Lecture 8

Maps

Maps: domain, codomain, range

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Definition.

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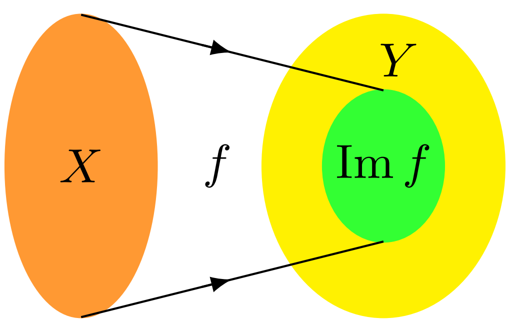

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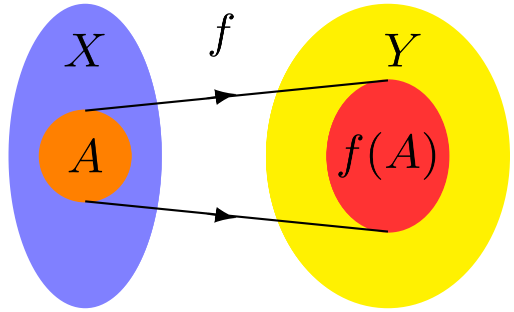

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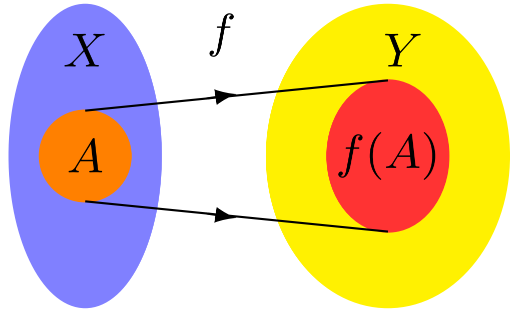

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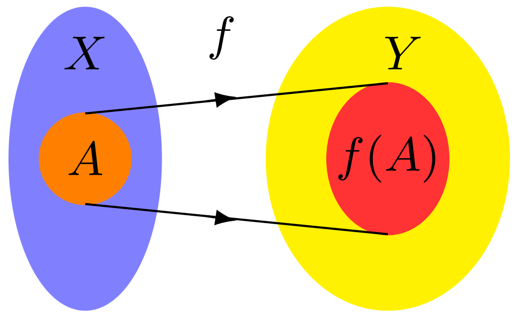

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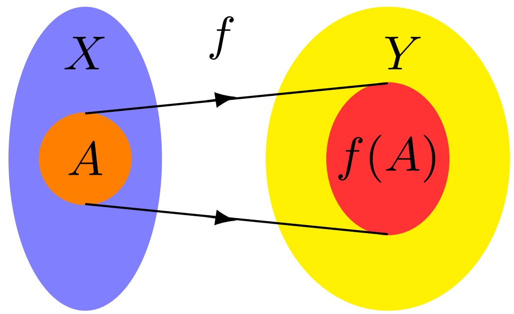

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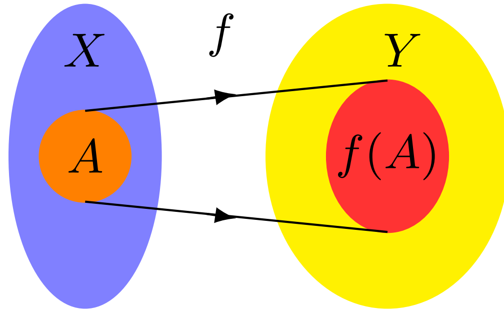

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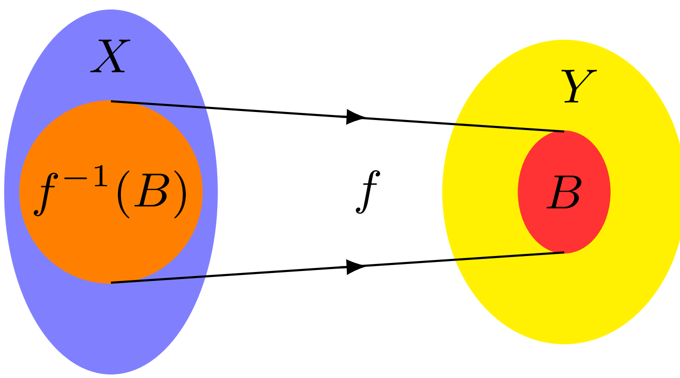

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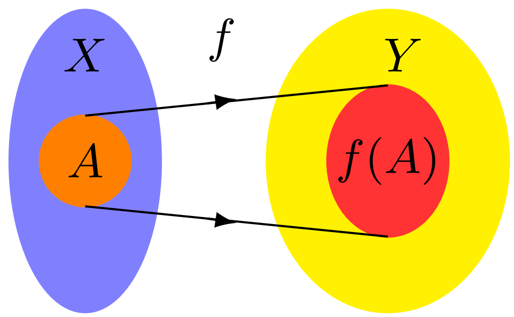

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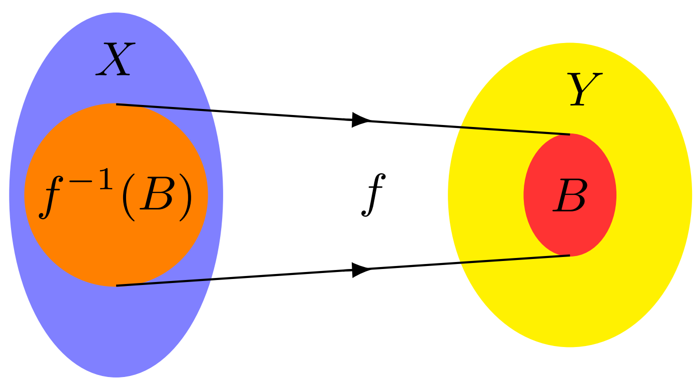

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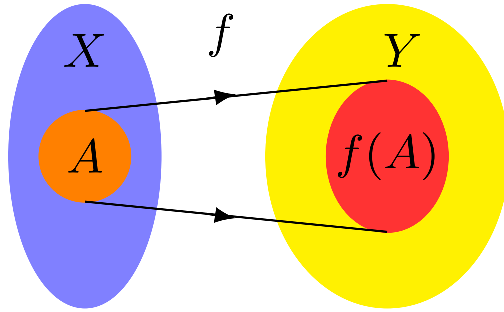

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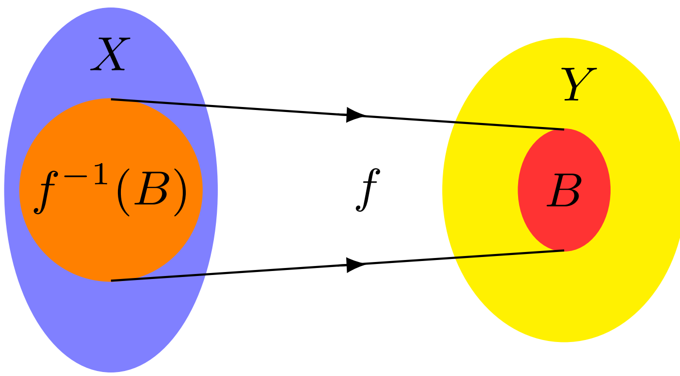

Warning: f^{-1} is **not** the inverse map!

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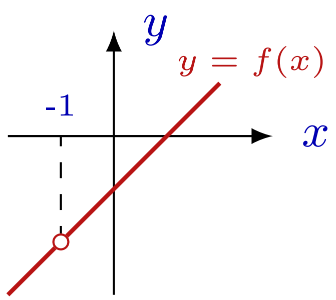

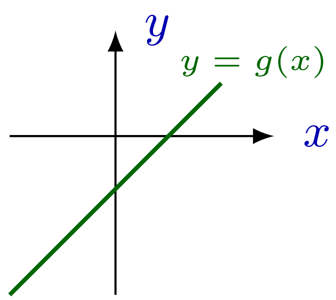

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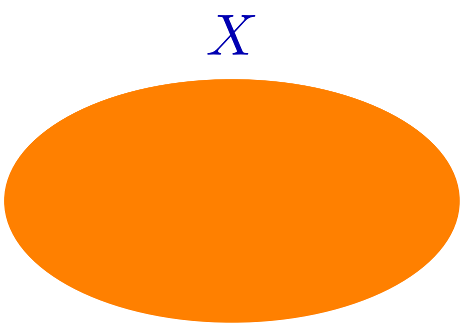

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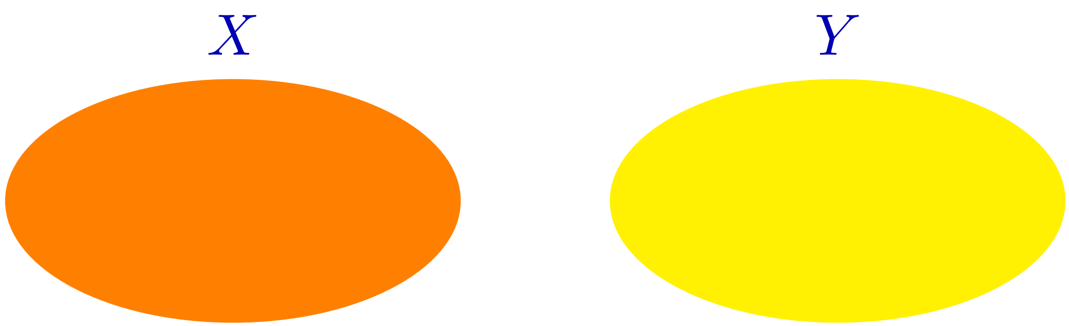

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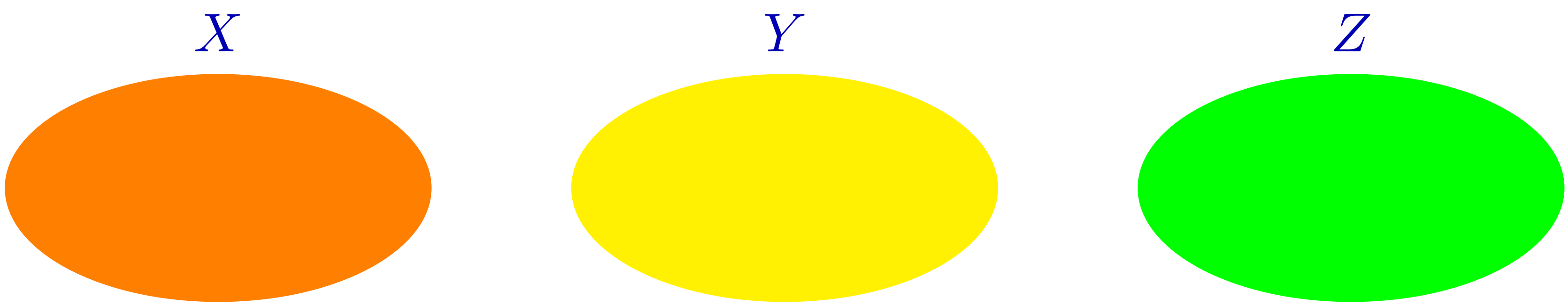

