Lecture 9

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For objects A, B of a category C, the set of all morphisms $A \to B$ of the category C will be denoted by $\mathrm{mor}_C(A, B)$.

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Main Example. The category of sets.

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The notion of isomorphism sets up a problem to classify objects in the category.

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However,
cardinalities of infinite sets and the **numbers of elements** of finite sets have quite different properties.

Hilbert hotel ”Infinity”

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Hilbert Hotel revisited.

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Any reasonable general construction converting a set to other set can be upgraded to a functor.

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Notation: $\mathcal{P}(X)$

By definition, $\mathcal{P}(X) = \{A \mid A \subset X\}$

Control question. Can $\mathcal{P}(X) = \varnothing$?

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$X \mapsto \mathcal{P}(X), (f : X \to Y) \mapsto (f_* : \mathcal{P}(X) \to \mathcal{P}(Y))$ is a functor.

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Corollary. $|\mathcal{P}(X)| = |\mathcal{M}ap(X, \{0,1\})| = 2^{|X|}$.

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Example 1. Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$.

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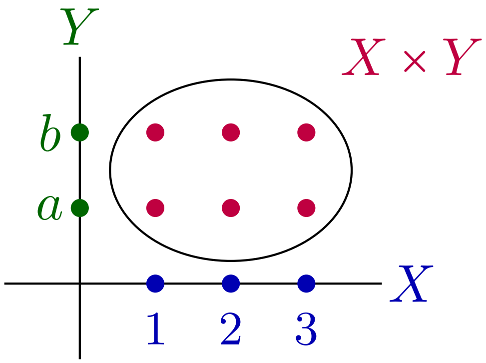

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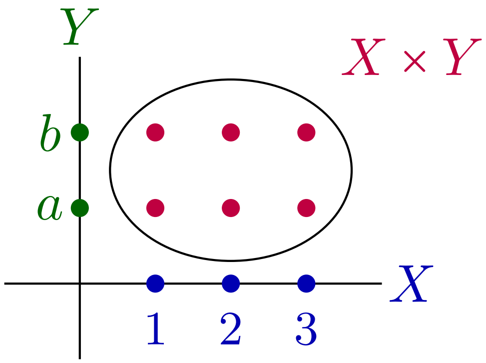

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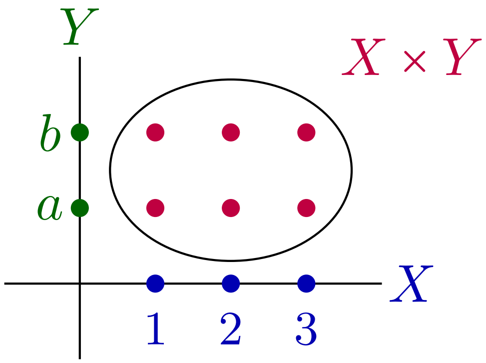

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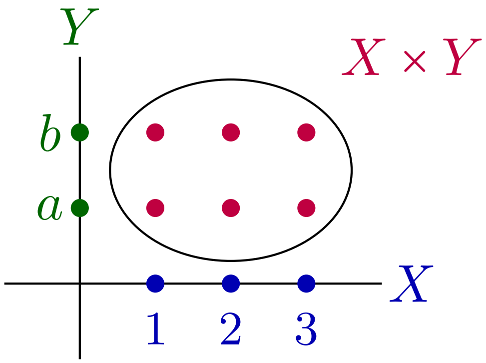

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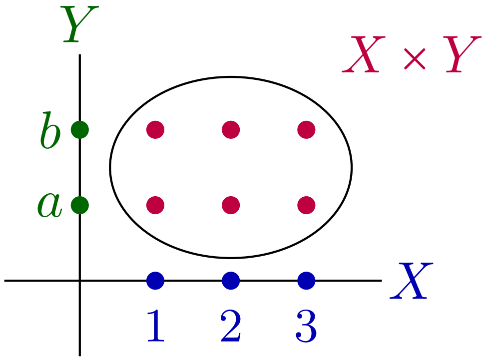

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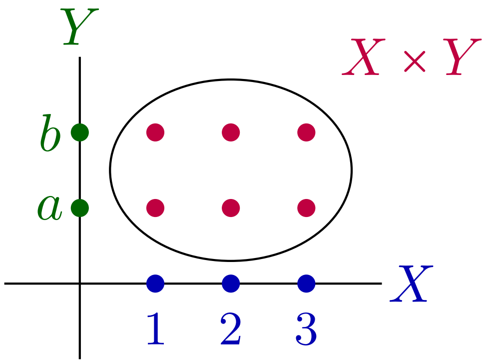

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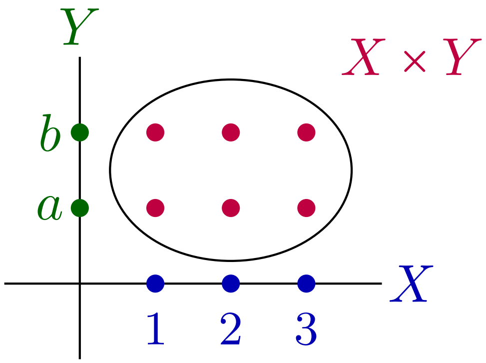

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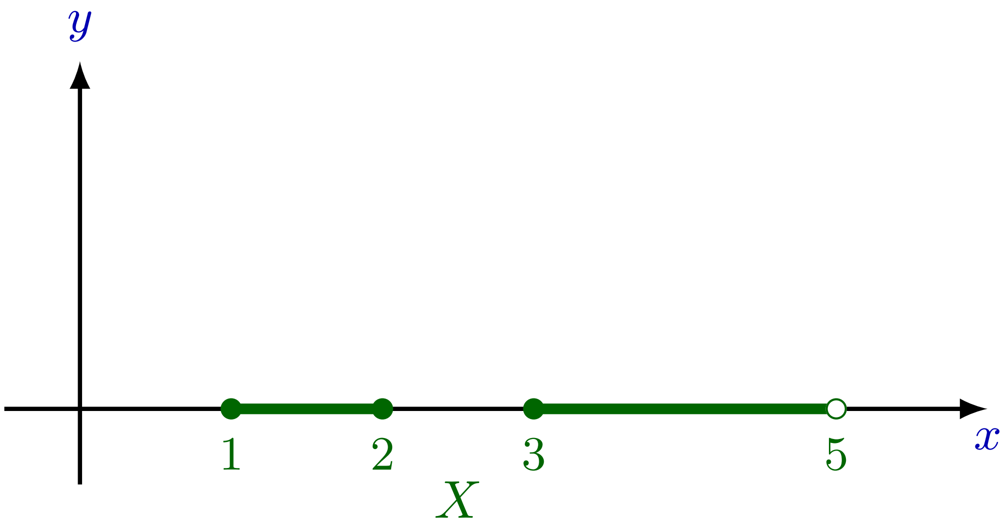

