Lecture 1

Predicates

Our goals

The main goal is to prepare ourselves for **advanced** mathematical courses.

Practical goals:

- Learn how to **read**, **understand**, and **write** mathematics.
- For this, first of all, learn the **basic language** needed in mathematics: **logic**.
- **Navigate** through any new mathematical environment.
  i.e. understand the **structure** of a mathematical subject
  (axioms, definitions, theorems, proofs).
- Learn the **second** basic math language: **set theory**
  (including everything about maps).
- Learn elements of the **third** basic math language: **category theory**
- Practice in **expressing** yourselves mathematically
  learn how to **speak mathematics,**
  how to **solve problems** and **write down solutions**,
- Learn how to **cope with difficulties** in math studies.

First steps in logic

In the section of Logic, we are going to study the following topics:

- propositions, predicates and statements
- logical connectives
- quantifiers
- structure of a theory
- basic methods of proofs

Propositions

What is a definition?

Definition. A **proposition** is a sentence that is either true or false.

A **definition** is an **agreement** about the meaning of words to be used.

In the definition of **proposition** above,

the meaning of the word "proposition" is given.

In our course, the word "proposition" will be used in **exactly** this meaning.

Let us read the **definition** of a **proposition** again.

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What is a proposition?

Definition. A **proposition** is a sentence that is either true or false.

What is a sentence?

A sentence is a grammatical unit, containing at least a subject and predicate. **Examples.**

1) $1+1=2$ Is this a proposition? Yes, this is a complete sentence, which is true.

2) $1+1=3$ Is this a proposition? Yes, this is a complete sentence, which is false.

3) $1+1$ Is this a proposition? No, this is **not** a complete sentence.

There is no predicate. Nothing is claimed.

4) $1+1$ is fun! Is this a proposition? No, this is a complete sentence,

but it has **no truth value** (neither true nor false)

until we would agree what 'fun' means.

5) This sentence is false. Is this a proposition? Hmm... This is a sentence.

Is it true? No! Is it false then? No! So it is **not** a proposition.

This is so called **paradox**, a self-contradictory statement, which is
neither true nor false. We are not going to study paradoxes in our course.

6) $x>1$ Is it a proposition? It is a sentence, whose truth value depends on x.
It is so-called **open sentence** or **predicate**.

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Predicates and statements

A **predicate** is a proposition **with parameter(s)**.

It is a complete sentence that includes words or symbols,
which may take **different values**.

These words or symbols are called **variables.**

For each combination of their values, the sentence **must** be either true or false.

Values of variables are taken from some **domains**,
which are either **explicitly described** or **silently assumed**.

The variables are said to be **free**
as long as **no** restrictions on their values are assumed.
For example, in the statement $x^2 \geq 0$ the variable x
may be silently assumed to be a real number,
or may be explicitly required to be a root of equation $x^2 - 3x + 2 = 0$.

The word **statement** is used to denote **either a proposition or a predicate**.
A statement without a free variable is a proposition,
a statement with a free variable(s) is a predicate.

Boolean functions

Each proposition is either **true** or **false**.
We call true and false **truth values** or **Boolean values**
and denote by symbols T or F.
A variable which may take only Boolean values, is called a **Boolean variable.**
A function on Boolean variables which takes Boolean values
is called a **Boolean function**.
So, a Boolean function is a function in n Boolean variables
which takes two Boolean values: $\underbrace{\{\mathsf{F},\mathsf{T}\} \times \cdots \times \{\mathsf{F},\mathsf{T}\}}_{n \text{ times}} \to \{\mathsf{F},\mathsf{T}\}$.

The main purpose is to **build** new **statements**
as **compositions** of old statements **with a Boolean function**.

Connectives and propositional forms

Logical connectives (or simply **connectives**) are

the simplest Boolean functions.

There are 5 commonly used connectives:

$\neg$	$\wedge$	$\vee$	$\implies$	$\iff$
negation	conjunction	disjunction	implication	equivalence

New statements, constructed from statements P and Q by the connectives, look as follows:

$\neg P$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
not P	P and/but Q	P or Q	P implies Q	P is equivalent to Q

Definition. An **expression** formed of **Boolean variables** and **connectives** is called **a propositional form.**

For example, $(P \implies Q) \wedge P$, $\neg P \vee Q$, $(P \wedge \neg Q) \vee (\neg P \wedge Q)$

are propositional forms in variables P and Q.

