcal{J}$ is a perfect set, will give the result.

Let $\mathcal{B}(p)$ denote the **immediate basin of** p, that is, the connected component of $\mathrm{Bas}(p)$ which contains p.

Theorem 7.8. *If p is an attracting fixed point, then $\mathcal{B}(p)$ contains a critical value.*

Proof. Suppose there is no critical value in $\mathcal{B}(p)$. Then if U is a simply connected neighborhood of p contained in $\mathcal{B}(p)$, we can construct a local inverse S_1 for $f|_U$ so that $S_1(p) = p$.

[2] To adapt this proof to a rational function $R(z)$, instead one sets J_0 to be the Julia set with infinity, all critical values, and all poles of R removed. Then, for a neighborhood U of a point $w \in \mathcal{K}$, let S_1, S_2, and S_3 be three local inverses of R^2 and take

$$g_n(z) = \frac{R^n(z) - S_1(z)}{R^n(z) - S_2(z)} \frac{S_3(z) - S_2(z)}{S_3(z) - S_1(z)}.$$

The family $\{g_n\}$ is normal if and only if $\{R^n\}$ is normal.

Observe that $U_1 = S_1(U)$ is a simply connected subset of $\mathcal{B}(p)$, so we can repeat the process using U_1 to construct S_2. Indeed, since there is no critical value in $\mathcal{B}(p)$, it can be repeated infinitely often, giving a family $\{S_k\}$ on U, which is normal (by Montel's theorem) since $S_k(U) \subset \mathcal{B}(p) \subset \mathcal{F}$.

But p is a repelling fixed point for each S_k, so the family cannot be normal. □

We can extend this result to the a count on attracting periodic orbits: if p is of period k, apply the theorem to f^k.

From Theorem 7.8, we get an upper bound of $2d - 2$ on the number of attracting orbits of a rational map f of degree d. Fatou and Julia showed (by means of a perturbation argument) that the number of neutral cycles is at most $4d - 4$. However, in 1987 Shishikura [Shi87] used quasiconformal surgery techniques to give the sharp bound of at most $2d-2$ non-repelling orbits for any rational map of degree d. This bound is commonly called the **Fatou-Shishikura inequality**. Applying the Fatou-Shishikura inequality gives the following.

Theorem 7.9. *The Julia set $\mathcal{J}_f$ is the closure of the repelling periodic orbits of f.*

Proof. To prove this, we merely combine the fact that $\mathcal{J} \subset \overline{\{\text{periodic points}\}}$ with the fact that there are only finitely many non-repelling periodic orbits. Since $\mathcal{J}$ is a perfect set, each point in it must be the accumulation of some sequence of repelling periodic points. □

8. Neutral periodic orbits

We have seen that repelling periodic orbits always lie in the Julia set, and that attracting orbits lie in the Fatou set. If p is part of a periodic cycle with multipler on the unit circle, p may either lie in the Julia set or in the Fatou set.

Theorem 8.1. *Let p be a fixed point with multiplier $f'(p) = \lambda$, $|\lambda| = 1$. Then $p \in \mathcal{F}$ if and only if the Schröder functional equation (SFE) $\phi(f(z)) = \lambda\phi(z)$ has an analytic solution in a neighborhood of p.*

Proof. If (SFE) has a solution in a neighborhood of p, then $p \in \mathcal{F}$ follows immediately.

Conversely, suppose $p \in \mathcal{F}$ and let U be the maximal domain such that $p \in U$ and $U \subset \mathcal{F}$. Since U is disjoint from J, its complement in $\widehat{\mathbb{C}}$ contains more than three points, so the universal cover $\widetilde{U}$ is conformally equivalent to $\mathbb{D}$.