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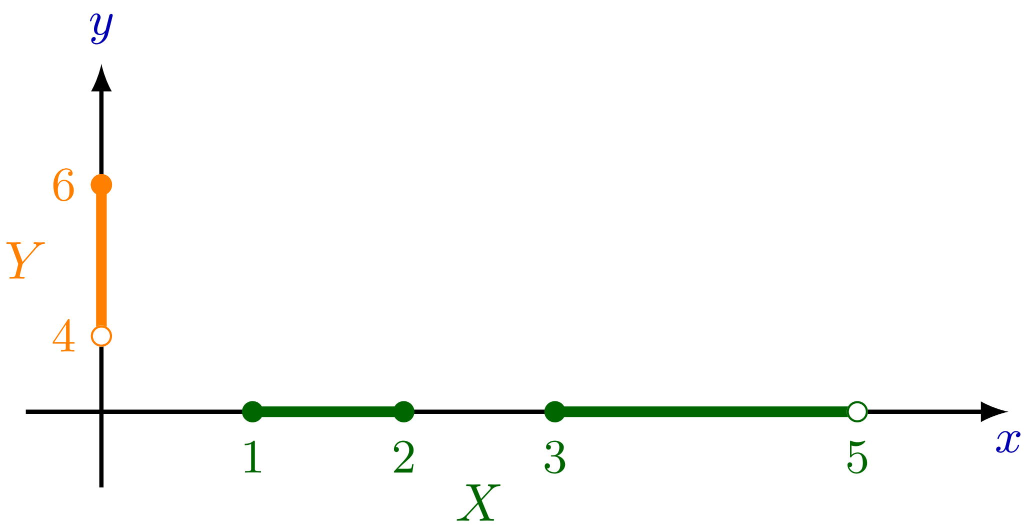

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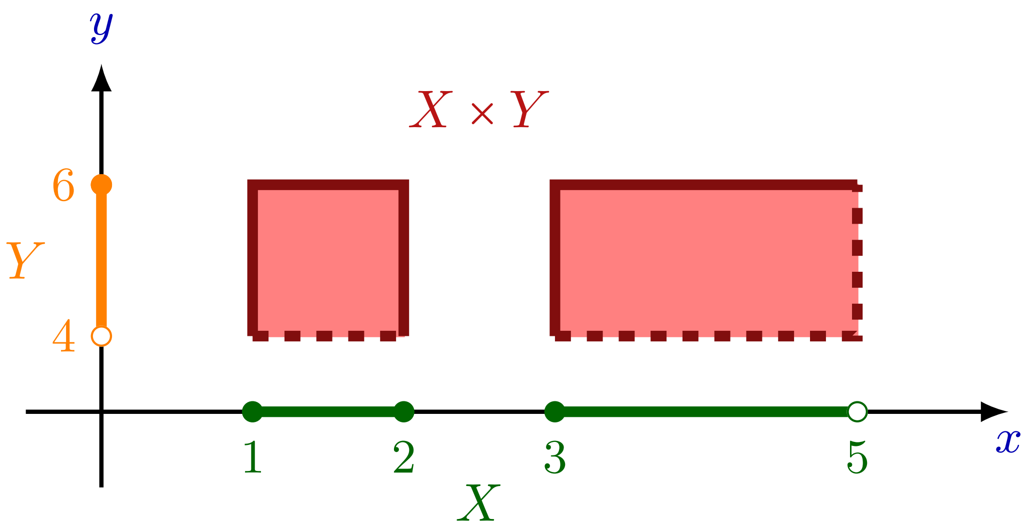

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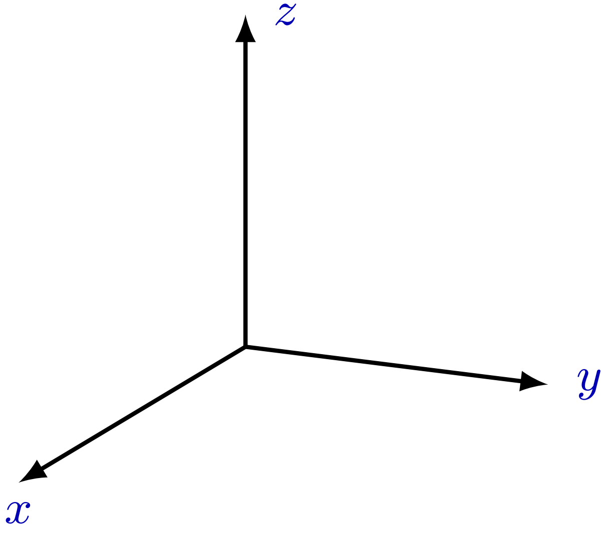

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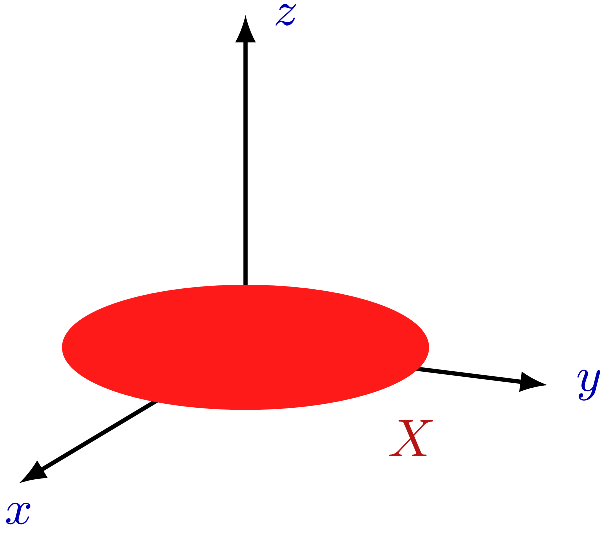

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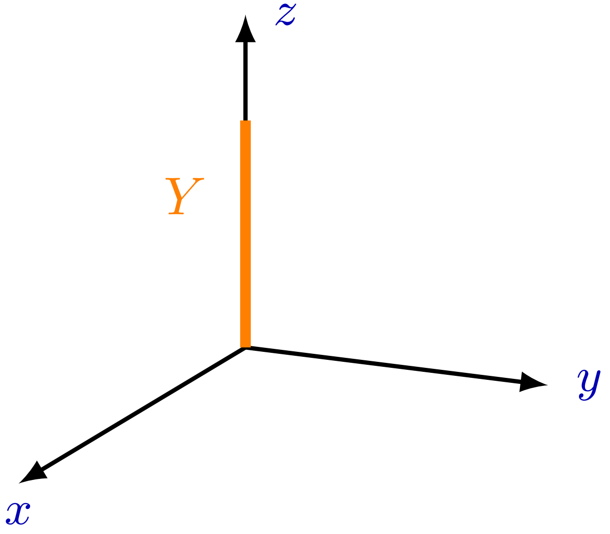

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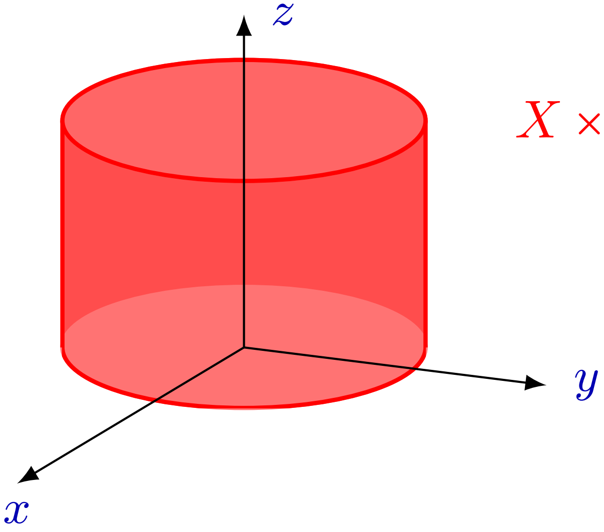

$X \times Y$ is a solid cylinder in $\mathbb{R}^3$

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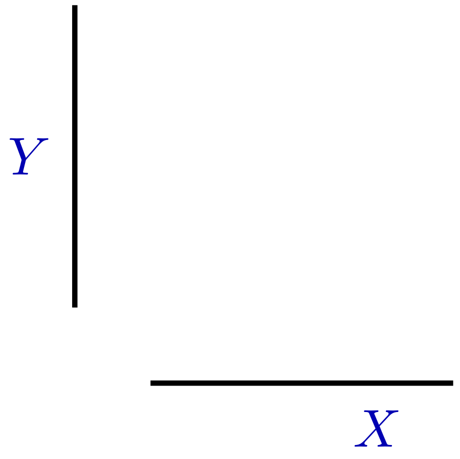