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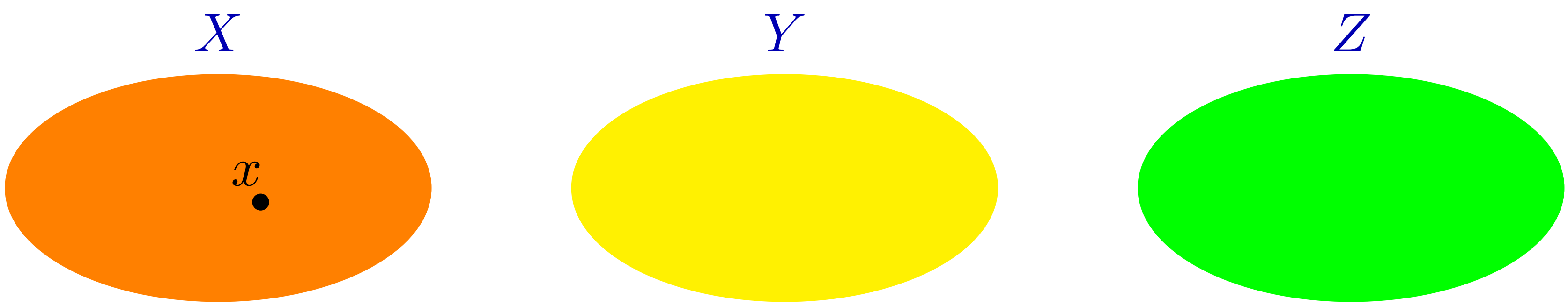

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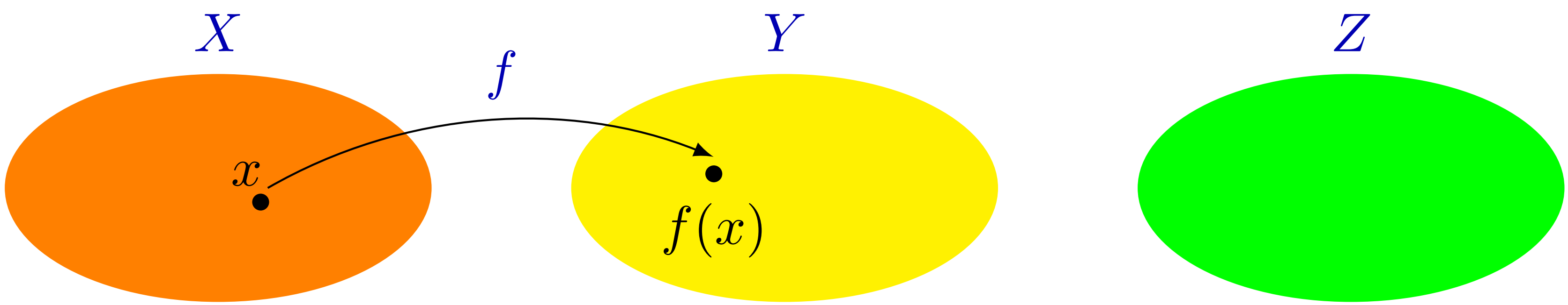

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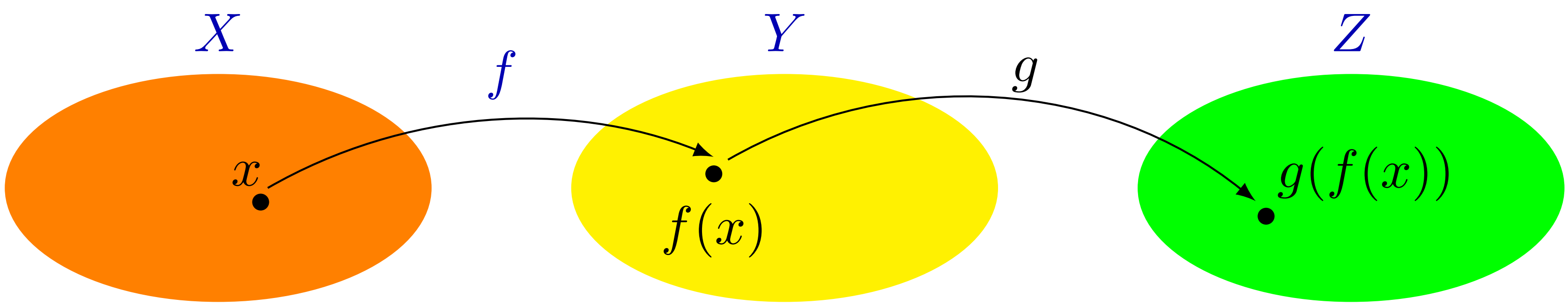

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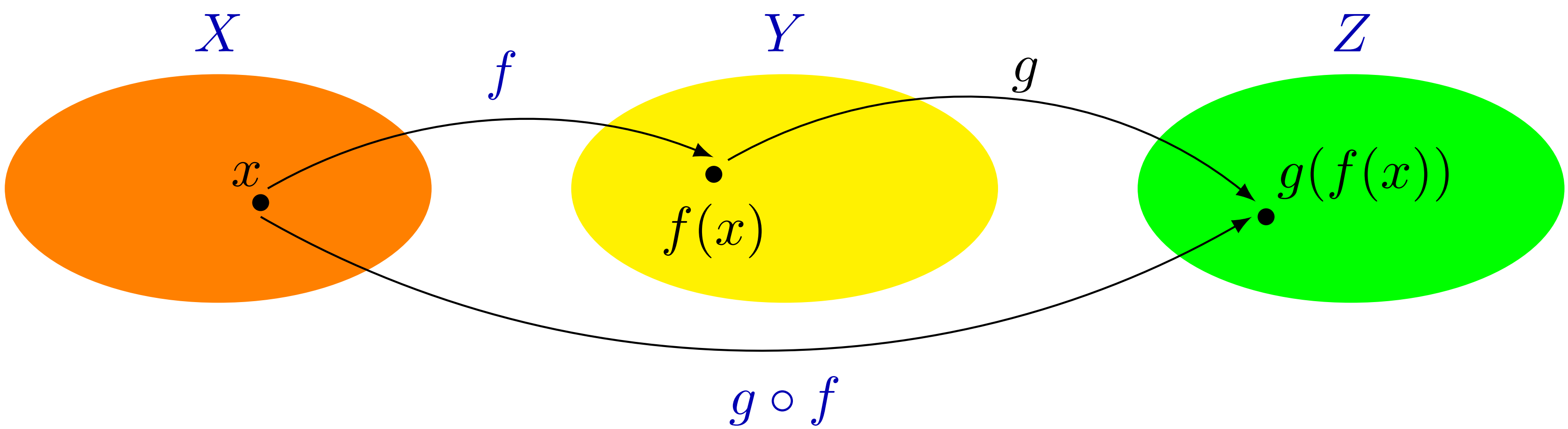

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Let X, Y be sets. Choose any $y_0 \in Y$ and define a map $f : X \to Y$ by $f(x) = y_0$ for all $x \in X$. This map is called a **constant** map.

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Or $\exists c \in Y \; \forall a \in X \; f(a) = c$.

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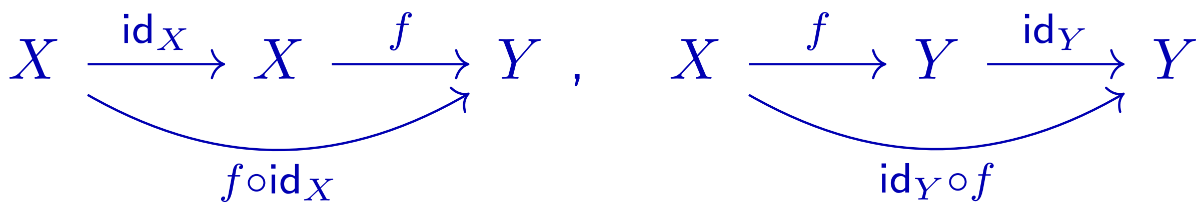

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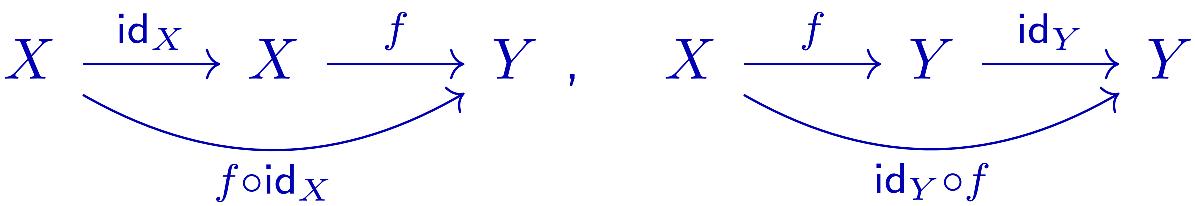

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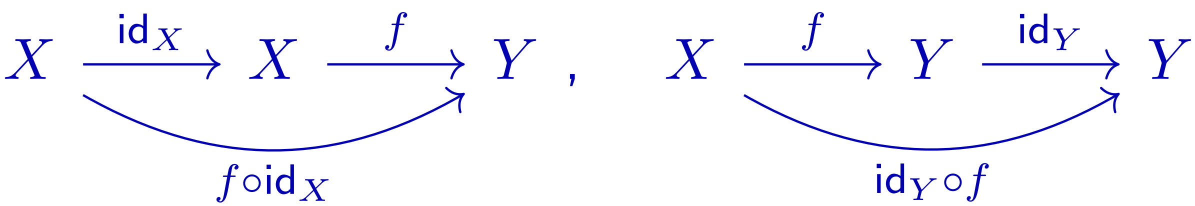

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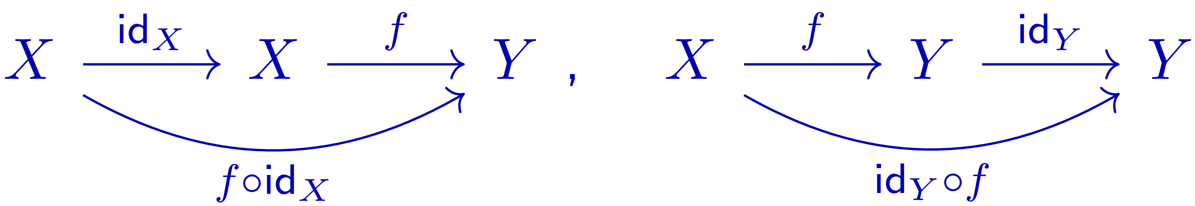

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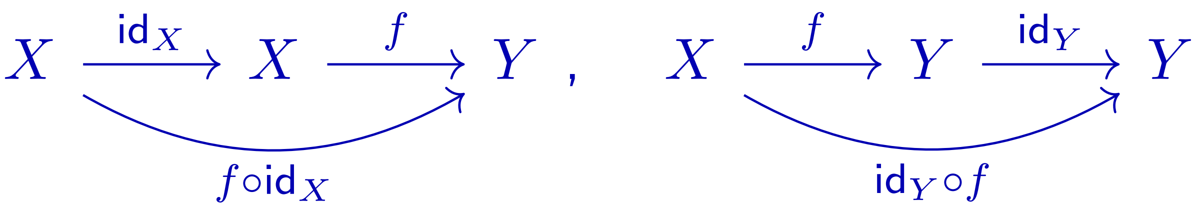