Thus, we have a cover $\varphi\colon \mathbb{D} \to U$ with $\varphi(0) = p$, and we can lift f to $\tilde{f}\colon \mathbb{D} \to \mathbb{D}$ with $\tilde{f}(0) = 0$. Since $|\tilde{f}'(0)| = |\lambda| = 1$, then by the Schwarz lemma, $\tilde{f}(z) = \lambda z$. The mapping φ is the solution ϕ to (SFE). □

Corollary 8.2. *If p is a periodic point with multiplier λ a root of unity, then $p \in \mathcal{J}$.*

Proof. Replacing f by an iterate if necessary, we may assume p is a fixed point. Suppose $\lambda^k = 1$ and $p \in \mathcal{F}$. Then if (SFE) has a solution ϕ in some neighborhood U of p, we have $\phi \circ f^k \circ \phi^{-1}$ is the identity on U. But since f is analytic, it is the identity on $\widehat{\mathbb{C}}$, and thus f is of degree 1, a contradiction. □

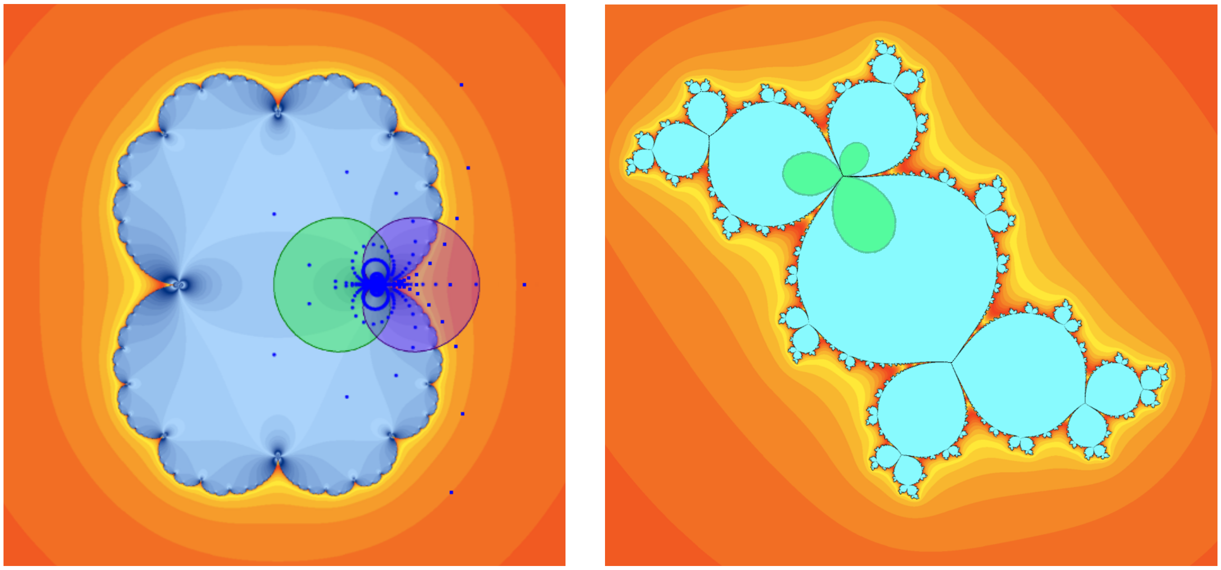


FIGURE 8.3. On the left is the "cauliflower" ($z \mapsto z^2 + 1/4$) with a cardiod-shaped attracting petal in green, a repelling petal in violet, and the forward orbits of several points shown. At right is the "fat rabbit" with attracting petals indicated in green.

Theorem 8.4 (Fatou-Leau flower theorem)**.** *Suppose $\lambda^n = 1$ with $\lambda^j \neq 1$ for $1 < j < n$. Let $f(z) = \lambda z + a_2 z^2 + \ldots$ be analytic in a neighborhood of the origin. If $f^n \neq Id$, there is an integer k for which there are nk **attracting petals** bounded by analytic curves which are tangent at the origin. The union of the petals is forward invariant, and any orbit in a petal is asymptotic to the origin. Any compact set within a petal converges uniformly to the origin under f^n.*

In addition, there are nk **repelling petals**; in each, every orbit eventually leaves the petal (alternatively, in each petal there is a branch of the inverse for which the petal is attracting).