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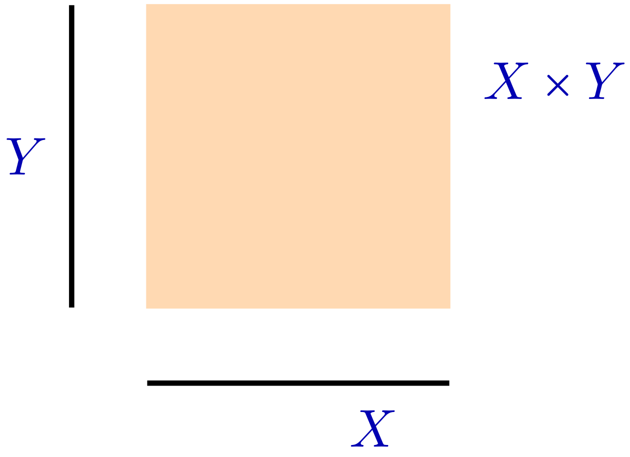

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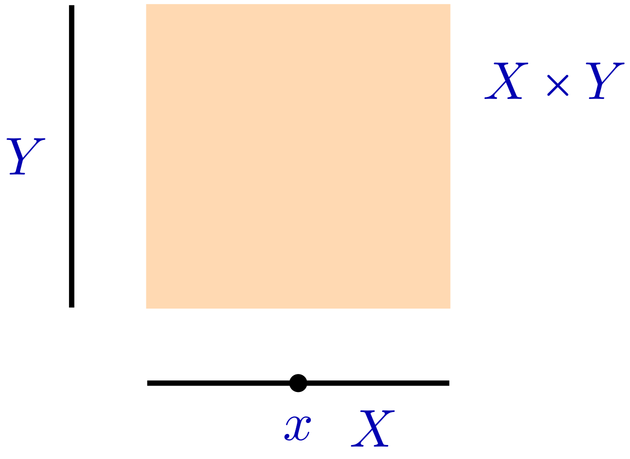

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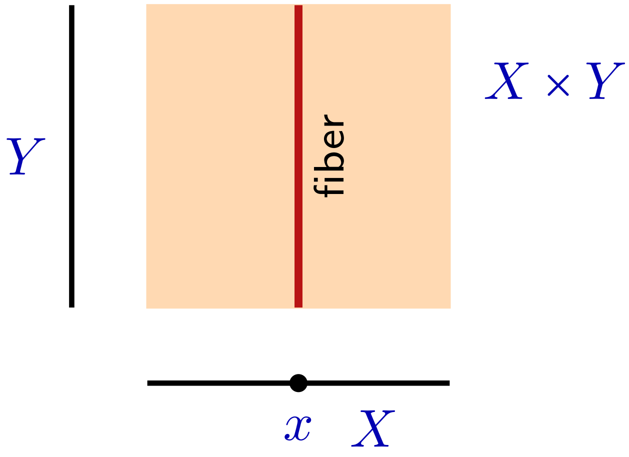

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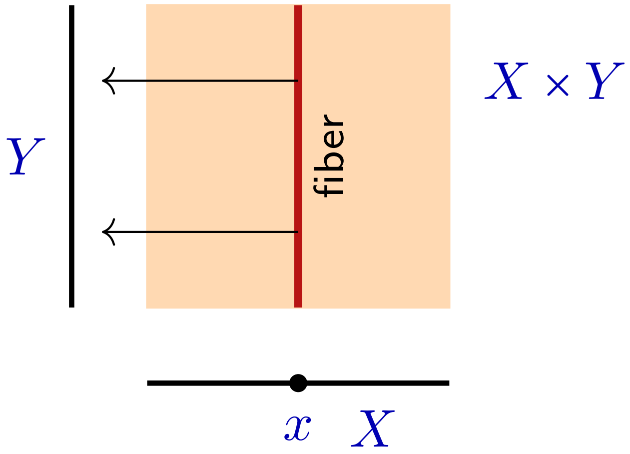

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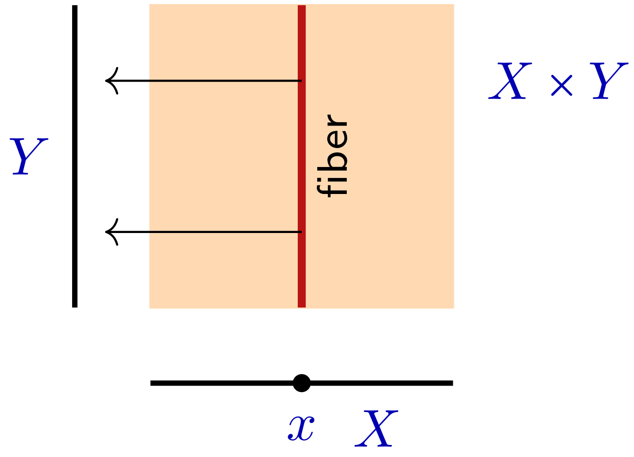

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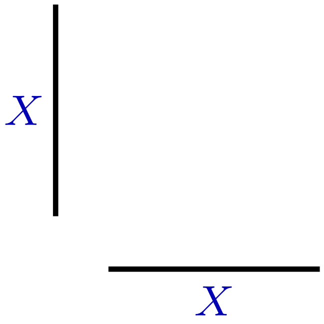

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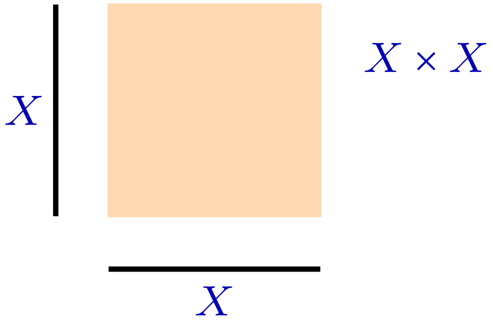

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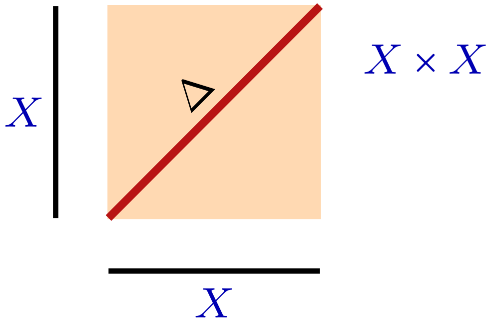

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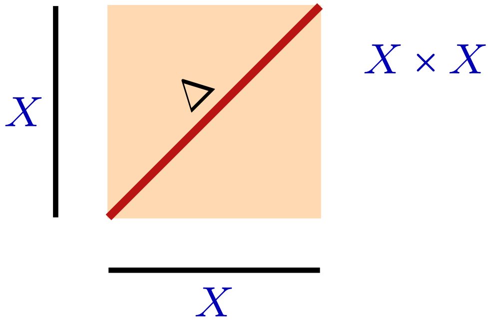

The diagonal is the image of $\mathrm{id}_X \odot \mathrm{id}_X$.