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Truth table for negation

A propositional form is described by the **truth table**.

Let P be a proposition. How to define $\neg P$?

$\neg P$ is a proposition which truth value is determined by the truth value of P as follows:

P	$\neg P$
T	F
F	T

A negation is a function $\{\mathsf{T},\mathsf{F}\} \to \{\mathsf{T},\mathsf{F}\}$ defined by $\mathsf{T} \mapsto \mathsf{F}$, $\mathsf{F} \mapsto \mathsf{T}$,

or by formulas $\neg(\mathsf{T}) = \mathsf{F}$ and $\neg(\mathsf{F}) = \mathsf{T}$,

or by $\neg\mathsf{T} = \mathsf{F}$ and $\neg\mathsf{F} = \mathsf{T}$.

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Truth table for connectives

Let P and Q be propositions.

The **truth values** of conjunction, disjunction, implication and equivalence are defined by the following **truth table**:

P	Q	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Keep in mind that this table is **the definition** of conjunction, disjunction, implication and equivalence.

Conjunction

The logical connective of conjunction has many **incarnations** in **human languages**.

in colloquial **English** $P \wedge Q$ can be expressed as follows:

P	and	Q
P	as well as	Q
P	also	Q
P	along with	Q
P	together with	Q
P	but	Q
P	though	Q
P	although	Q
P	moreover	Q
P	nevertheless	Q

These words express emotional **nuances** in our anticipation of the relation between P and Q.

Even in a mathematical context, it may be **uneasy to recognize** a conjunction under these words.

Disjunction

Colloquial equivalents of logical connection of disjunction $\vee$ are **or** and **either**:
$P \vee Q$ is pronounced as "P or Q" or "P either Q".

Keep in mind that the disjunction is **not** exclusive,
that is it takes true value in the case when one of P, Q is true
and also in the case when P, Q are both true.

Warning. The colloquial construction "either...or..." is **not** a disjunction.

"Either P or Q" is called an **exclusive disjunction**,
it means P and not Q or Q and not P.

In a formula, "either P or Q" is written as $(P \wedge \neg Q) \vee (Q \wedge \neg P)$.

Exercise. Construct a truth table for the propositional form "either P or Q" and convince yourself that "either P or Q" has true value only if exactly one of P, Q is true.

Priority of connectives

Connectives, like arithmetic operations, differ by their **priorities**:

$$\neg \quad \wedge \quad \vee \quad \Longrightarrow \quad \Longleftrightarrow$$

high ⟷ low

Example 1. Let P, Q, R, S be propositions.
What is the order of the logical operations in the formula
$P \wedge \neg Q \implies R \vee S$?

$$\boxed{\boxed{P \wedge \boxed{\neg Q}} \Longrightarrow \boxed{R \vee S}}$$

You may use parentheses to prevent misunderstanding of the formula:

$$(P \wedge (\neg Q)) \implies (R \vee S)$$

Parentheses do matter!

$P \wedge (\neg (Q \implies R) \vee S)$ is a **non-equivalent** proposition.

Priority of connectives

Example 2. Determine the truth value of the following:

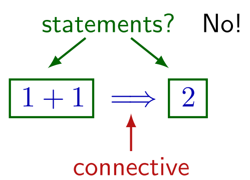
$$\neg(2 < 1) \implies 1 = 2 \vee 1 + 1 = 2$$

Solution.

$\boxed{\neg \boxed{(2<1)}} \implies \boxed{\boxed{1=2} \vee \boxed{1+1=2}}$ is ⓣ

$(2<1)$: F; $\neg(2<1)$: T; $1=2$: F; $1+1=2$: T; $1=2 \vee 1+1=2$: T

Warning. Is the following correct: $1 + 1 \implies 2$? No! Why?



- Use appropriate mathematical symbols: $1 + 1 = 2$
- A logical **connective** may **not** connect **numbers**!

Examples of compound propositions

Example 1.

$\boxed{\pi = 3.14} \vee \boxed{e < 3}$ ⓣ or F?

$\pi = 3.14$: F; $e < 3$: T

Remark. Logical disjunction is **not** exclusive. Compare:

3 is prime **or** 5 is prime. ⓣ or F? **Both** 3 and 5 are prime.