ROTATION DOMAINS. In 1942, Carl Siegel [Sie42] found a full measure subset of the unit circle Λ such that whenever the multiplier at p is in Λ, the Schröder functional equation has a solution, so $p \in \mathcal{F}$.

Theorem 8.5. *Let p be a fixed point for f with multiplier $\lambda = e^{2\pi i\theta}$, where θ is irrational. Suppose also there exist constants a and b such that*

$$\left|\theta - \frac{p}{q}\right| > \frac{a}{q^b} \qquad \textit{for all rationals } \frac{p}{q}.$$

Then (SFE) has a solution and $p \in \mathcal{F}$.

The condition above is roughly that "θ is badly approximated by rationals." For example, the golden mean $\frac{1+\sqrt{5}}{2}$ is such a number, as is any number for which the terms in its continued fraction expansion are bounded. In this case, the topological disk around p on which f is conjugate to an irrational rotation is called a **Siegel disk**.

The Julia set of a polynomial with a Siegel disk can be extremely complicated, despite the fact that the dynamics on the Siegel disk is conjugate to a "simple" rigid rotation. For example, there

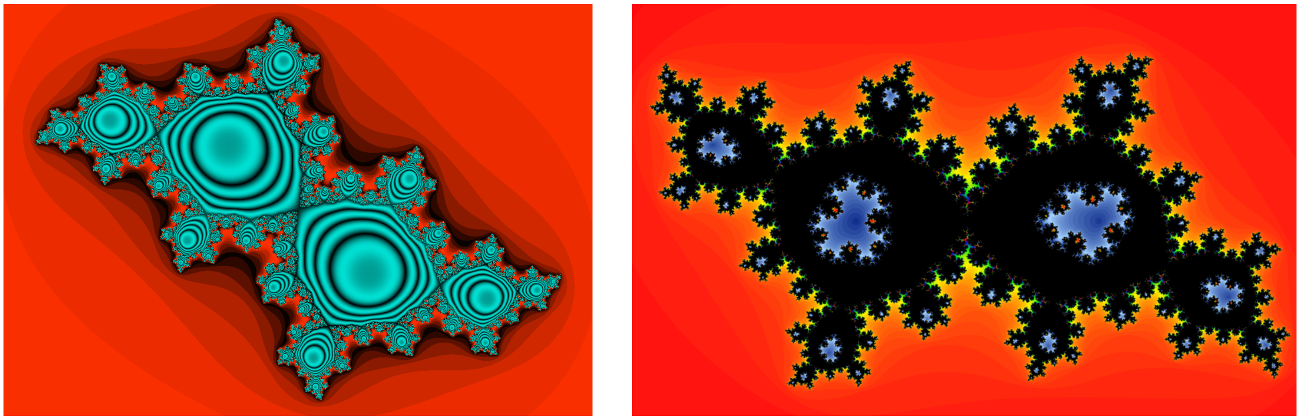


FIGURE 8.6. On the left is the Julia set of a polynomial which has a Siegel disk with a rotation number of the golden mean (the Siegel disk and its preimages are shown in blue; the light-dark bands indicate orbits within the Siegel disk). On the right is a rational map with a Hermann ring with a golden-mean rotation ($\mathrm{Bas}(\infty)$ is in reds and yellows, $\mathrm{Bas}(0)$ in shades of blue, with the Hermann ring and its preimages shown in black).

are quadratic polynomials with Siegel disks whose Julia set is a non-computable set (see [BY09]), as well as Julia sets with no interior but non-zero Lebesgue measure (see Section 10).

Related to Siegel disks are Herman rings. A **Herman ring** is a domain which is conformally equivalent to an annulus, on which the dynamics is conjugate to an irrational rotation. Thus, a Herman ring is a subset of the Fatou set. Michel Herman showed their existence in 1979.

Siegel disks and Herman rings are collectively referred to as **rotation domains**.