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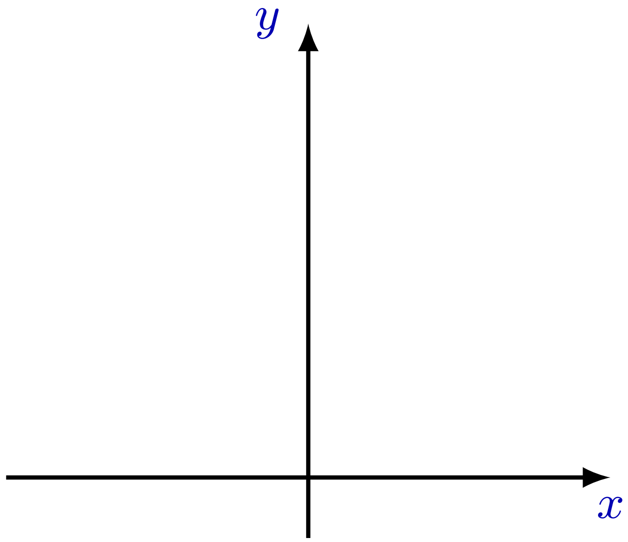

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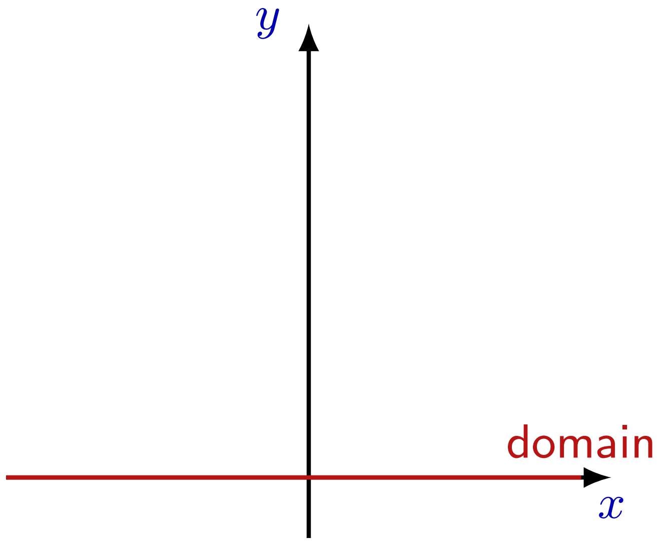

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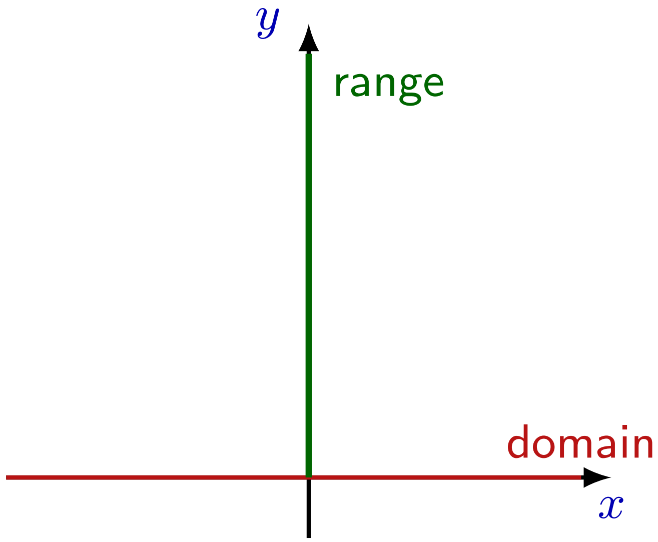

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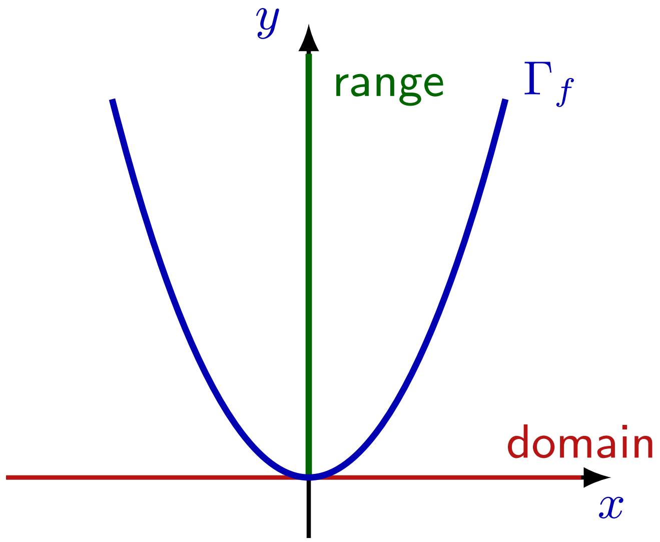

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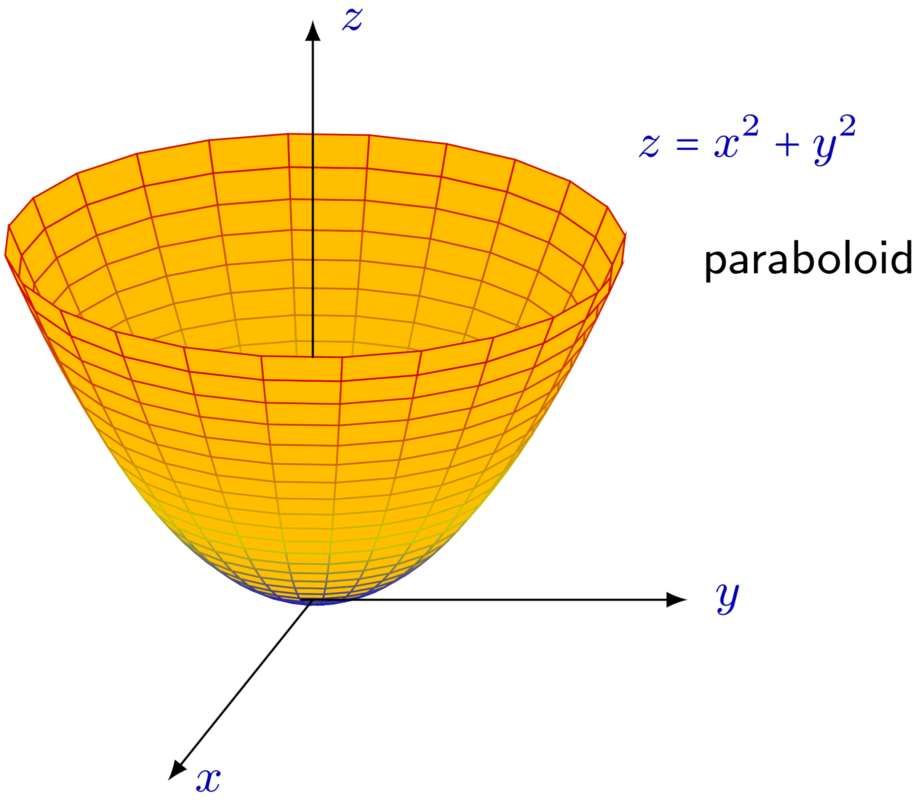

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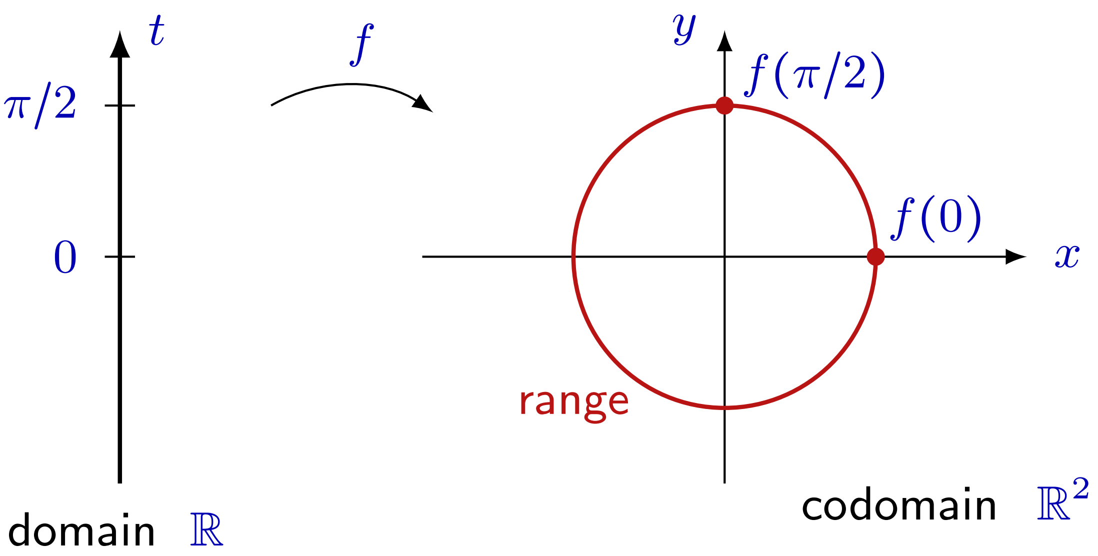