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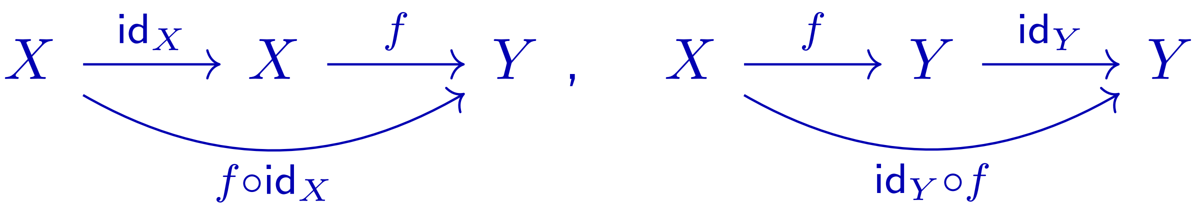

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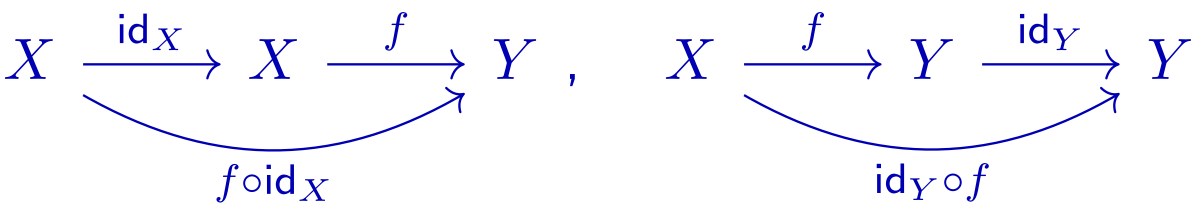

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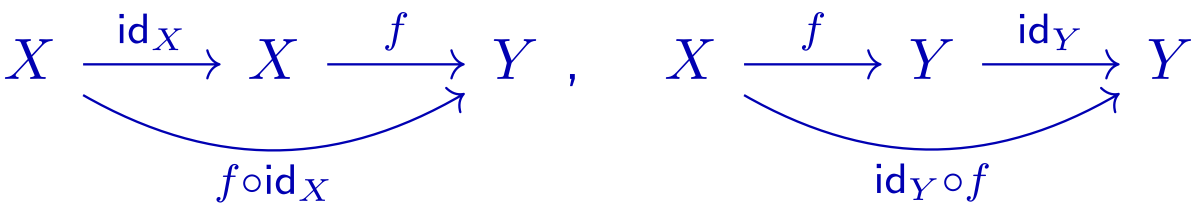

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$\forall\, y \in \operatorname{Im} f$ the equation $y = f(x)$ has **at most one** solution.

Injective or not?

Injective or not?

Example.

Injective or not?

Example.

-
-
-

Injective or not?

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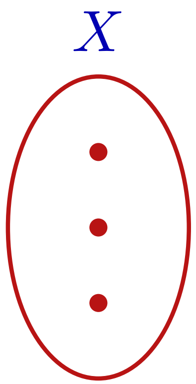

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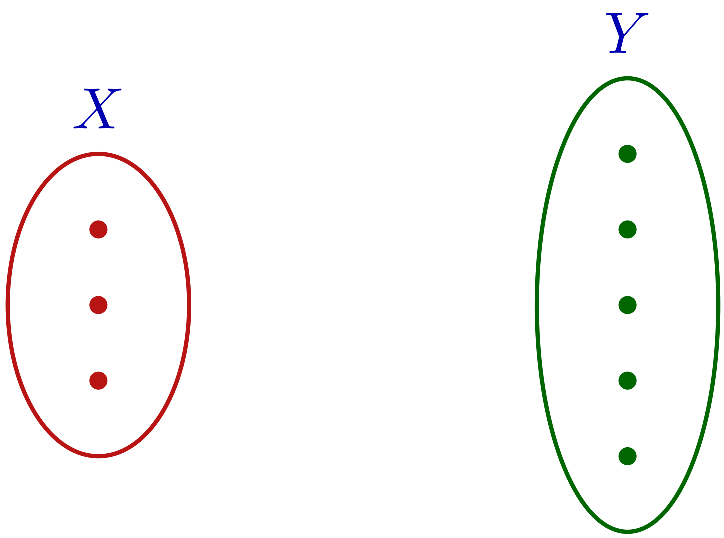

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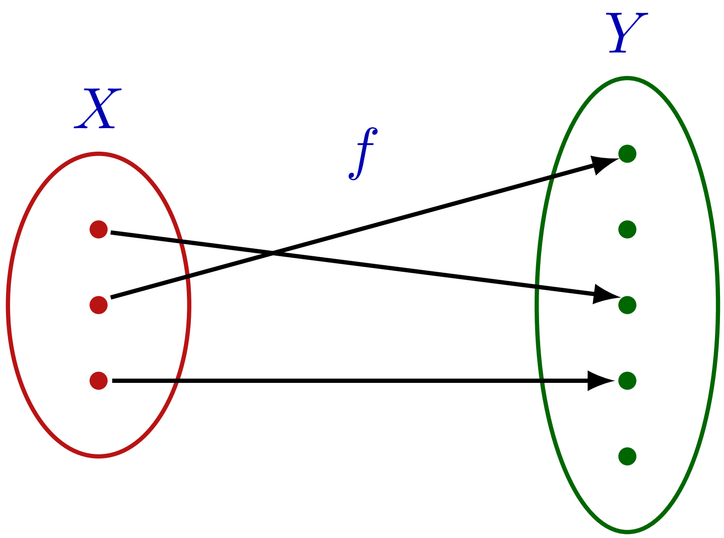

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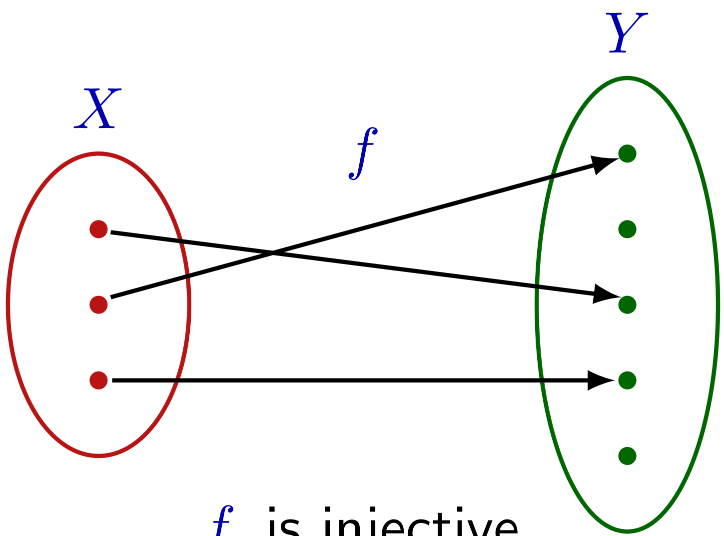

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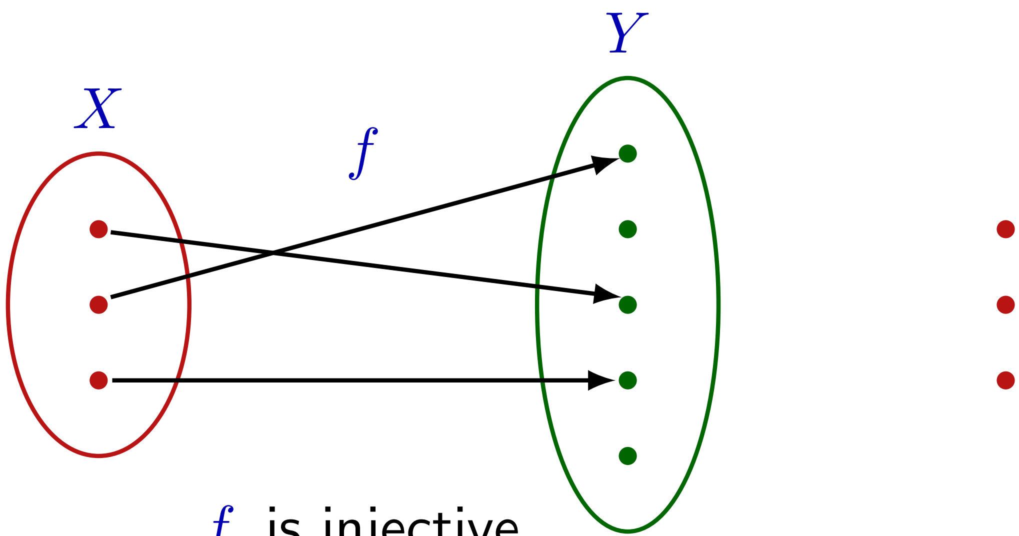

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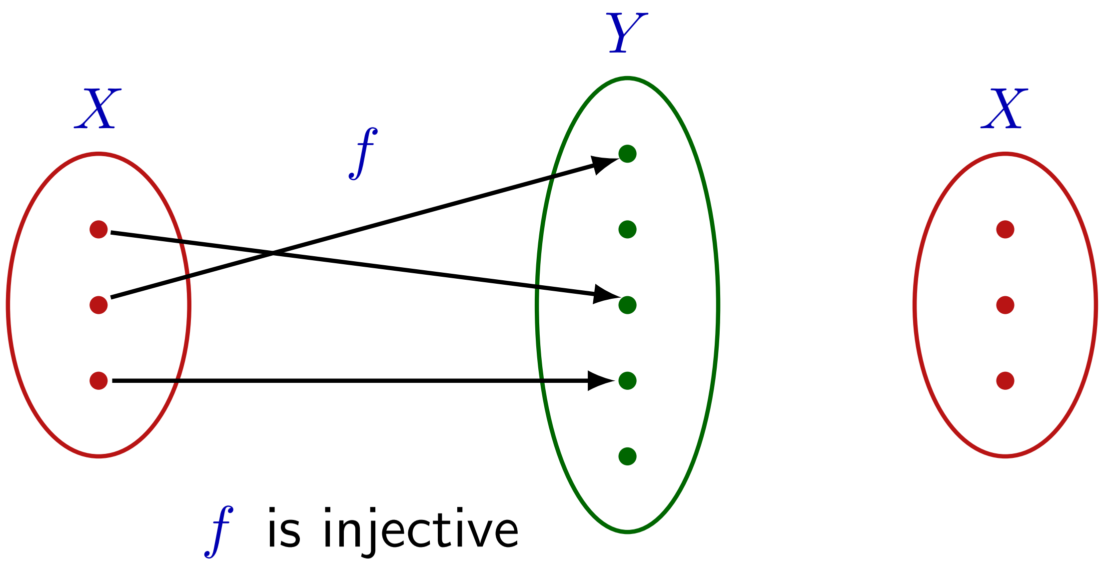

