MAT 548, Homework 6, due Friday, Oct 9

From Lee's textbook: please do **questions 6-9, 6-13ab, 6-16** at the end of Chapter 6.

Comments:

(i) Some of you have been asking about the notion of transversality for two maps: see Question 6-13. For this class, the notion "map transverse to submanifold" is more important, due to the transversal preimage theorem 6.30 and a more transparent picture. But talking about two transverse maps is also useful sometimes (example: parameterized curves). Note that 6-13c (not assigned) contains a mistake.

(ii) Question 6-16 looks very long, but every part is very short once you get the main idea!

More questions:

1. (a) Show that the set of lines in $\mathbb{R}^n$ is a smooth manifold. (The lines do not have to pass through the origin and are not oriented.)

(b) Let $M \subset \mathbb{R}^n$ be a smooth submanifold. Show that almost every line in $\mathbb{R}^n$ intersects M transversely. What does "almost every" mean here?

2. Let M be a manifold with boundary. Show that there is a smooth non-negative function $\pi : M \to \mathbb{R}$ such that 0 is a regular value of π, and $\partial M = \pi^{-1}(0)$.

Hint: Use partition of unity. Don't forget to prove regularity.

3. Let $M \subset \mathbb{R}^n$ be a smooth manifold, and $X \subset M$ a smooth submanifold. Define the normal bundle to X in M by

$$N^M X = \{(p, v) : p \in X, \quad v \in T_p M, \quad v \perp T_p X\}.$$

Show that $N^M X$ is a smooth manifold, and that the projection $\sigma : N^M X \to X$, $\sigma(p, v) = p$ defines a (smooth) vector bundle (that is, there are local trivializations). What is the dimension of this manifold and the rank of this vector bundle?

Caution: as we discussed in class, you can't see the normal bundle in slice charts (because the chart maps don't have to preserve the inner product). It will be helpful to carefully read the normal bundle theorem in Guillemin–Pollack and work out the details.

4. The following technical lemma generalizes Parametric Transversality Theorem. We'll need it soon.

Let $f : N \to M$ be a smooth map, $X \subset M$ a closed submanifold (all have empty boundary). Further, $S = B_\epsilon(0) \subset \mathbb{R}^k$ is a parameter space, and $F : N \times S \to M$ is a smooth map transverse to X, such that $F_{s_0} = f$ for some $s_0 \in S$.

Let $C \subset N$ be a closed subset, and suppose that $f \pitchfork X$ on C: this means that for every $p \in C \cap f^{-1}(X)$, the subspaces $d_p f(T_p N)$ and $T_{f(p)} X$ span $T_{f(p)} M$.

By modifying F appropriately, prove that there is a smooth map $G : N \times S \to M$ transverse to X, such that $G_{s_0} = f$, and $G_s(p) = f(p)$ for all s if $p \in C$. By Sard's theorem, it follows that f can be deformed to a function g_s transverse to X, without changing f on the subset where transversality already holds.

Hint: Use partition of unity or bump functions. First show that if $f \pitchfork X$ on C, then transversality holds in an open neighborhood of C.

Note: correction added, in the first version S was an arbitrary smooth manifold (which creates topological difficulties). The important corollary of this lemma is that you can find g_s which is homotopic to f and transverse to X. If you started with a bigger parameter space, you can always restrict to a small ball in a chart and assume $S = B_\epsilon(0) \subset \mathbb{R}^k$.