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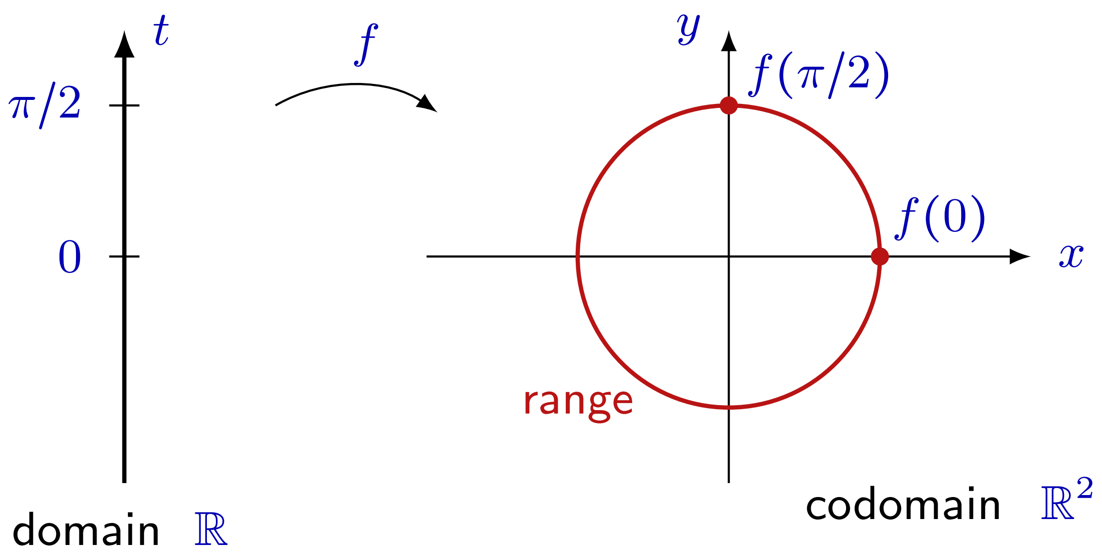

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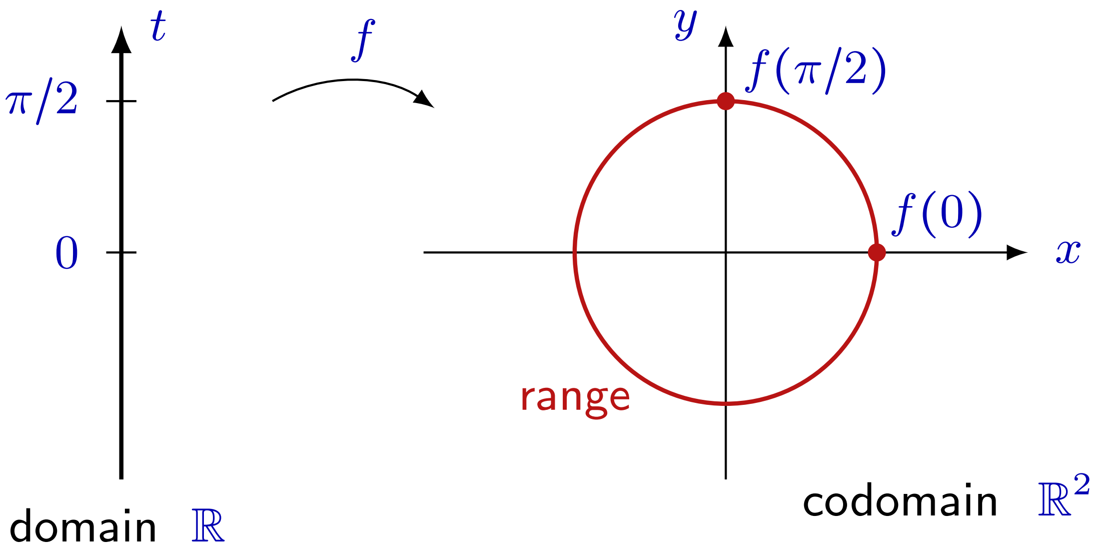

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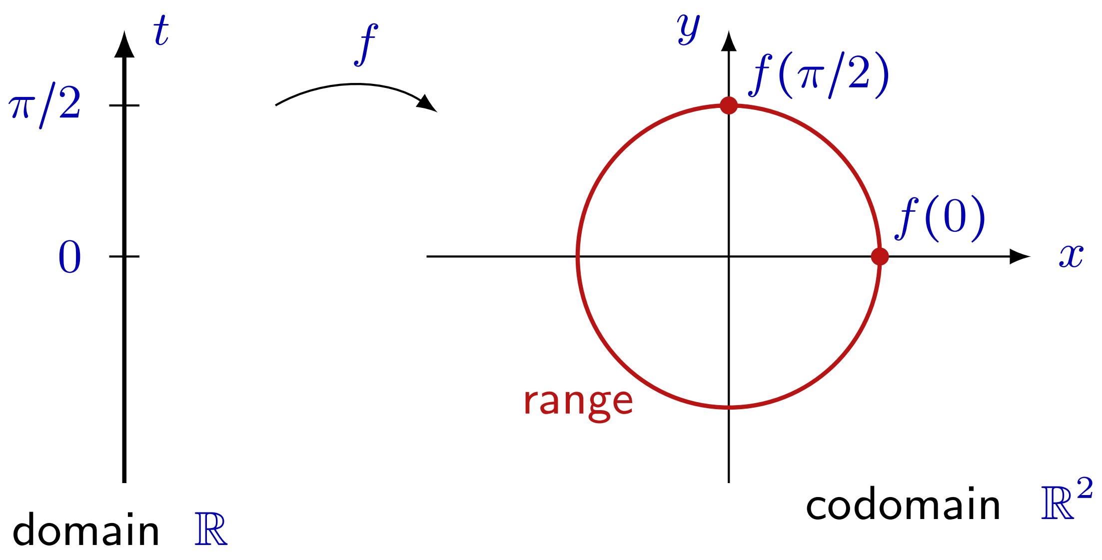

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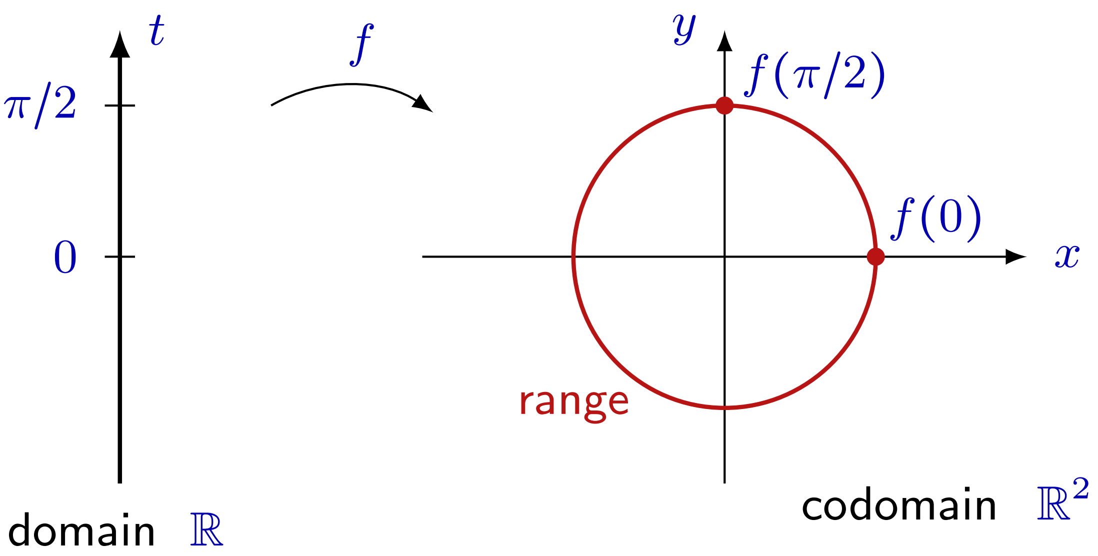

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Γ_f is a curve in $\mathbb{R}^3$. It is called **helix**.