I'll merry Jane **or** I'll merry Joan. But **not both**! **Or** is exclusive here.

Control question:
Is the sentence "I'll merry Jane **or** I'll merry Joan" a proposition? No. Why?

Example 2.

$\boxed{3 \text{ is even}} \implies \boxed{3 \text{ is odd}}$ ⓣ or F?

3 is even: F; 3 is odd: T

Equivalent propositional forms

Definition. Two propositional forms are said to be **equivalent** if they take **the same truth values** for all possible values of variables.

Propositions P and Q are equivalent, iff the proposition $P \iff Q$ is true.
Notation. Therefore we write $P \iff Q$ also for saying that propositions P and Q are equivalent.

Example 1. $\neg(\neg P)$ is equivalent to P for any proposition P, that is, $\neg(\neg P) \iff P$.

Proof.

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

the same

Therefore, $\neg(\neg P) \iff P$. □

Tautology and contradiction

Definition. A **tautology** is a propositional form which is **always true.**
A **contradiction** is a propositional form which is **always false.**

Here the word **always** means **for all values of the arguments**.

Examples. $P \vee (\neg P)$ is a tautology (**Law of excluded middle**)
$P \wedge (\neg P)$ is a contradiction (**Law of consistency**)

Indeed,

P	$\neg P$	$P \vee (\neg P)$	$P \wedge (\neg P)$
T	F	T	F
F	T	T	F
		tautology	contradiction

Algebraic relations

Conjunction and **disjunction** have **properties**
similar to properties of **addition** and **multiplication**.

Like addition and multiplication, each of them is **commutative** and **associative**:

For any statements P, Q and R

$$P \wedge Q \iff Q \wedge P, \quad (P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$$
$$P \vee Q \iff Q \vee P, \quad (P \vee Q) \vee R \iff P \vee (Q \vee R)$$

Besides, there are **two** (!) **distributivities**:

$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$$

$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$$

Each of these equivalences is easy to prove. Say, by an appropriate truth table.
Associativity allows to define **multiple** conjunction and disjunction
$P \wedge Q \wedge R$ and $P \vee Q \vee R$.

When are they true?

Denials

Definition. Let P be a statement.
A **denial** of P is any statement which is equivalent to $\neg P$.

Example. Construct a denial for the proposition "All integers are even."

Note that the proposition is **false**, therefore it's denial should be **true**.

Here are several possible **denials**:

It is not true that all integers are even.

Not all integers are even.

Some integers are not even.

There are integers which are not even.

Some integers are odd.

There are integers which are odd.

De Morgans' laws

A key to finding a useful denial is the following theorem:

Theorem (De Morgans' laws). Let P and Q be propositions. Then

$$\neg(P \wedge Q) \iff \neg P \vee \neg Q,$$
$$\neg(P \vee Q) \iff \neg P \wedge \neg Q.$$

Proof is by truth table. Let us prove the first identity:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

the same

The second identity is proven similarly, or deduced from the first one:

$\neg(P \vee Q) \iff \neg(\neg\neg P \vee \neg\neg Q) \iff \neg(\neg\neg P \vee \neg\neg Q) \iff \neg\neg(\neg P \wedge \neg Q) \iff \neg P \wedge \neg Q$ $\square$

Conjunction, disjunction and negation

Theorem. Any propositional form is equivalent to a propositional form containing only $\neg$, $\wedge$, $\vee$ as connectives.

A required form can be easily constructed after analyzing the following.

Example. Let P and Q be propositions. Let R be a propositional form taking the following truth values depending on P and Q:

P	Q	R
T	T	F
T **T**	F **F**	T **T**
F **F**	T **T**	T **T**
F	F	F

$P \wedge \neg Q$	$\neg P \wedge Q$
F	F
T	F
F	**T**
F	F

S
F
T
T
F

Construct a propositional form in terms of P, Q and $\neg$, $\wedge$, $\vee$ equivalent to R.

Solution. Choose all rows yielding **T**.

They are **T** **F** **T**, **F** **T** **T**.

Construct the corresponding conjunctions:

$P \wedge \neg Q$, $\neg P \wedge Q$. What are the truth values of them?

Construct disjunction out of all these conjunctions:

$S = (P \wedge \neg Q) \vee (\neg P \wedge Q)$. What are the truth values of it? Done!