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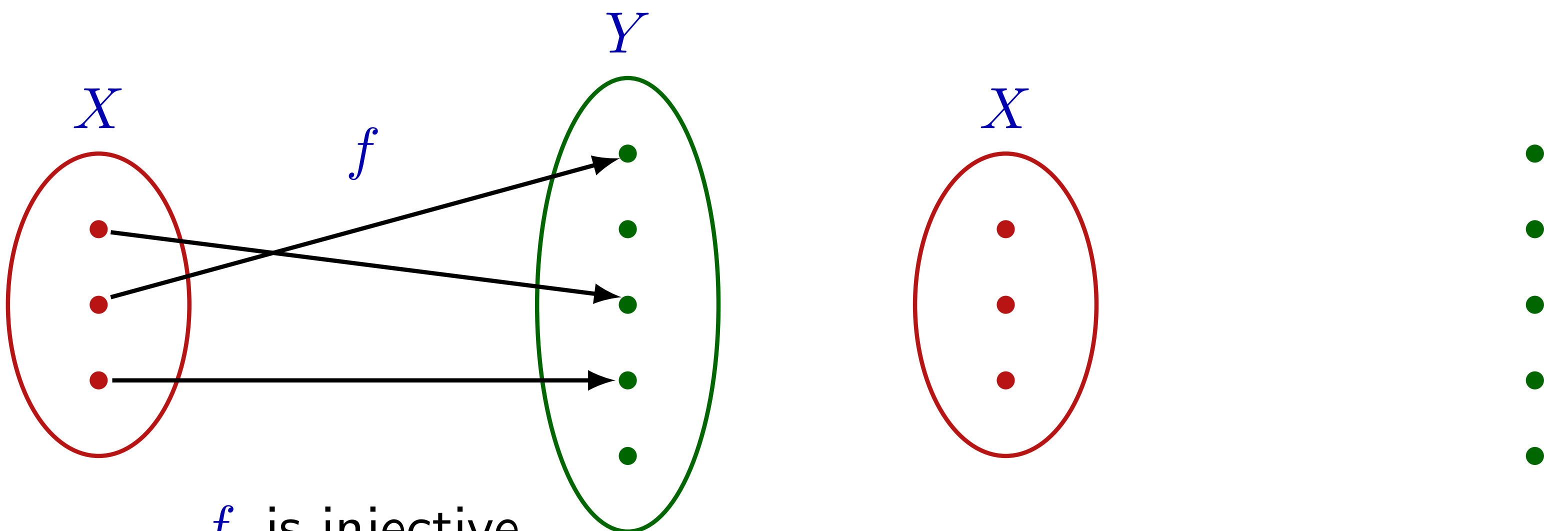

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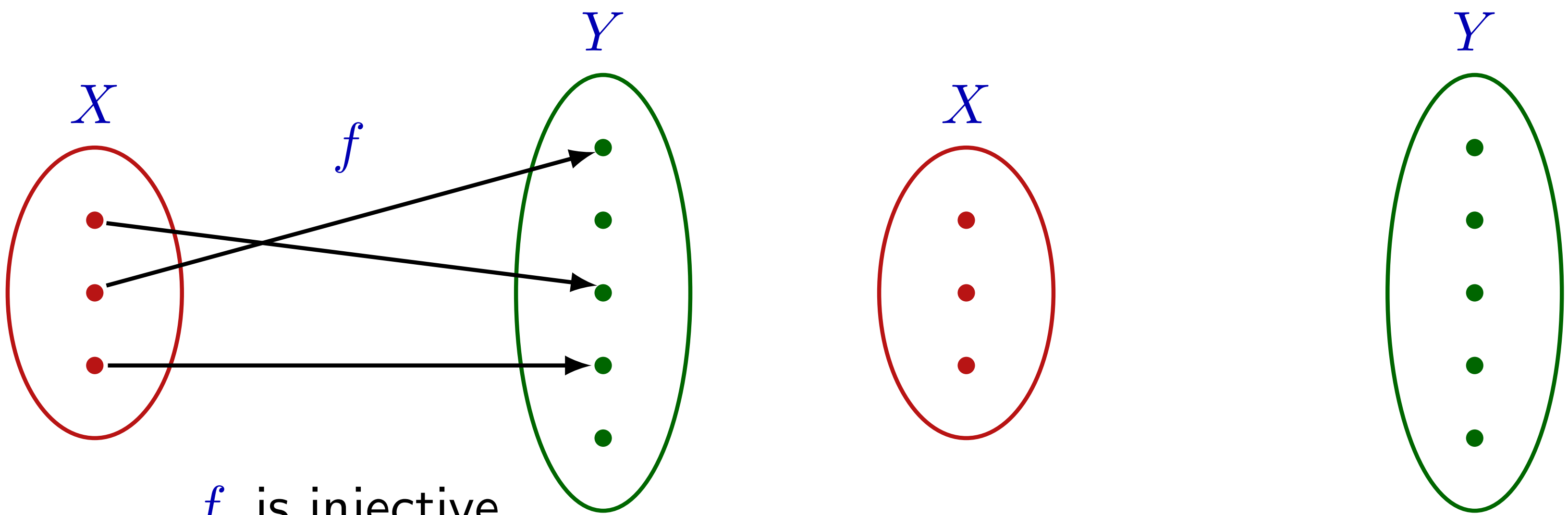

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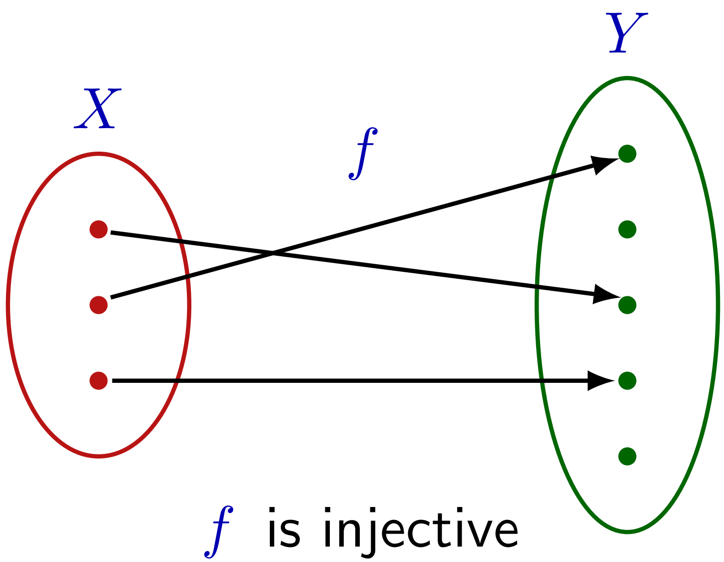

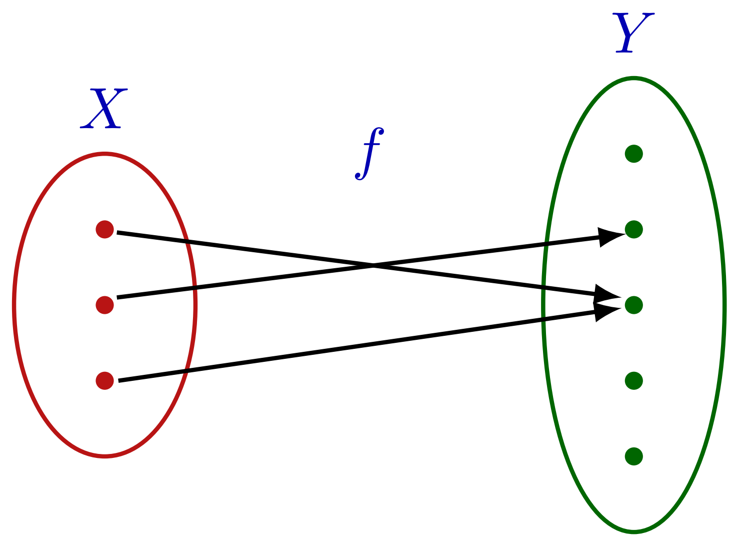

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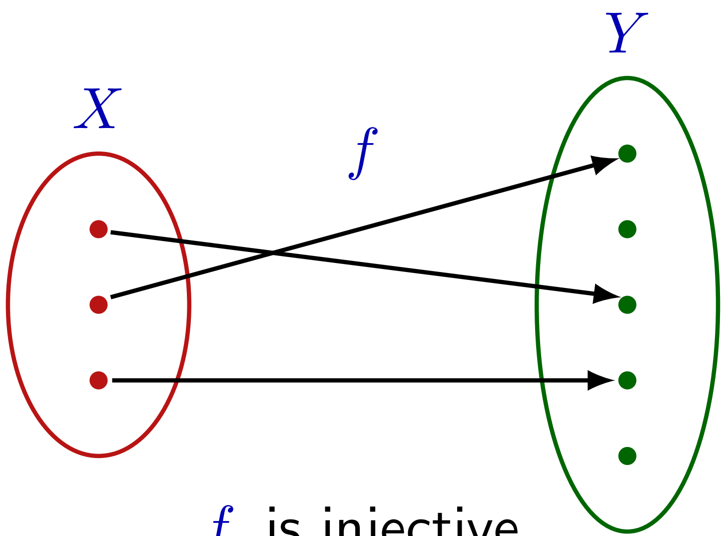

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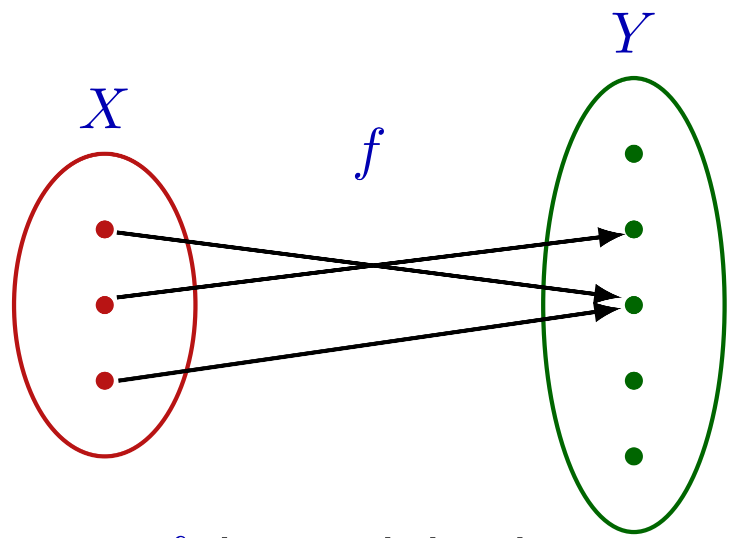

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Non-constant linear function is injective

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Theorem.

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Theorem. A linear map $f : \mathbb{R} \to \mathbb{R}$

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Remark. If $a = 0$, then the map $f(x) = b$ is a **constant** map, it is **not** injective.

