MAT 540, Homework 12, due Thursday, Nov 21

1. Let N, N be two connected closed surfaces, $p : M \to N$ a finite-sheeted covering. Find and prove a relation between the Euler characteristic $\chi(M)$ and $\chi(N)$, in terms of the number of sheets of the covering.

Note: The same relation holds for CW-complexes in general, but we only established the invariance of the Euler characteristic for triangulated surfaces (as a side product of our proof of the classification theorem). For this problem, you can assume without proof that the surface has a triangulation with "sufficiently small" simplices (so that each simplex lies in an evenly covered neighborhood). Making simplices can be achieved by the process of "barycentric subdivision", see e.g. Hatcher. p.120. All triangulations are finite (the surfaces are compact).

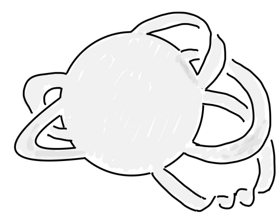
2. (a) Let Σ_g and Σ_h be closed connected orientable surfaces of genus g and h, respectively. For what g, h does there exist a covering $\Sigma_g \to \Sigma_h$? Explain how to construct a covering (a somewhat informal explanation is fine) or show that it doesn't exist.

(b) For an integer $p > 0$, let $L(p, 1)$ be a lens space (see HW 10 Question 3). For what m, n does there exist a submersion $L(n, 1) \to L(m, 1)$? Prove your answer: for each pair m, n, construct a smooth submersion or show that it doesn't exist.

3. (a) Show that any two continuous maps $f : \mathbb{RP}^n \to T^k$ are homotopic if $n > 1$. Here, $T^k = (S^1)^k$ is the k-dimensional torus, $k > 0$, and $\mathbb{RP}^n$ is the n-dimensional real projective space.

(b) Show that any two continuous maps $f : \mathbb{CP}^m \to \mathbb{RP}^n$ are homotopic if $n > 2m > 0$.

4. The figure shows a compact surface with boundary embedded in $\mathbb{R}^3$: this surface can be represented as a disk with bands (rectangles) attached, some of the bands are twisted. Identify this surface as one of the standard surfaces from the classification theorem (a connected sum of tori with holes or a connected sum of projective planes with holes); specify the number of holes and the number of tori or projective planes in the connected sum.



5. Let X be any topological space, and let $\phi_0, \phi_1 : \partial D^n \to X$ be two homotopic continuous maps, where D^n is the n-disk. Show that

$$X \sqcup_{\phi_0} D^n \sim X \sqcup_{\phi_1} D^n,$$

that is, attaching a cell via two homotopic maps produces homotopy equivalent spaces. (This fact justifies some of homotopy equivalences we constructed informally for the fundamental group calculations of specific spaces, like in Question 2d Homework 9. It is most useful in practice when X is a CW complex and the maps ϕ_i are cell attachments.)

Hint: you can find a space Y such that both of your spaces are subsets of Y, and Y deformation retracts on each of them.

6. Find a 3-fold path-connected covering $p : \tilde{X} \to X$ of the space $X = S^1 \vee S^1$ and a loop $\gamma : I \to X$, $\gamma(0) = \gamma(1) = x_0$, such that when you lift the path γ to $\tilde{X}$, you can get a closed loop or an open path, depending on its initial point in $\tilde{X}$. In other words: consider $\tilde{x}_1$, $\tilde{x}_2 \in \tilde{X}$ such that $p(\tilde{x}_1) = p(\tilde{x}_2) = x_0$, and let $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ be two lifts of γ, such that $\tilde{\gamma}_1(0) = \tilde{x}_1$ and $\tilde{\gamma}_2(0) = \tilde{x}_2$. Find an example where $\tilde{\gamma}_1(1) = \tilde{x}_1$, so $\tilde{\gamma}_1$ is a closed loop, but $\tilde{\gamma}_2(1) \neq \tilde{x}_2$ (the path $\tilde{\gamma}_2$ is not a loop).

7. Given an arbitrary group G, show that there is a path-connected CW complex X with $\pi_1(X, x_0) = G$.

Use the fact that every group is a quotient of a free group (possibly on infinitely many generators) by a normal subgroup; it follows that every group can be given by generators and relations. For simplicity, you may assume that there are finitely many generators and finitely many relations; then your CW complex will be finite.