

# 1 Calculation of the complex orientation of a two-fold branched covering space of a surface. 13.7.80

Let  $A$  be a non-singular real algebraic surface, let its set  $A_{\mathbb{R}}$  of real points be zero homologous in its complexification  $A_{\mathbb{C}}$ .

Let  $p : Y_{\mathbb{C}} \rightarrow B_{\mathbb{C}}$  be a two-fold covering space branched over  $A_{\mathbb{C}}$ . Everything is equivariant with respect to the complex conjugation involution. Let  $B_p = p(Y_{\mathbb{R}})$  and let  $[Y_{\mathbb{R}}] \mapsto 0 \in H_2(Y_{\mathbb{C}}; \mathbb{Z}/2)$ .

**Lemma 1.** *Then  $A_{\mathbb{R}}$  divides  $A_{\mathbb{C}}$  (if  $H_3(Y_{\mathbb{C}}; \mathbb{Z}/2) = 0$ ).*

**Proof.** Consider the homomorphism

$$\text{Inv}H_2(Y_{\mathbb{C}}; \mathbb{Z}/2) \rightarrow H_1(A_{\mathbb{C}}; \mathbb{Z}/2) / \text{in}^!H_3(Y_{\mathbb{C}}; \mathbb{Z}/2)$$

It maps  $[Y_{\mathbb{R}}]$  to 0. Therefore if  $[Y_{\mathbb{R}}] = 0$  in  $Y_{\mathbb{C}}$ , then  $[A_{\mathbb{R}}] = 0$  in  $A_{\mathbb{C}}$ .  $\square$

**Comment (April 2026).** In this proof we consider the homomorphism which is defined on the subspace  $\text{Inv}H_2(Y_{\mathbb{C}}; \mathbb{Z}/2) \subset H_2(Y_{\mathbb{C}}; \mathbb{Z}/2)$  of classes invariant under the non-trivial automorphism of covering  $Y_{\mathbb{C}} \rightarrow B_{\mathbb{C}}$ . The cycle  $[Y_{\mathbb{R}}]$  is invariant,  $Y_{\mathbb{R}}$  is the preimage of a half  $B_{\mathbb{R}+}$  of  $B_{\mathbb{R}}$  bounded by  $A_{\mathbb{R}}$ . By assumption,  $Y_{\mathbb{R}}$  realizes  $0 \in H_2(Y_{\mathbb{C}}; \mathbb{Z}/2)$ . Therefore its image in  $H_1(A_{\mathbb{C}}; \mathbb{Z}/2) / \text{in}^!H_3(Y_{\mathbb{C}}; \mathbb{Z}/2) = H_1(A_{\mathbb{C}}; \mathbb{Z}/2)$  also is 0. But it is realized by  $A_{\mathbb{R}}$ .

$Y_{\mathbb{C}}$  and  $Y_{\mathbb{R}}$  can be embedded in some linear bundles over  $B_{\mathbb{C}}$  and  $B_{\mathbb{R}}$  respectively. Namely, let  $\xi : \mathbb{C}M \rightarrow B_{\mathbb{C}}$  be an equivariant linear bundle, whose zeros is  $A_{\mathbb{C}}$ . Since the homology class  $[A_{\mathbb{C}}] \in H_2(B_{\mathbb{C}}; \mathbb{Z}/2)$  is zero,  $[A_{\mathbb{C}}] \in H_2(B_{\mathbb{C}}; \mathbb{Z}/2)$  is divisible by 2. Let the  $\frac{[A_{\mathbb{C}}]}{2}$  be the class  $c \in H_2(B_{\mathbb{C}})$  (it is unique since  $\text{Tors } H_2(B_{\mathbb{C}}) = 0$ ), and let  $\eta : \mathbb{C}N \rightarrow B_{\mathbb{C}}$  be the corresponding linear bundle. Then there exists an isomorphism  $\eta \otimes \eta \rightarrow \xi$ . Consider the composition  $\eta \rightarrow \eta \otimes \eta \rightarrow \xi : z \mapsto z \otimes z$ . Fiberwise, this is squaring. Since  $\text{Tors } H_2(B_{\mathbb{C}}) = 0$ , the class  $c$  is invariant under the complex conjugation, and it can be lifted to  $\mathbb{C}N$ . Hence the map  $\eta \rightarrow \xi$  can be chosen equivariantly.

Though, the following construction is more convenient.  $\mathbb{C}N \rightarrow \mathbb{C}M$  is a two-fold covering branched over  $B_{\mathbb{C}}$  (which is embedded into  $\mathbb{C}M$  as the zero section), and involution  $\text{conj} : \mathbb{C}M \rightarrow \mathbb{C}M$  can be lifted in two ways to  $\mathbb{C}N$ . Let us choose in  $\mathbb{C}M$  a non-zero real (equivariant) section with zeros  $A_{\mathbb{C}}$ . Its

preimage under  $\mathbb{C}N \rightarrow \mathbb{C}M$  is  $Y_{\mathbb{C}}$ . Then  $Y_{\mathbb{R}}$  is the fixed point set of one of the liftings of  $\text{conj}_{B_{\mathbb{C}}}$  on  $Y_{\mathbb{C}} \cap \mathbb{R}N$ .

Let  $d$  a simple arc in  $B_{\mathbb{R}} \setminus \text{Int } B_p$  connecting points of  $\partial B_p$ . Orient  $d$ , some of its neighborhoods in  $\mathbb{R}N$  and some of its neighborhood in  $B_{\mathbb{R}}$ . Then we get local intersection numbers (in its boundary points) with  $A_{\mathbb{R}}$  and with  $\mathbb{C}Y$  equipped with their complex orientations.

*The complex orientations of  $A_{\mathbb{R}}$  and  $\mathbb{R}Y$  can be chosen so that the intersection numbers of  $A_{\mathbb{R}}$  with  $d$  and of  $\mathbb{R}Y$  with  $d$  were equal.*

**Proof.** Denote the initial and final points of  $d$  by  $d_0$  and  $d_1$  and connect these points by a simple arc  $l$  in one of the connected components of  $A_{\mathbb{C}} \setminus A_{\mathbb{R}}$ . Say, in  $A_{\mathbb{C}+}$ . Construct a disk  $\Delta$  in  $B_{\mathbb{C}} \setminus A_{\mathbb{C}}$  with boundary  $d \cup l$ . Orient  $\Delta$  according to the orientation of  $d$ . We may assume that  $\Delta$  is transverse to  $A_{\mathbb{C}+}$ .

The union  $\Delta \cup \text{conj } \Delta$  is a disk membrane on  $A_{\mathbb{C}}$  (possibly with self-intersections). Lift it to  $Y_{\mathbb{C}}$ . Denote the preimage by  $\Sigma$ . Orient  $\Sigma$ . Under the projection  $Y_{\mathbb{C}} \rightarrow Y_{\mathbb{C}} \text{ conj}$ , the sphere  $\Sigma$  is two-fold branched covering