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Let X be a cw-complex with involution $c : X \to X$,
which is **cellular** in the sense that $c(cell) = cell$.

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which is **cellular** in the sense that $c(cell) = cell$.

Let $F = fix(c)$ and $X' = X/_c$;
and assume that each open cell is either contained in F or does not meet F.

Let $C_*(X)$ denote chain complex of cellular chains with coefficients in $\mathbb{Z}/_2$.

Extra differentials

Denote by θ the map $1 + c_* : C_*(X) \to C_*(X)$.

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The homology sequence over ∂ is called the **Smith sequence**.

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The differential d does not respect the grading by dimension, therefore the homology of $(C_*(X), d)$ is not graded,

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$H(C_*(X), d)$ is isomorphic $H_*(F)$.

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Exercise. Explicitly describe homomorphisms of this sequence.

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Definition. Let V be a vector space over $\mathbb{Z}/_2$, and $q : V \times V \to \mathbb{Z}/_2$ be a bilinear form. A vector $\gamma \in V$ is said to be **characteristic vector** of q, if $q(\alpha, \alpha) = q(\alpha, \gamma)$.