

MAT 548, Homework 8, due Friday, Oct 30

1. Prove the following very easy but very useful facts. Let X be a topological space.

(a) If X is simply connected, $a, b \in X$, then any two paths in X connecting a and b are homotopic. (Be careful: the homotopy of paths must preserve the endpoints.)

(b) X is contractible if and only if id_X is homotopic to a constant map.

(c) A path-connected space X is contractible if and only if for every path-connected space Y , any two maps $f, g : X \rightarrow Y$ are homotopic.

By definition, a contractible space is homotopy equivalent to a point. Parts (b) and (c) give equivalent definitions.

2. Let X be a closed smooth manifold of $\dim(X) > 0$. (Recall that “closed” means compact without boundary). Prove that X cannot be contractible.

Hint: engineer a map from X to another space that is not null-homotopic.

3. For $n \geq 1$, show that the following statements are equivalent if X is path-connected.

(1) For every $0 < k \leq n$, any two continuous maps $S^k \rightarrow X$ are homotopic.

(2) For every $0 < k \leq n$, any two continuous maps $(S^k, s_0) \rightarrow (X, x_0)$ are homotopic, i.e. $\pi_k(X, x_0) = 0$. Here, $s_0 \in S^k$ and $x_0 \in X$ are fixed basepoints.

(3) For every $0 < k \leq n$, every continuous $g : S^k \rightarrow X$ can be extended to a continuous map $G : D^{k+1} \rightarrow X$, such that $G|_{S^k} = g$.

Each of the statements above can be taken as a definition of an n -connected space X . (For a 1-connected space, a more common term is *simply connected*.) For $n > 1$, the sphere S^n is $(n-1)$ -connected but not n -connected (and therefore not contractible), since $\pi_n(S^n) = \mathbb{Z}$.

Note: For an arbitrary X and $n = k = 0$, the statements above are equivalent to the path-connectedness of X . The set $\pi_0(X, x_0)$ can be defined by analogy with π_n for $n > 0$; $\pi_0(X, x_0)$ does not have a group structure. Feel free to prove the $n = k = 0$ case if you are interested, but do not submit this part.

4. Recall that for any collection of basepointed spaces (X_α, x_α) , $x_\alpha \in X_\alpha$, the *wedge sum* $\vee_\alpha X_\alpha$ is the quotient of the disjoint union $\sqcup_\alpha X_\alpha$ obtained by identifying all $x_\alpha \in X_\alpha$ to a single point.

The goal of this question is to develop more intuition about homotopy equivalence. Each of the topological spaces below is homotopy equivalent to a (finite) wedge sum of some spheres, possibly of different dimensions. (Some of the spaces may be homotopy equivalent to a single sphere or a circle, or be contractible.) Find the corresponding wedge sum in each case, and explain briefly what the homotopy equivalences look like and what the required homotopies do. Arguing informally (“by picture”) is fine. No formulas are required, but please think through the details and describe your homotopies carefully (otherwise it’s easy to make a mistake with discontinuous maps). Some of the homotopy equivalences will be given by deformation retractions.

(a) the complement $T^2 \setminus \{p\}$ of a point p in a torus T^2 ;

(b) the complement of k lines through the origin in $\mathbb{R}^n$, $n \geq 3$;

(c) the union $S^2 \cup D$ of the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ and the disk $D = \{x^2 + y^2 \leq 1, z = 0\}$.

(d) the union $S^2 \cup I$ of the sphere S^2 in $\mathbb{R}^3$ and a segment I connecting two opposite points of S^2 ;

(e) the sphere S^n , $n \geq 2$, with k distinct points identified, that is, the quotient S^n/K for $K = \{x_1, x_2, \dots, x_k\} \subset S^n$.

Note: Different wedge sums of spheres are not homotopy equivalent to each other (although we cannot prove it yet), so the answer is unique in each case.

5. (a) Let (X, x_0) and (Y, y_0) be two topological spaces with basepoints, and let pr_X, pr_Y be the projections of the product $X \times Y$ on its factors. Let γ be loop in $(X \times Y, (x_0, y_0))$. Show that the map

$$\gamma \mapsto (\text{pr}_X(\gamma), \text{pr}_Y(\gamma))$$

descends to homotopy classes of loops and gives a group isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Corollary: the fundamental group of the n -torus $T^n = (S^1)^n$ is given by $\pi_1(T^n) = \mathbb{Z}^n$.

(b) Show that $\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$, $n > 1$. It suffices to explain briefly why your proof from (a) works for any n .

(c) Let α, β be two loops in $(X \times Y, (x_0, y_0))$ such that

$$\alpha(s) = (a(s), y_0), \quad \beta(s) = (x_0, b(s)).$$

It follows from (a) that the classes of α and β commute in $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy between $\alpha * \beta$ and $\beta * \alpha$.

6. Use the fundamental group and its functorial properties to show that the Möbius band cannot retract onto its boundary circle.

Hint: you'll need to think of the maps on π_1 induced by the inclusion and a hypothetical retraction, not only about the groups themselves.

7. Find the fundamental group of the following spaces. Use homotopy equivalences to familiar spaces or methods from smooth topology (transversality) via smooth approximation. Please *do not* use the Seifert–van Kampen theorem even if you know it.

(a) $X = \mathbb{R}^3 \setminus (C \cup L)$ is the complement of the union of the z -axis L and the circle $C = \{x^2 + y^2 = 1, z = 0\}$.

(b) Z is the complement of a finite set of disjoint lines in $\mathbb{R}^4$.

8. Let $x_0 \in \mathbb{R}^2 - \{0\}$, and consider a (continuous) map $\gamma : [0, 1] \rightarrow \mathbb{R}^2 - \{0\}$, $\gamma(0) = \gamma(1) = x_0$.

Explain how to define an integer-valued *winding number* $W_0(\gamma)$ of the loop γ about 0. (See Homework 7.)

We know that $\pi_1(\mathbb{R}^2 - \{0\}, x_0) = \mathbb{Z}$. The usual convention is that the positive generator $1 \in \mathbb{Z}$ corresponds to the counterclockwise loop about the origin. Show that

$$[\gamma] = W_0(\gamma) \cdot 1 \in \pi_1(\mathbb{R}^2 - \{0\}, x_0).$$

Note: there is a similar statement for the winding number of a map of S^n to $\mathbb{R}^{n+1} - \{0\}$ and the corresponding element of $\pi_n(\mathbb{R}^{n+1} - \{0\}, x_0)$, with essentially the same proof. The required part of the question only concerns $n = 1$.