Quadratic function is not injective

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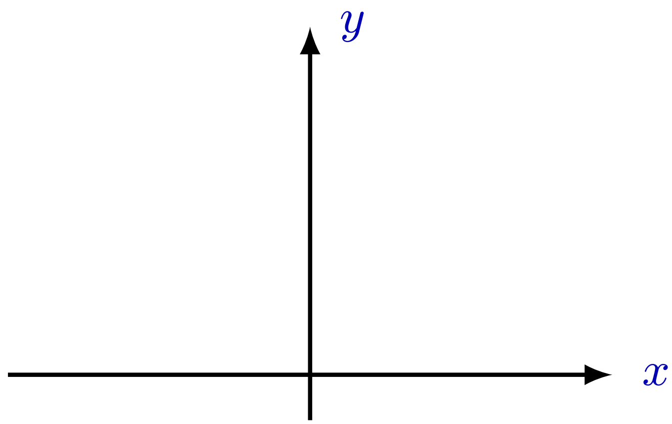

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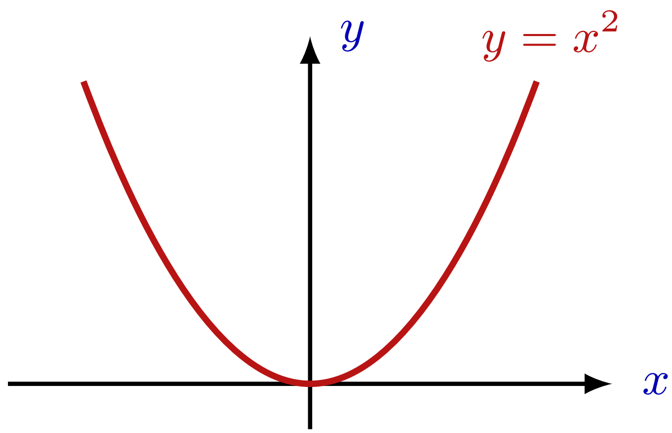

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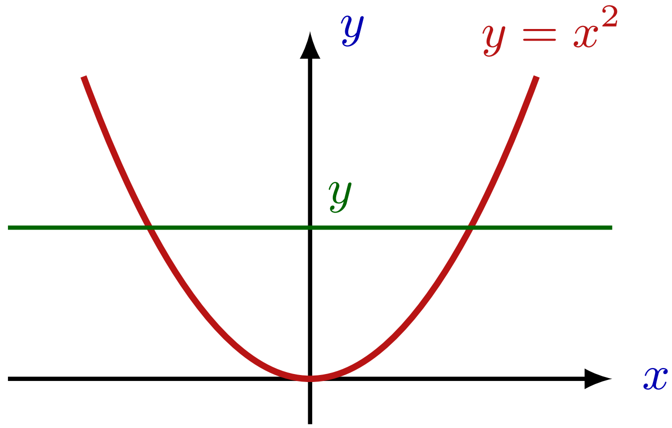

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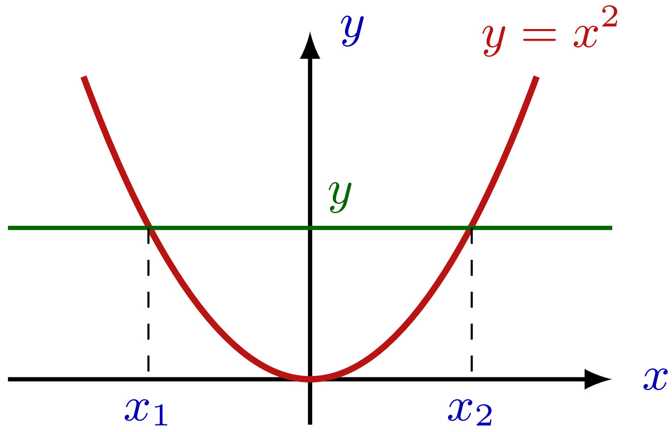

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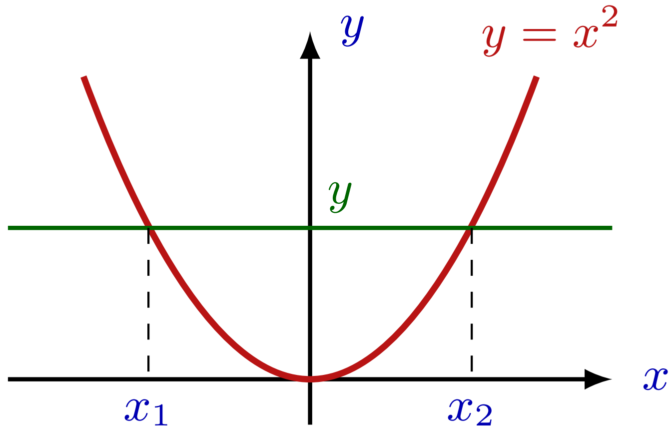

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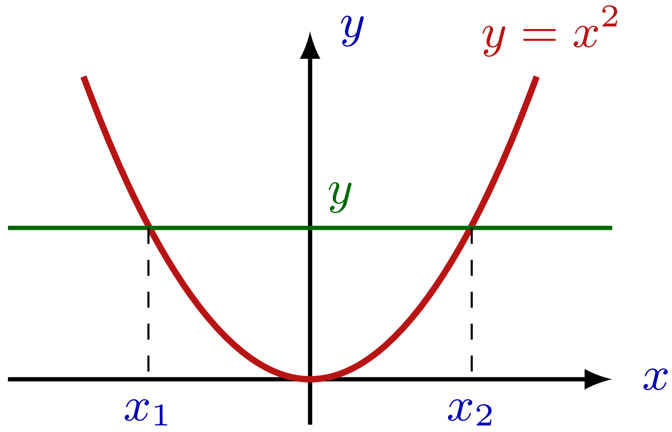

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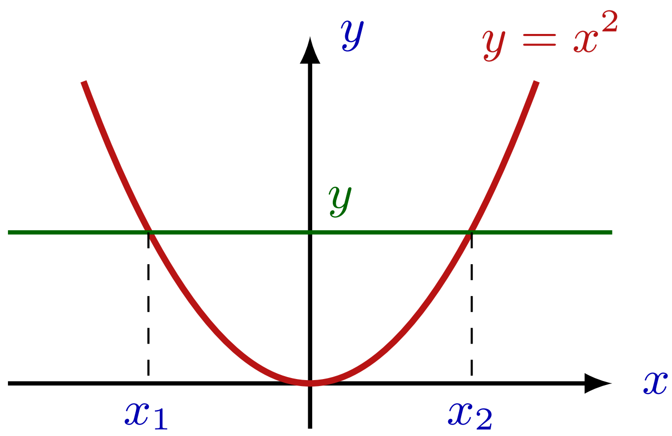

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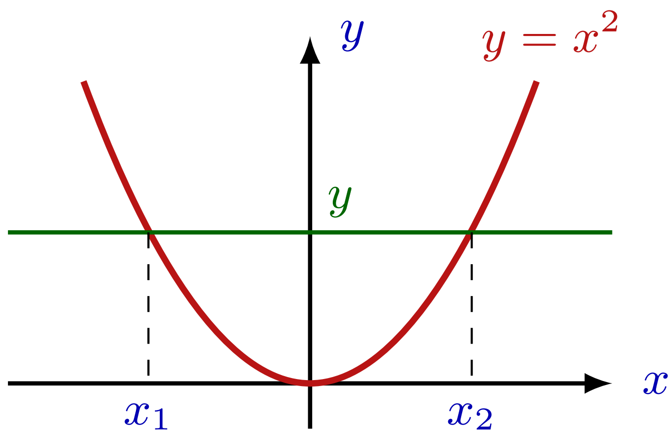

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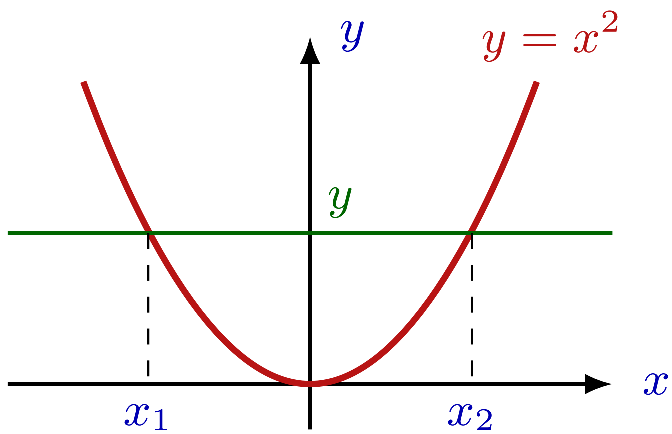

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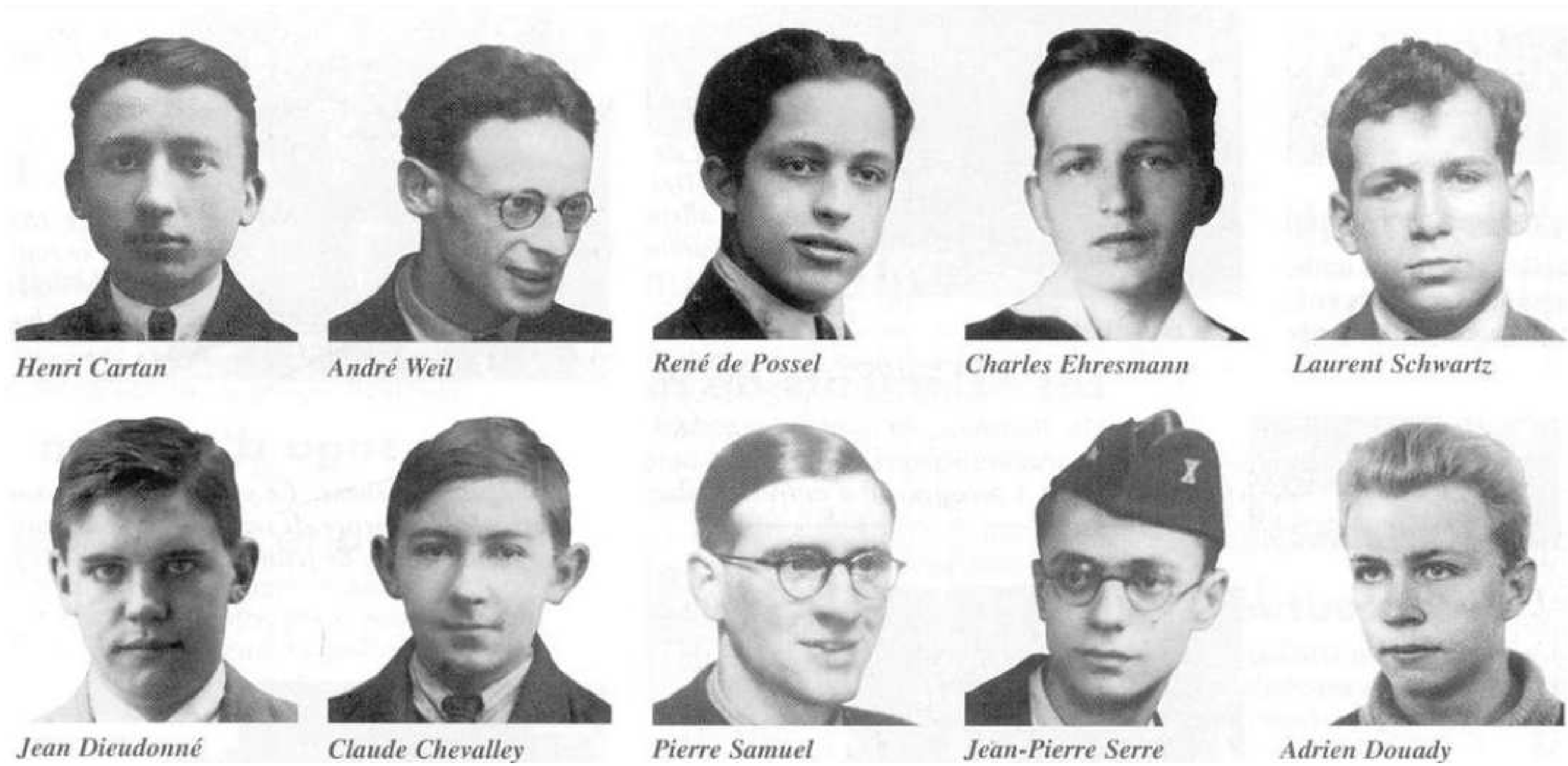
Remark: The restriction $f\big|_{\mathbb{R}_+}$ is injective.