Because of the maximum principle, Herman rings cannot occur for polynomials. Shishikura has shown that the degree of a rational map with a Herman ring must be at least three. For any odd degree at least 3, we can construct a Blaschke product f which sends the unit circle to itself by an orientation-preserving diffeomorphism with any desired rotation number ρ. If ρ is Diophantine (i.e., badly approximated by rationals), then f will have a Herman ring.

For example, the map $e^{2\pi i t}z^2\dfrac{z-4}{1-4z}$, with $t \approx 0.6151732$, has a Herman ring with rotation number $(\sqrt{5}-1)/2$. See Figure 8.6.

CREMER POINTS. The condition of a Diophantine rotation number for the existence of a Siegel disk is sufficient, but not sharp. For the map $z^2+\lambda z$, if λ is **Brjuno** (i.e., if the convergents p_n/q_n of λ satisfy $\sum(\log q_{n+1})/q_n < \infty$), then the map will have a Siegel disk about zero.

Yoccoz (see [Hub93], [Yoc95]) showed that this condition is sharp for quadratic polynomials: if it fails to hold, the map cannot be linearized in a neighborhood of the fixed point. Indeed, it has the *small cycles property*: every neighborhood of the origin contains infinitely many periodic orbits.

A point with this property is called a **Cremer point**, and is of necessity in the Julia set.

We know of no good means of making a picture of a map with a Cremer point, although topological models exist and such Julia sets are always computable [BBY07] when they have empty interior (which is necessarily true for quadratics). Julia sets with Cremer points are not locally connected sets [Sul83] and can have positive Lebesgue measure (see Section 10). See [BBCO10] for more details and further references.

9. Fatou-Sullivan classification of Fatou components

In 1983, Dennis Sullivan [Sul83] classified the possibilities for a Fatou component which is eventually periodic. While most of these possibilities were known to Fatou and Julia, Sullivan's work completed the classification.

Let U be an eventually periodic such a component of the Fatou set. Then, after passing to an iterate, we can view U as being fixed. There are only four possibilities:

(1) U is the immediate basin of an attracting (or super-attracting) point p.
(2) U is the immediate basin of one petal of a parabolic point.
(3) U is a Siegel disk.
(4) U is a Herman ring.

Furthermore, Sullivan [Sul85] showed that every Fatou component is eventually periodic, that is, there are *no wandering domains* for rational maps[3].

Theorem 9.1 (Sullivan's Non-Wandering theorem)**.** *Every Fatou component U for a rational map is eventually periodic. That is, there exist integers $n \geq 0$ and $p \geq 1$ so that the forward image $f^n(U)$ is mapped onto itself by f^p. In particular, it follows that every Fatou component is either a branched covering or a biholomorphic copy of some periodic Fatou component, which necessarily belongs to one of the four types described above.*

10. A Julia set of positive measure

Earlier, we noted that either $\mathcal{J}$ has no interior or it is the entire Riemann sphere. However, recently Buff and Chéritat [BC12] showed that there is a quadratic polynomial with a Julia set of positive measure which is not the entire Riemann sphere.

They did this by constructing a sequence of perturbations of Siegel disks with an increasingly complicated boundary, such that the loss of measure in the filled Julia set is controlled. The Siegel disks become "digitated", with deep channels entering towards the center of the disk. Informative pictures of the process can be found on Arnaud Chéritat's web page [Che].

More precisely, Buff and Chéritat showed the following.

Theorem 10.1. *Let $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$. Then*

- *there exists α such that P_α has a fixed point of Cremer type and $\mathcal{J}_{P_\alpha}$ has positive measure;*
- *there exists β such that P_β has a Siegel disk and $\mathcal{J}_{P_\beta}$ has positive measure.*

11. Conclusion

In this brief note, I hope I have given you some idea of the beauty and complexity related to holomorphic dynamics in one variable. Of course, due to limited space many details and interesting, relevant topics were omitted. I hope I have inspired the reader to follow up and learn more about this vibrant and exciting area of mathematics.

[3] This result is not valid for transcendental maps, since $z + \sin(2\pi z)$ has a wandering domain.

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