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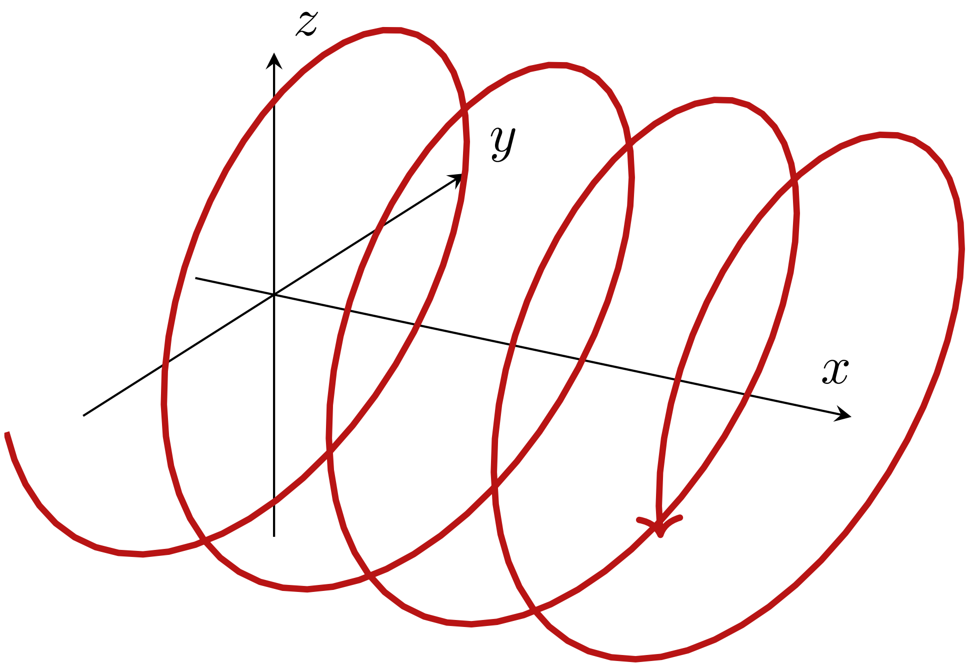

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We will deal mostly with **binary** relations on a **single** set.

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$\mathcal{P}(X \times X)$ is a huge set!

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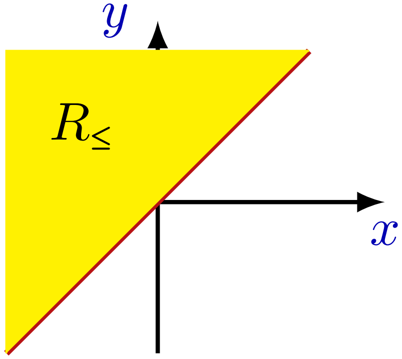

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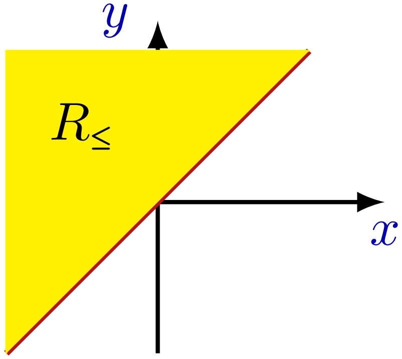
The relation $R_\leq$ is a subset of the plane: $R_\leq = \{(x,y) \in \mathbb{R}^2 \mid x \leq y\} \subset \mathbb{R}^2$,

so we may draw the **graph** of $R_\leq$.



$$\forall x, y \in \mathbb{R} \quad \underbrace{(x,y) \in R_\leq}_{x \leq y} \text{ or } \underbrace{(y,x) \in R_\leq}_{y \leq x} .$$

Relation of inclusion

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Example 2.

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Inclusion $\subset$

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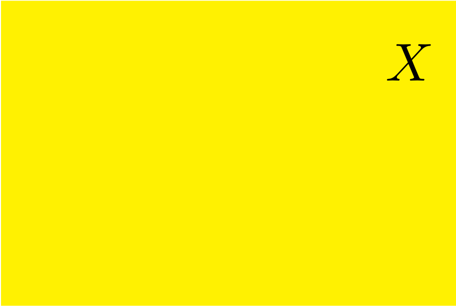

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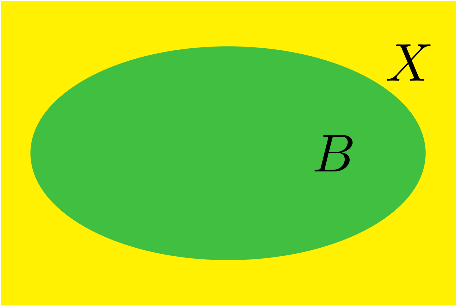

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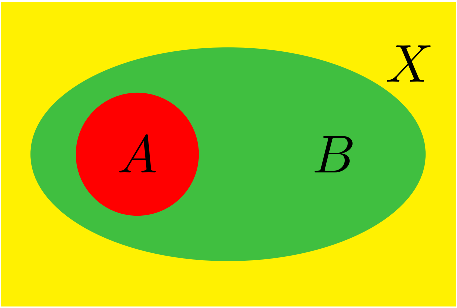

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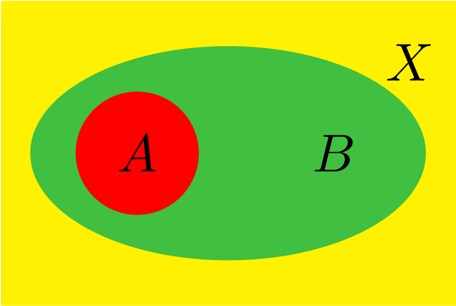

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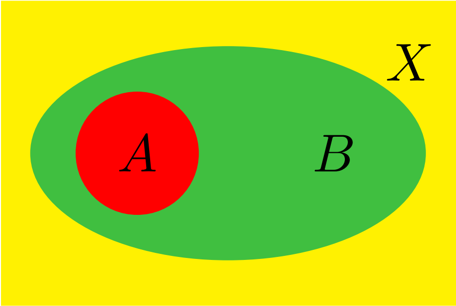

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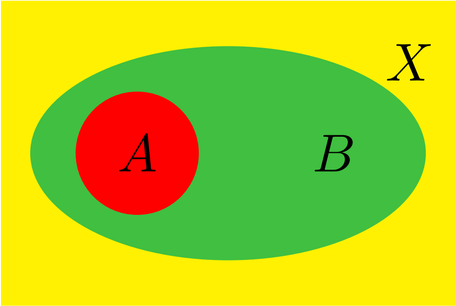