Bourbaki


Henri Cartan
André Weil
René de Possel
Charles Ehresmann
Laurent Schwartz
Jean Dieudonné
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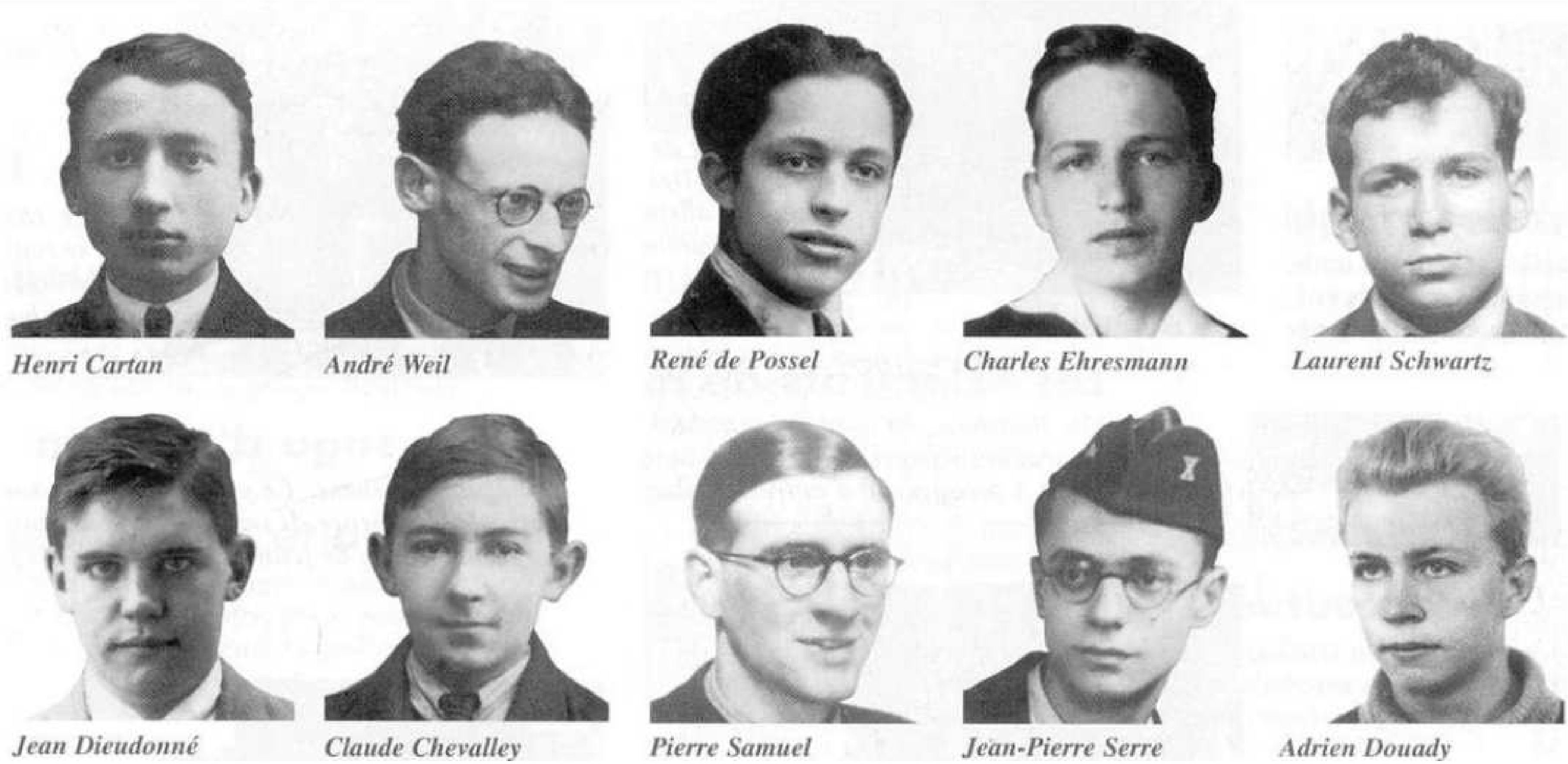
Pierre Samuel
Jean-Pierre Serre
Adrien Douady

Bourbaki



Nicolas Bourbaki

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-
-
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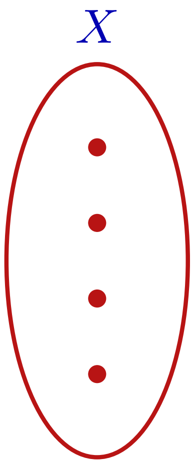

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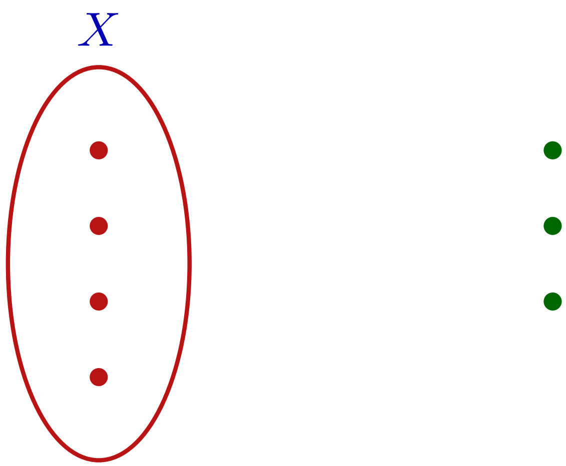

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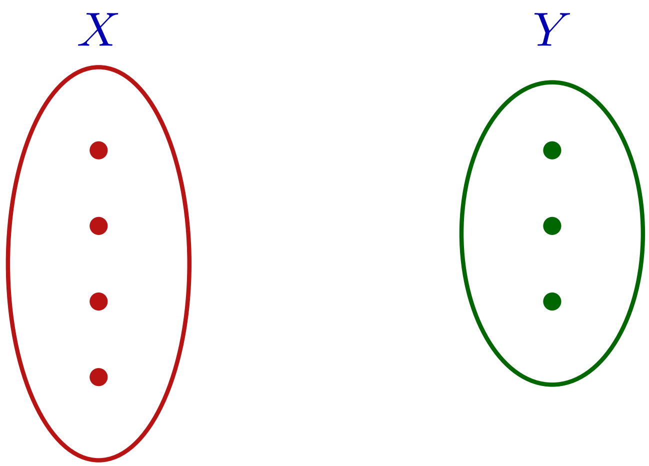

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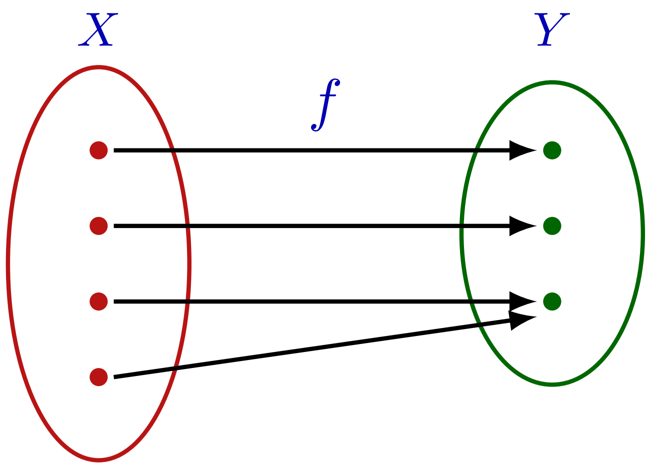

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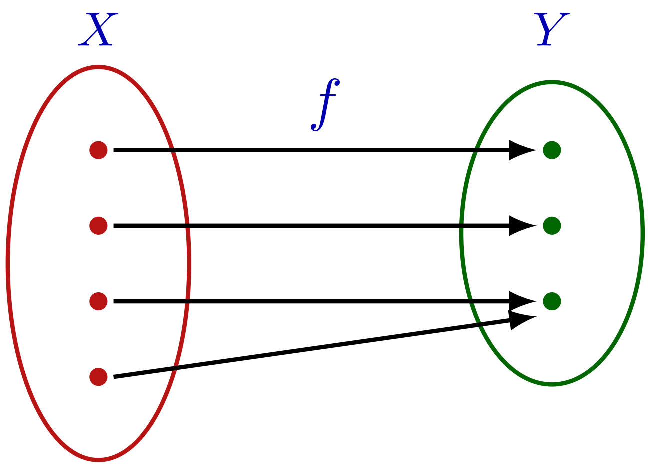

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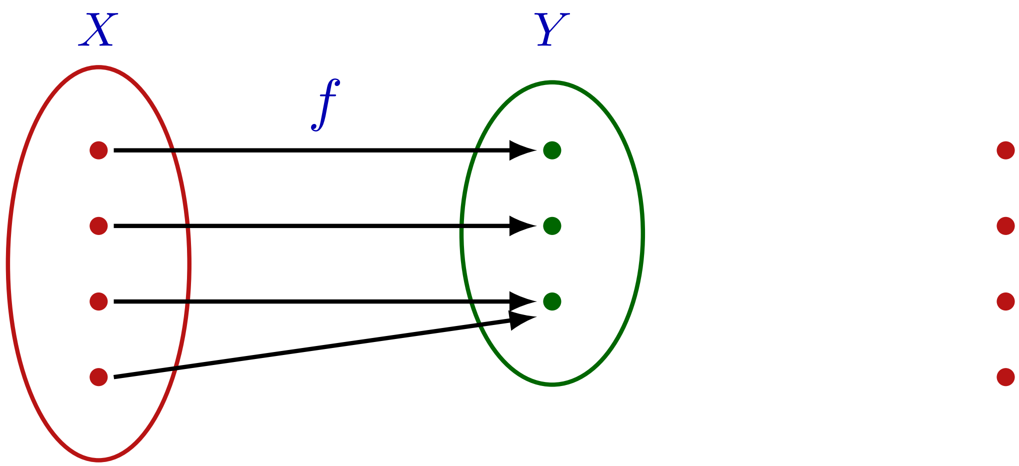

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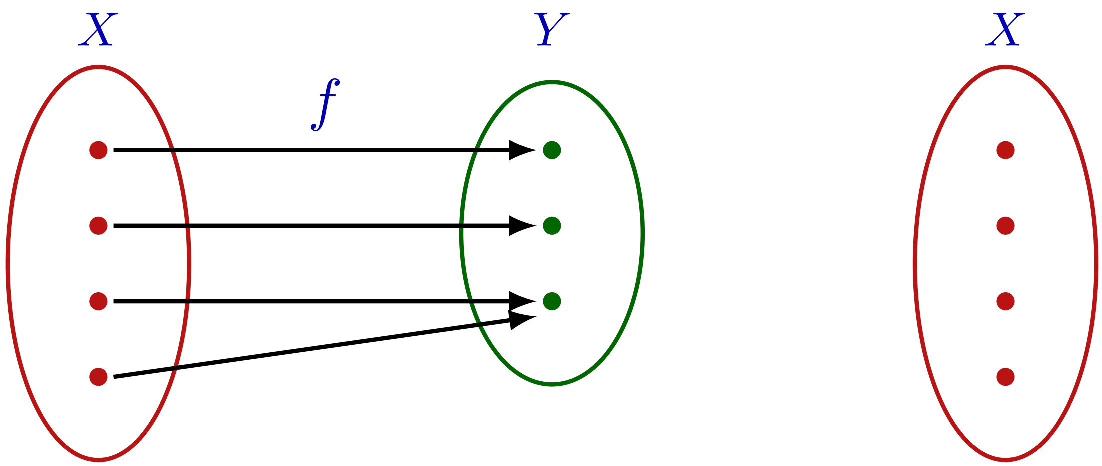