Construction of a disjunctive normal form

The construction on the preceding slide works
for any Boolean function, which is not a contradiction.

The result is a propositional form
It is called the **disjunctive normal form** of the function.

For a Boolean function $B(P_1, \ldots, P_n)$, it is a **disjunction** of some number
of conjunctions of variables P_i or their negations $\neg P_i$.
Each of the conjunction involves each of the variables (with or without negation).
The disjunctive normal form is **unique up to permutations** of the conjunctions. **Control questions:**

- When the construction doesn't work?
- What to do if it doesn't work?
- What is a **conjunctive normal form** of a Boolean function?
- When a conjunctive normal form doesn't exist?
- What is the denial of a conjunctive normal form?

What are implications for?

So far we dealt with only three connectives: $\neg$, $\wedge$ and $\vee$. They did **a great job!**

Any non-trivial Boolean function have been **canonically** presented as
two propositional forms involving only these three connectives.

Do we need any other connectives? Do we need $\Longrightarrow$ and $\Longleftrightarrow$?

If yes, then what is their **purpose**?

In order to answer, we need to study b $\Longrightarrow$ and $\Longleftrightarrow$.

Implication as a disjunction

Theorem. Let P and Q be propositions.
Then $P \implies Q$ is equivalent to $\neg P \vee Q$.

In formula: $(P \implies Q) \iff (\neg P \vee Q)$.

I.e., the normal **conjunctive** form for $P \implies Q$ is $\neg P \vee Q$.
Control question. Where is a **conjunction** in this normal **conjuctive** form?
Proof. Notice that $P \implies Q$ is false if and only if P is true and Q is false,
and $\neg P \vee Q$ is false also if and only if P is true and Q is false. □
This can be seen on the **truth tables:**

P	Q	$P \implies Q$	P	$\neg P$	Q	$\neg P \vee Q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	F	T	F	T

the same

Negation of an implication

Theorem. Let P and Q be propositions. Then
$\neg(P \implies Q) \iff P \wedge \neg Q$

Proof. Since $P \implies Q$ is equivalent to $\neg P \vee Q$,
then $\neg(P \implies Q)$ is equivalent to $\neg(\neg P \vee Q)$,
which is equivalent (by De Morgan) to $\neg\neg P \wedge \neg Q$,
which is equivalent to $P \wedge \neg Q$, as required. □

Thus, the **disjunctive normal form** of $\neg(P \implies Q)$ is $P \wedge \neg Q$.

Control question. Where is a **disjunction** in this **dijunctive** normal form?

Since the disjunctive normal form consists of a single conjunction,
$\neg(P \implies Q)$ is **true** for **only one** combination of truth values:
P is true and Q is false.
Thus $P \implies Q$ is **false** for only this one combination of truth values:
when P is true while Q is false.
When P is false, then $P \implies Q$ is true. Counter-intuitive?

Modus ponence

$P \wedge (P \implies Q)$ is equivalent to $P \wedge Q$. **Proof.**

$P \wedge (P \implies Q) \iff P \wedge (\neg P \vee Q) \iff (P \wedge \neg P) \vee (P \wedge Q) \iff T \vee (P \wedge Q) \iff P \wedge Q$. □

Context: building a theory

Logic is used in the context of a theory.
Some statements in a theory are accepted to be true (and called **axioms**).
Some statements have been deduced from axioms and have become **theorems**.

What does it mean, to prove a proposition P? Say, let $A_1, \ldots, A_n$ be the list of axioms.
To prove a proposition P means

to prove the proposition $(A_1 \wedge \cdots \wedge A_n) \implies P$.

As soon as one proves P, it can be added to the set of axioms.
By Modus Ponence, $A \wedge (A \implies P)$ is equivalent to $A \wedge P$.

Conditional sentences

Definition. Let P and Q be statements.
The implication $P \implies Q$ is called a *conditional sentence*.

Notations: $P \implies Q$ $\quad Q \impliedby P$ $\quad \begin{matrix} Q \\ \Uparrow \\ P \end{matrix}$ $\quad \begin{matrix} P \\ \Downarrow \\ Q \end{matrix}$

Terminology. $P \implies Q$

P	Q
assumption	conclusion
antecedent	consequent
sufficient condition	necessary condition

The conditional statement $P \implies Q$ admits various **readings**:

P implies Q

If P, then Q — Q, when (whenever) P

P only if Q — Q if P

P is sufficient for Q — Q is necessary for P

Examples

Example 1. Read the sentence

$$\underbrace{\boxed{x > 5}}_{P} \implies \underbrace{\boxed{x > 3}}_{Q}$$

$x > 5$ implies $x > 3$.

