MAT 540, Homework 7, due Wednesday, Oct 16

1. Show that the following statements are equivalent. (Each of them can be taken as a definition of an n-connected space X.) In (2), $s_0 \in S^k$ and $x_0 \in X$ are fixed basepoints.

(1) For every $k \leq n$, any two continuous maps $S^k \to X$ are homotopic.

(2) For every $k \leq n$, any two continuous maps $(S^k, s_0) \to (X, x_0)$ are homotopic, i.e. $\pi_k(X, x_0) = 0$, for every $x_0 \in X$.

(3) For every $k \leq n$, every continuous $g : S^k \to X$ can be extended to a continuous map $G : D^{k+1} \to X$, such that $G|_{S^k} = g$.

Strictly speaking, the statement about $\pi_k(X, x_0)$ is valid for $k \geq 1$. For $k = 0$, the statements above are equivalent to path-connectedness of X.

This question is a generalization of question 5 p.38 in Hatcher Chapter 1; feel free to think about the case $n = 1$ (the general proofs are essentially the same).

2. (a) Let (X, x_0) and (Y, y_0) be two topological spaces with basepoints, and let π_x, π_y be the projections of the product $X \times Y$ on its factors. Let γ be loop in $(X \times Y, (x_0, y_0)$. Show that the map

$$\gamma \mapsto (\pi_x(\gamma), \pi_y(\gamma))$$

descends to homotopy classes of loops and gives a group isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

(b) Show that $\pi_n(X \times Y, (x_0, y_0)) \simeq \pi_n(X, x_0) \times \pi_n(Y, y_0), \ \ n > 1$. It suffices to explain briefly why your proof from (a) works for any n.

(c) Let α, β be two loops in $(X \times Y, (x_0, y_0))$ such that

$$\alpha(s) = (a(s), 0), \qquad \beta(s) = (0, b(s)).$$

It follows from (a) that the classes of α and β commute in $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy between $\alpha * \beta$ and $\beta * \alpha$.

3. Let X be the complement of a finite set of disjoint lines in $\mathbb{R}^4$, $x_0 \in X$. Show that $\pi_1(X, x_0) = 0$.

Optional but recommended: can you prove the same if the lines are not necessarily disjoint?

4. Recall that $\mathbb{R}\mathrm{P}^k$ is the space of lines through the origin in $\mathbb{R}^{k+1}$, with the quotient topology. Thinking of inclusions $\mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots \subset \mathbb{R}^k \subset \mathbb{R}^{k+1}$, we get the standard inclusions

$$\mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2 \subset \cdots \subset \mathbb{R}\mathrm{P}^{k-1} \subset \mathbb{R}\mathrm{P}^k \subset \ldots.$$

(a) Let M be an m-dimensional smooth manifold, $f : M \to \mathbb{R}\mathrm{P}^n$ a continuous map, $m < n$. Show that f is homotopic to a map whose image lies in $\mathbb{R}\mathrm{P}^m \subset \mathbb{R}\mathrm{P}^n$. (Argue by induction, gradually reducing the dimension of the projective space you're mapping into.)

(b) Show that $\pi_1(\mathbb{R}\mathrm{P}^n, x_0) = \pi_1(\mathbb{R}\mathrm{P}^2, x_0)$ for all $n \geq 2$. Assume that $x_0 \in \mathbb{R}\mathrm{P}^2$.

Note for questions 3 and 4: we don't yet have any tools to prove that the fundamental group of a space is non-trivial or that two maps cannot be homotopic. However, approximations by smooth maps (Lee 6.26, 6.29) together with Sard's theorem and transversality give a powerful tool for finding homotopies (compare Homework 5 question 1). Note that for questions such as 3 and 4b, some care is required to keep the basepoint fixed (use the relative version of the approximation theorems, in an appropriate setup).

5. (a) Show that the complement of a small open disk U in $\mathbb{R}\mathrm{P}^2$ is homeomorphic to the Möbius band. It is convenient to think of $\mathbb{R}\mathrm{P}^2$ as the quotient of a 2-disk D^2 with the opposite points on the boundary ∂D identified. Make a convenient choice of a small disk U (you don't have to prove that the resulting complement is independent of this choice).

(b) Recall the the Klein bottle K is the quotient space of the square $[0,1] \times [0,1]$ under the identifications $(x, 0) \sim (x, 1)$, $(y, 0) \sim (1 - y, 1)$. Show that the Klein bottle can be represented as the union of two Möbius bands sharing a common boundary.

Informally, this means that the Klein bottle can be obtained by gluing two Möbius bands along their boundaries. We assume that a particular "nice" identification is used for gluing, and will not check that the result is independent of a choice of this identification.

(c) Show that the Klein bottle is homeomorphic to the connected sum of two projective planes,

$$K = \mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2.$$

(The statement also true smoothly; for this question, you don't have to explain the smooth structures.)