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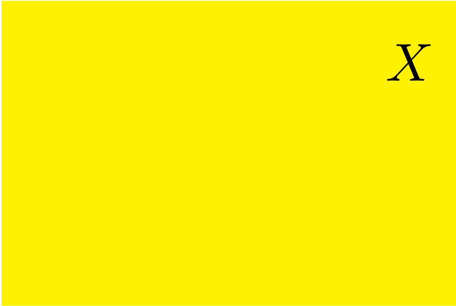

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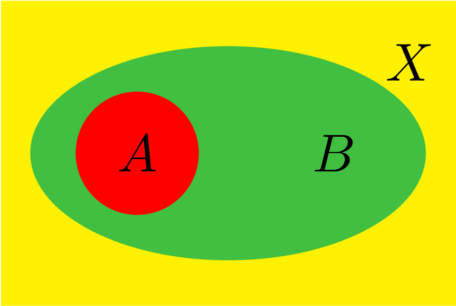

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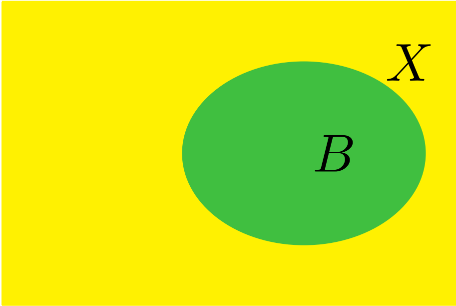

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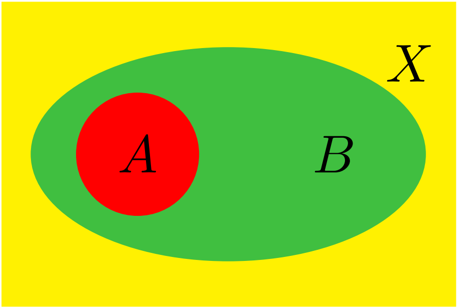

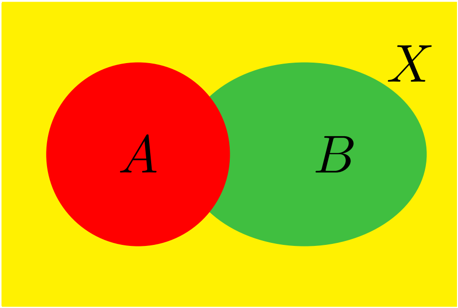

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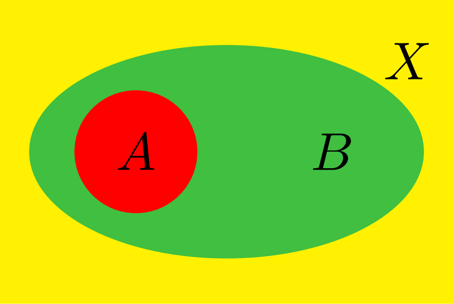

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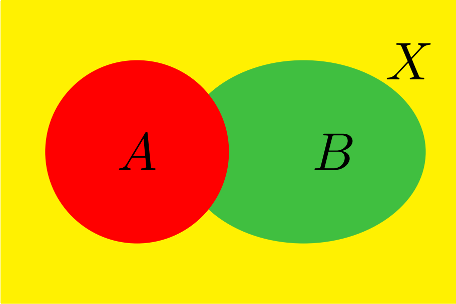

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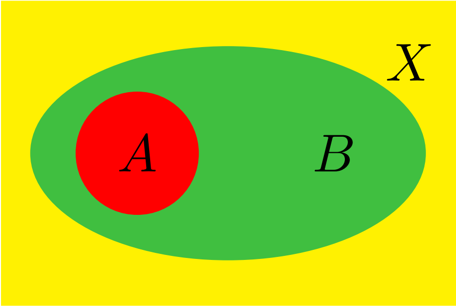

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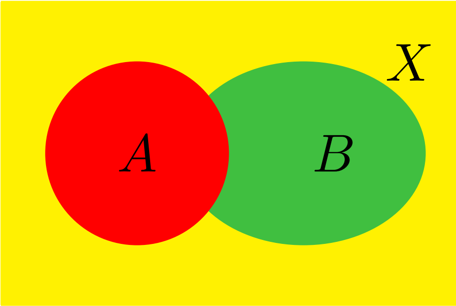

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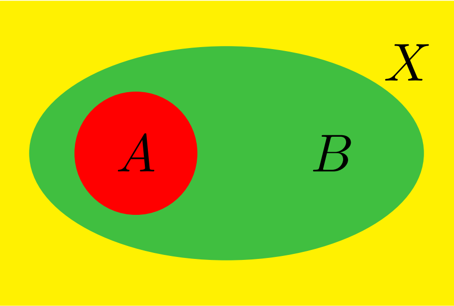

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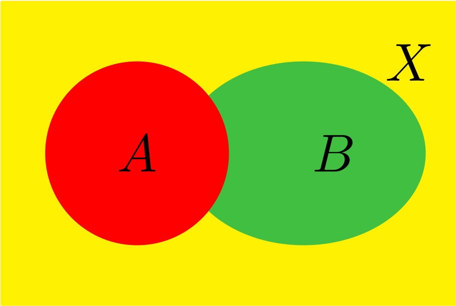

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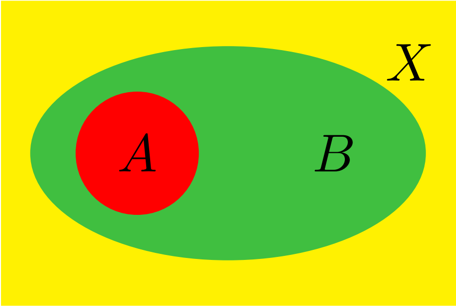

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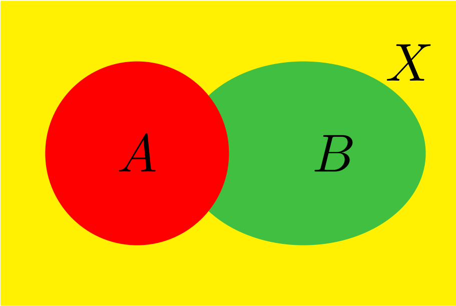

$(A, B) \notin R_\subset$ since

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Is it true that $\forall A, B \in \mathcal{P}(X) \quad \underbrace{(A, B) \in R_\subset}_{A \subset B}$ or $\underbrace{(B, A) \in R_\subset}_{B \subset A}$?

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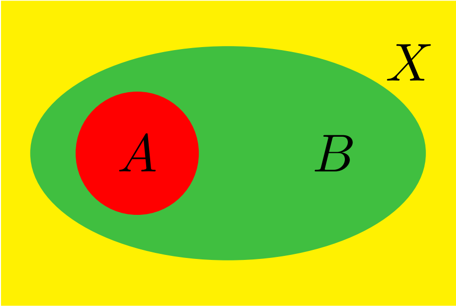

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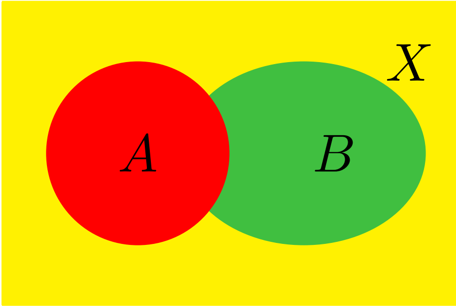

$(A, B) \notin R_\subset$ since

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Is it true that $\forall A, B \in \mathcal{P}(X) \quad \underbrace{(A, B) \in R_\subset}_{A \subset B}$ or $\underbrace{(B, A) \in R_\subset}_{B \subset A}$? No!

Relation of divisibility

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$$a \mid b \iff b = a \cdot k \text{ for some } k \in \mathbb{N}.$$

$2 \mid 6$ since $6 = 2 \cdot 3$,

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$\forall a \in \mathbb{N} \quad 1 \mid a$ and $a \mid a$.

