RECURRENCES AND GENERATING FUNCTIONS

ROBERT HOUGH

Problem 1. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geqslant 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

Find a formula for T_n of the form $A_n + B_n$ as the sum of two well-known sequences.

Problem 2. Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integers n.

Problem 3. Show that the next integer above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

Problem 4. Let $x_k = k$ for $k = 1, 2, ..., 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geqslant 2006$. Show that the sequence has 2005 consecutive terms that are divisible by 2006.

Problem 5. Let $f(m,1) = f(1,n) = 1$ for $m \geqslant 1, n \geqslant 1$ and let, for $m > 1$ and $n > 1$,

$$f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1)$$

and

$$S(n) = \sum_{a+b=n} f(a,b).$$

Prove for $n \geqslant 2$

$$S(n+2) = S(n) + 2S(n+1).$$

Problem 6. Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer, and $x^{(0)} = 1$. Prove that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$

Problem 7. Calculate

$$s_{i,n} = \sum_{k \equiv i \bmod 3} \binom{n}{k}.$$

Problem 8. Show that

$$x + \frac{2}{3}x^3 + \frac{2}{3}\frac{4}{5}x^5 + \frac{2}{3}\frac{4}{5}\frac{6}{7}x^7 + \cdots = \frac{\arcsin x}{\sqrt{1-x^2}}.$$

Problem 9. Let

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geqslant 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

Problem 10. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, ..., x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leqslant m$. Show that $f(m,n) = f(n,m)$.