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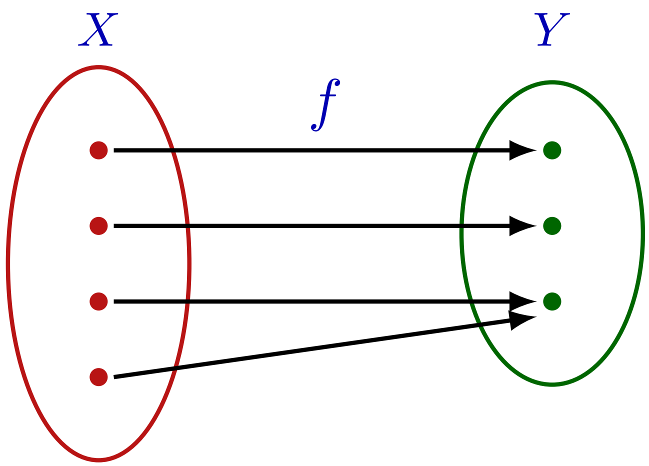

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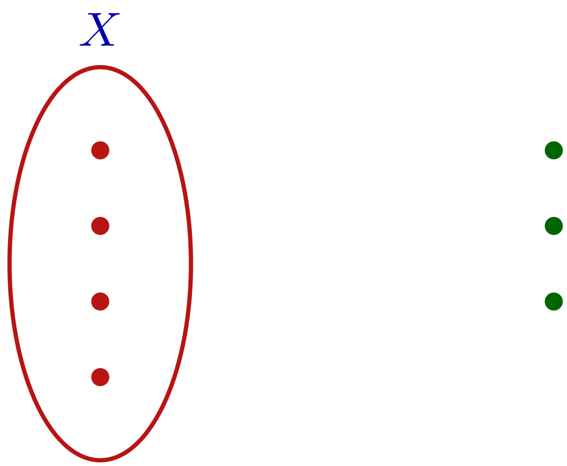

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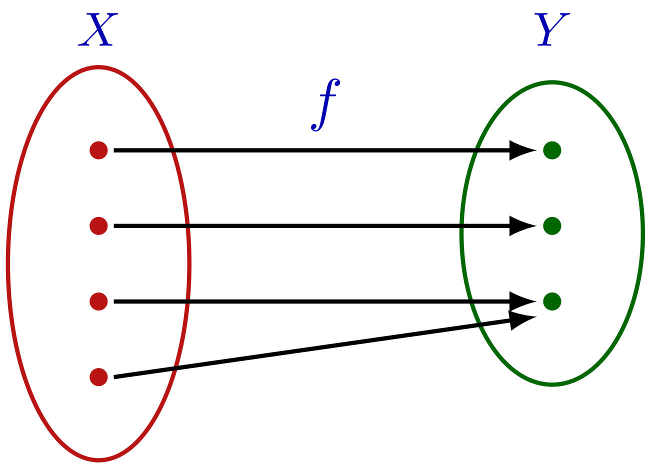

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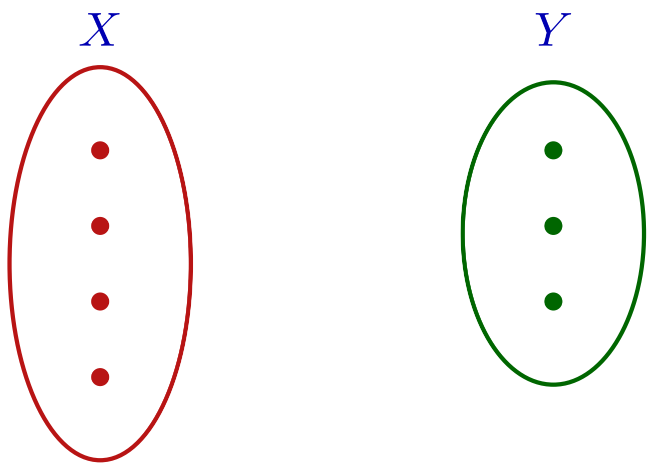

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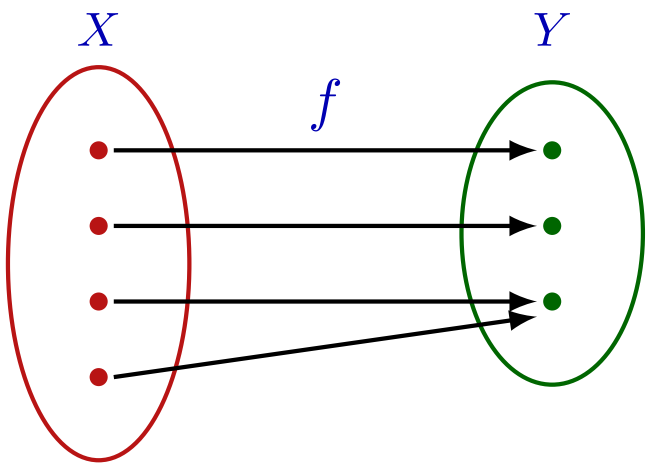

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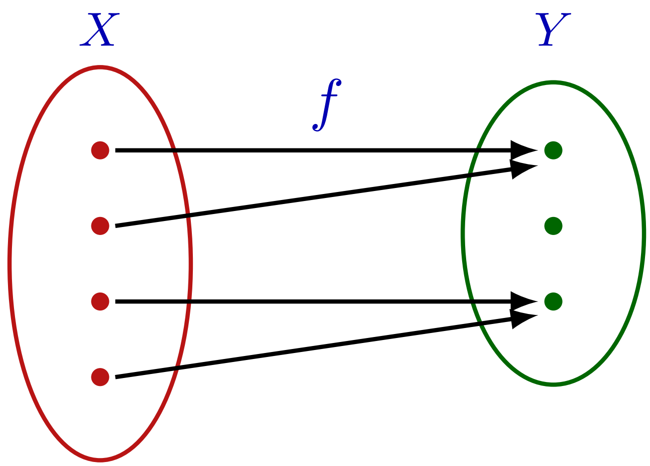
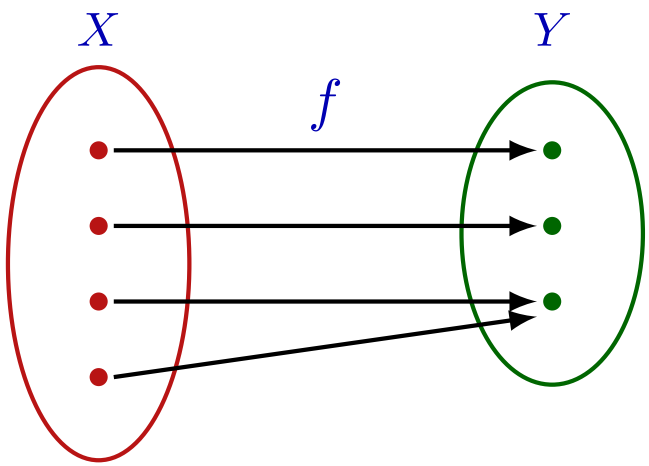


Sujective or not?

Example.



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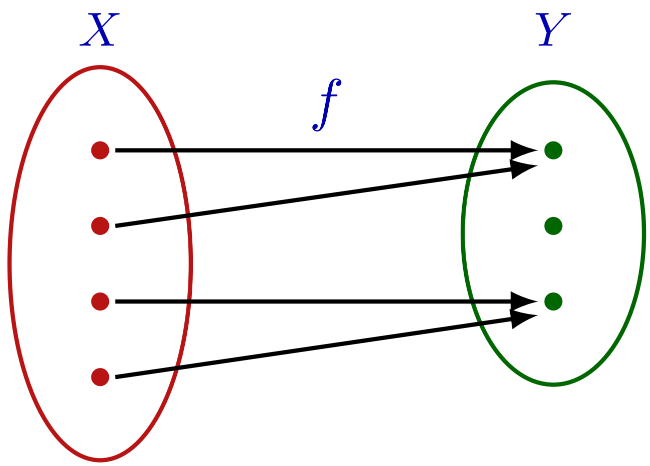

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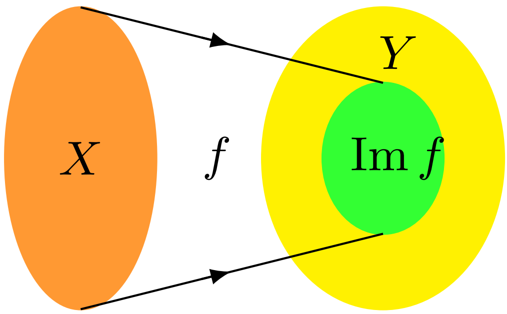

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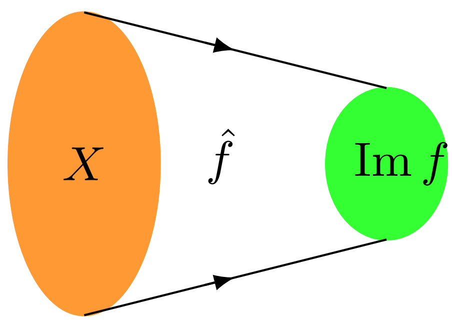

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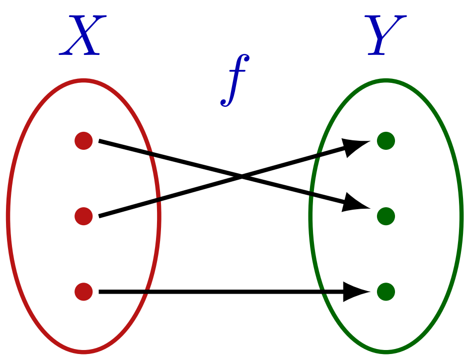

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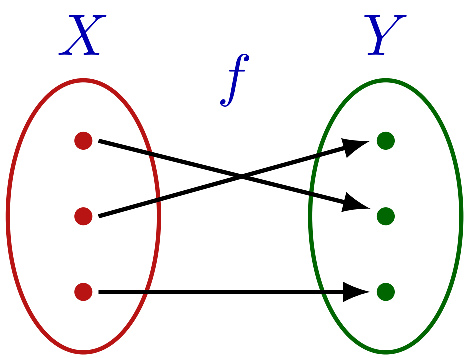

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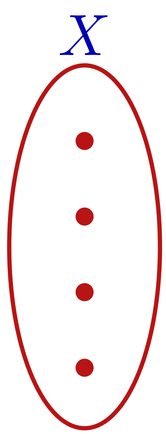