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$5 \equiv 2 \mod 3$

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$5 \equiv 2 \mod 3$ since $3 \,|\, (5-2)$

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$-4 \equiv 20 \mod 3$ since $3 \,|\, \underbrace{(-4-20)}_{-24}$

Relation of congruence modulo 3

Example 4. Define a relation of **congruence** modulo 3 on $\mathbb{Z}$ as follows:

$$a \equiv b \mod 3 \iff 3 \mid (a-b) .$$

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$5 \equiv 2 \mod 3$ since $3 \mid (5-2)$

$-4 \equiv 20 \mod 3$ since $3 \mid \underbrace{(-4-20)}_{-24}$

$16 \equiv 16 \mod 3$

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$-4 \equiv 20 \mod 3$ since $3 \mid \underbrace{(-4-20)}_{-24}$

$16 \equiv 16 \mod 3$ since $3 \mid \underbrace{(16-16)}_{0}$

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$-4 \equiv 20 \mod 3$ since $3 \mid \underbrace{(-4-20)}_{-24}$

$16 \equiv 16 \mod 3$ since $3 \mid \underbrace{(16-16)}_{0}$

$2019 \equiv 0 \mod 3$

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$16 \equiv 16 \mod 3$ since $3 \,|\, \underbrace{(16-16)}_{0}$

$2019 \equiv 0 \mod 3$ since $3 \,|\, (2019-0)$

Criteria for divisibility by 3 and 9

Lemma.

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Lemma. A number is divisible by 3

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Lemma. A number is divisible by 3 iff the sum of its digits is divisible by 3.

Criteria for divisibility by 3 and 9

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Proof.

Criteria for divisibility by 3 and 9

Lemma. A number is divisible by 3 iff the sum of its digits is divisible by 3.

Proof. Let a number N is written with digits $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$.

Criteria for divisibility by 3 and 9

Lemma. A number is divisible by 3 iff the sum of its digits is divisible by 3.

Proof. Let a number N is written with digits $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$. Then

Criteria for divisibility by 3 and 9

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$$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$$

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$$= a_n \cdot (\underbrace{99\dots9}_{n} + 1) + a_{n-1} \cdot (\underbrace{99\dots9}_{n-1} + 1) + \cdots + a_2 \cdot (99 + 1) + a_1(9 + 1) + a_0$$

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$$= \underbrace{(a_n \cdot 99\ldots9 + a_{n-1} \cdot 99\ldots9 + \cdots + a_2 \cdot 99 + a_1 \cdot 9)}_{\text{divisible by } 3}$$

$$+ (a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0) .$$

Criteria for divisibility by 3 and 9

Lemma. A number is divisible by 3 iff the sum of its digits is divisible by 3.

Proof. Let a number N is written with digits $a_0, a_1, a_2, \dots, a_{n-1}, a_n$. Then

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$$= a_n \cdot (\underbrace{99\dots9}_{n}+1) + a_{n-1} \cdot (\underbrace{99\dots9}_{n-1}+1) + \cdots + a_2 \cdot (99+1) + a_1(9+1) + a_0$$

$$= \underbrace{(a_n \cdot 99\dots9 + a_{n-1} \cdot 99\dots9 + \cdots + a_2 \cdot 99 + a_1 \cdot 9)}_{\text{divisible by } 3}$$

$$+(a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0).$$

Therefore, N is divisible by 3 iff
the sum $a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0$ of its digits is divisible by 3.

Criteria for divisibility by 3 and 9

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Proof. Let a number N is written with digits $a_0, a_1, a_2, \dots, a_{n-1}, a_n$. Then

$$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \dots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$$

$$= a_n \cdot (\underbrace{99\dots9}_{n}+1) + a_{n-1} \cdot (\underbrace{99\dots9}_{n-1}+1) + \dots + a_2 \cdot (99+1) + a_1(9+1) + a_0$$

$$= \underbrace{(a_n \cdot 99\dots9 + a_{n-1} \cdot 99\dots9 + \dots + a_2 \cdot 99 + a_1 \cdot 9)}_{\text{divisible by } 3}$$

$$+(a_n + a_{n-1} + \dots + a_2 + a_1 + a_0)\,.$$

Therefore, N is divisible by 3 iff

the sum $a_n + a_{n-1} + \dots + a_2 + a_1 + a_0$ of its digits is divisible by 3.

Remark. The same proof proves that,

Criteria for divisibility by 3 and 9

Lemma. A number is divisible by 3 iff the sum of its digits is divisible by 3.

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Properties of relations

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$\leq$ on $\mathbb{R}$	$\equiv \mod 3$ on $\mathbb{Z}$	$\subset$ on $\mathcal{P}(X)$	divisibility on $\mathbb{N}$
reflexive $x \leq x$	reflexive $a \equiv a \mod 3$	reflexive $A \subset A$	reflexive $a \mid a$
antisymmetric $x \leq y \wedge y \leq x$ $\implies x = y$	symmetric $a \equiv b \mod 3$ $\implies b \equiv a \mod 3$	antisymmetric $A \subset B \wedge B \subset A$ $\implies A = B$	antisymmetric $a \mid b \wedge b \mid a$ $\implies a = b$
transitive $x \leq y \wedge y \leq z$ $\implies x \leq z$	transitive $a \equiv b \mod 3 \wedge$ $b \equiv c \mod 3$ $\implies a \equiv c \mod 3$	transitive $A \subset B \wedge B \subset C$ $\implies A \subset C$	transitive $a \mid b \wedge b \mid c$ $\implies a \mid c$
total $\forall x, y \in \mathbb{R}$ $x \leq y \vee y \leq x$			

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$$x \sim_\Sigma y \iff \exists S \in \Sigma \quad x, y \in S\,.$$

or in words:

$x \sim_\Sigma y$ iff x and y belong to the same element of the partition Σ.

For any partition Σ of X, the relation $\sim_\Sigma$ has the following three properties.

Reflexivity. $\forall a \in X \quad a \sim_\Sigma a\,.$

Indeed, each $a \in X$ belongs to some element of Σ. Therefore $a \sim_\Sigma a$. ☐

Symmetry. $\forall a, b \in X \quad a \sim_\Sigma b \implies b \sim_\Sigma a\,.$ Obvious. ☐

Transitivity. $\forall a, b, c \in X \quad a \sim_\Sigma b \wedge b \sim_\Sigma c \implies a \sim_\Sigma c\,.$

If $a \sim_\Sigma b$ and $b \sim_\Sigma c$, then there are $S, T \in \Sigma$ such that $a, b \in S$ and $b, c \in T$.
Hence $b \in S \cap T$. Therefore $S \cap T \neq \varnothing$ and $S = T$. ☐

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Together this gives us that $[a] = [b]$. $\square$

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Exercise. Verify that these two maps are really inverse to each other.

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That is, $[x] = f^{-1}f(x)$.

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It is called the **injective quotient** of f.

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with vertical arrows $A \xrightarrow{h} B$, $X \xrightarrow{g} Y$ on the left of the first and second diagrams and $A \xrightarrow{h} B$ in the third.

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Identity morphism:

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Which morphisms are inverse?

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$$X \xrightarrow{f} Y \;=\; f(X) \,\cup\, (Y \smallsetminus f(X))\,.$$

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Families Γ and Δ of sets are equivalent
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Categories in Linear Algebra.

Categorial generalizations of injective and surjective

Working with maps, we often consider elements of the source and target sets. However often elements may be avoided and everything can be done on the categorical level, replacing elements by compositions of maps.

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A morphism $f : A \to B$ in a category C is called **monic**
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Let X, Y be posets, a map $X \to Y$ is called
monotonic, or **monotone**, or **increasing** if $a \prec b \implies f(a) \prec f(b)$.

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If $a \prec b$ is false, then there is no morphism $a \to b$.