If $x > 5$, then $x > 3$.

$x > 3$, when (whenever) $x > 5$.

$x > 5$ only if $x > 3$.

$x > 3$ if $x > 5$.

$x > 5$ is sufficient for $x > 3$.

$x > 3$ is necessary for $x > 5$.

Examples

Example 2. True of false:

$$\underset{F}{\boxed{1>2}}\ \overset{\Leftarrow}{\text{if}}\ \underset{T}{\boxed{2>1}} \quad \text{is F}$$

Example 3. True of false:

$$\underset{F}{\boxed{1>2}}\ \overset{\Rightarrow}{\text{only if}}\ \underset{T}{\boxed{2>1}} \quad \text{is T}$$

Example 4. Let x be a real number. Let P be the predicate "$x = 1$", and Q be the predicate "$x^2 - 3x + 2 = 0$".

Determine the truth values of the following statements:

P if Q — P is sufficient for Q

P only if Q — P is necessary for Q

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Truth value of a predicate

The statements like $x = 1 \Leftarrow x^2 - 3x + 2 = 0$ look like a **predicate**
and a priori take different truth values for different values of x.

How can one speak about a **single** truth value?

However,
it may happen that **for all** the values of the variables the sentence holds true.

Then we say that the statement is true.

Otherwise we say that the statement is false.

If we speaks about truth value of a predicate
without mentioning values of variables,
it means that we are speaking truth value for all values of the variables.

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Examples

Example 4 (cont.) How are P ($x = 1$) and Q ($x^2 - 3x + 2 = 0$) related?

$x^2 - 3x + 2 = 0 \iff (x-1)(x-2) = 0 \iff x = 1 \vee x = 2$.

Therefore, Q is equivalent to the predicate $x = 1 \vee x = 2$.

Now we are ready to analyze the given statements:

P if Q means
$x = 1 \Longleftarrow x = 1 \vee x = 2$, which is false.

P only if Q means
$x = 1 \Longrightarrow x = 1 \vee x = 2$, which is true.

P is sufficient for Q means
$x = 1 \Longrightarrow x = 1 \vee x = 2$, which is true.

P is necessary for Q means
$x = 1 \Longleftarrow x = 1 \vee x = 2$, which is false.

Causation VS implication

The structure "**If** ... **then** ..." appears both in **language** and **logic**.

In **language**, "**If** ... **then** ..." means **causation**.

In **logic**, "**If** ... **then** ..." means **implication**.

Compare:

If it rains, **then** I take an umbrella. **No** truth value

cause (comes first) — effect (comes after)

If $e < 3$, **then** $\pi > 3$. It's a **true** statement.

assumption (not a cause) — conclusion (not an effect)

In mathematics, there is **no** causation and there are **no** tenses (past, present, future).

Hidden implications

Often an implication is **hidden**.

Example 1. Vertical angles are congruent.

If two angles are vertical, **then** they are congruent.

Example 2. One can circumscribe a circle around a regular polygon.

If a polygon is regular, **then** one can circumscribe a circle around it.

Example 3. A differentiable function is continuous.

If a function is differentiable, **then** it is continuous.

Converse, contrapositive, inverse

Let P and Q be statements. Consider the statement $P \implies Q$. Then

$Q \implies P$ is called the *converse* statement,

$\neg Q \implies \neg P$ is called the *contrapositive* statement,

$\neg P \implies \neg Q$ is called the *inverse* statement.

Theorem. A statement and its contrapositive are equivalent.

The converse and inverse of a statement are equivalent.

Proof is either by truth table or using already proven logical identities:

$P \implies Q \iff \neg P \vee Q \iff Q \vee \neg P \iff \neg Q \implies \neg P$.

So the implication $P \implies Q$ and its contrapositive $\neg Q \implies \neg P$ are equivalent for any propositions P and Q.

Therefore, we get also that $Q \implies P \iff \neg P \implies \neg Q$, that is the converse is equivalent to the inverse.