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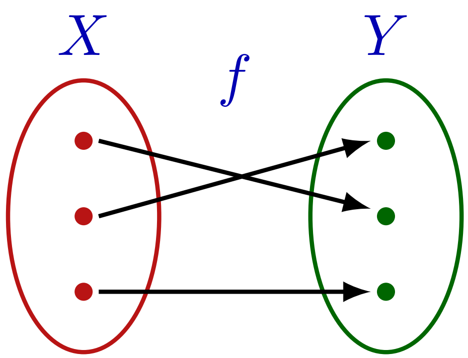

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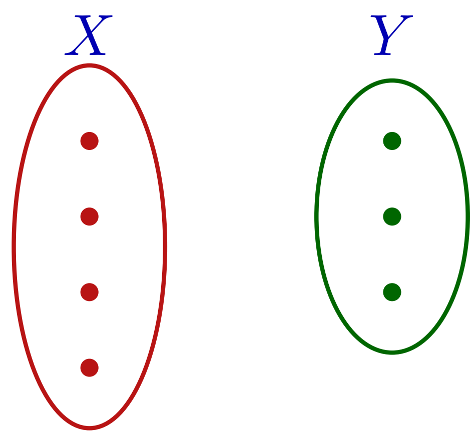

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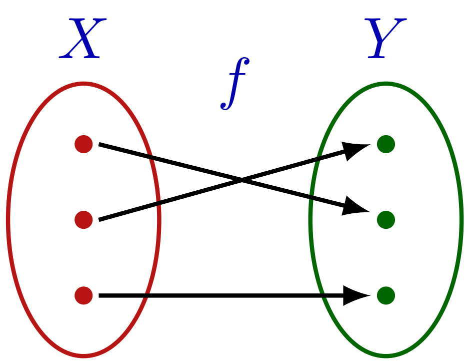

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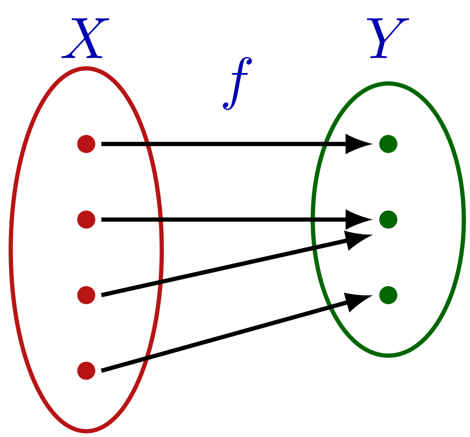

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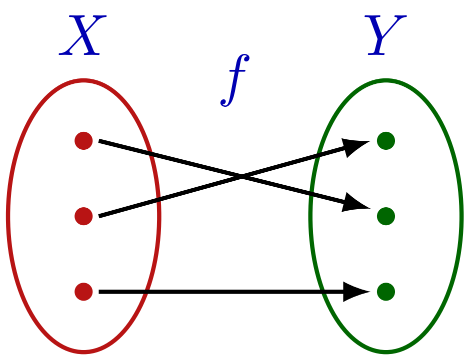

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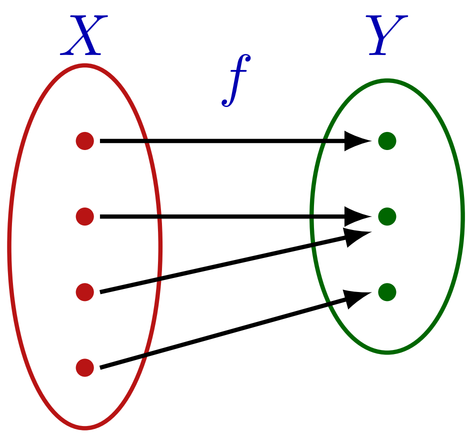

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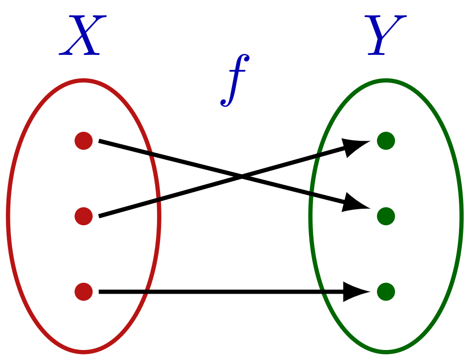

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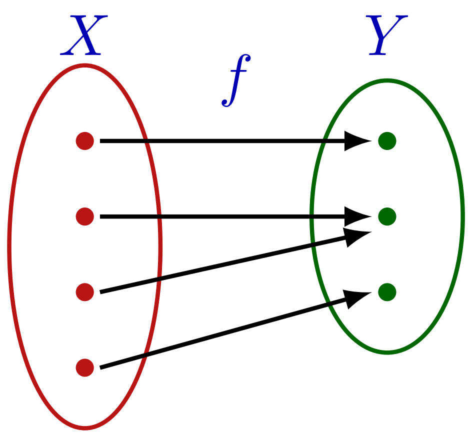

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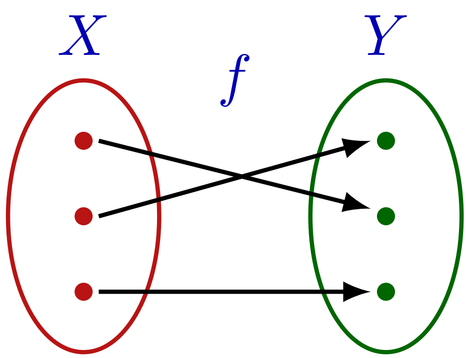

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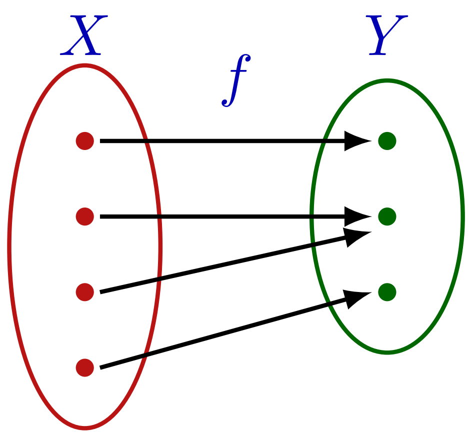

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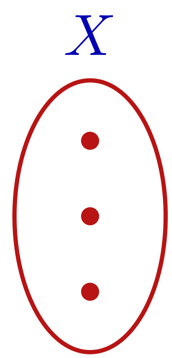

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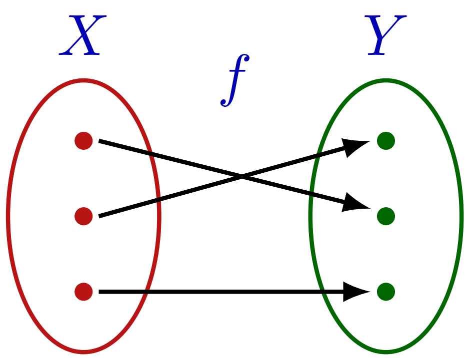

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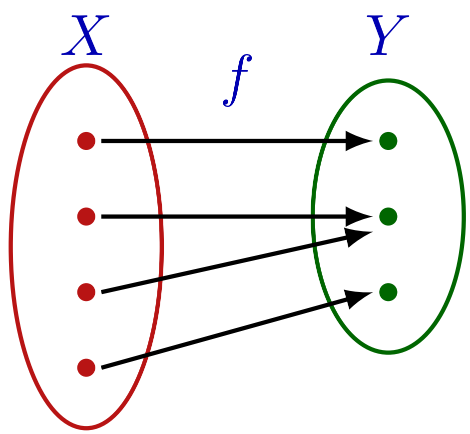

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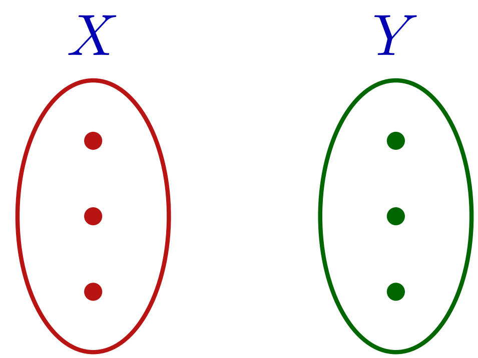

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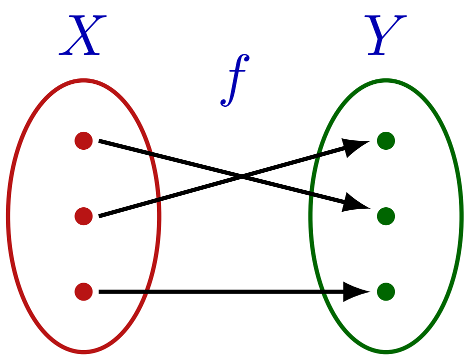

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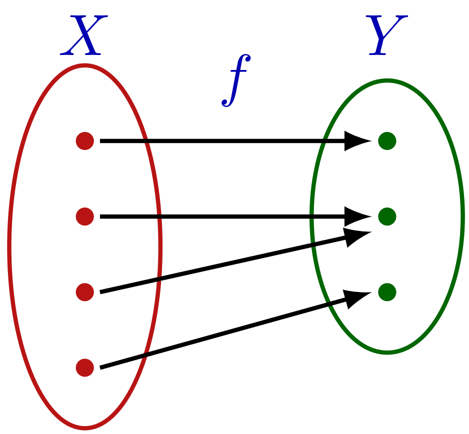

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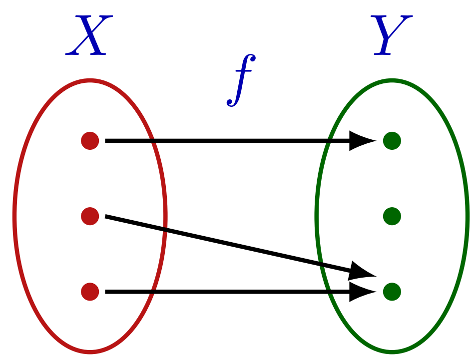

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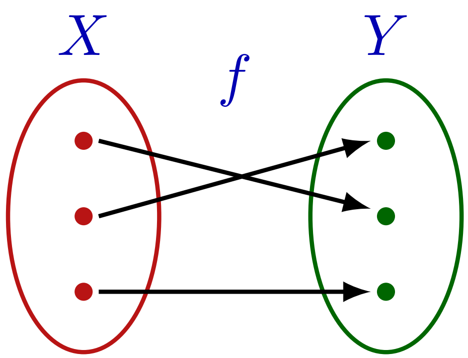

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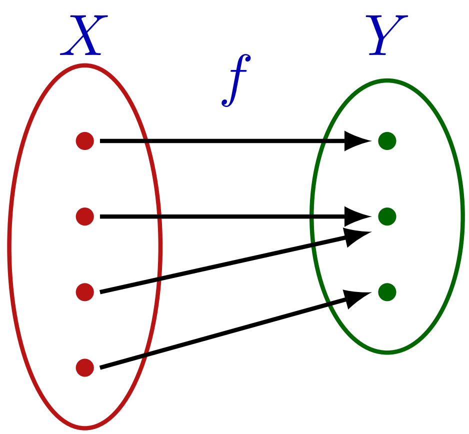

f is not bijective
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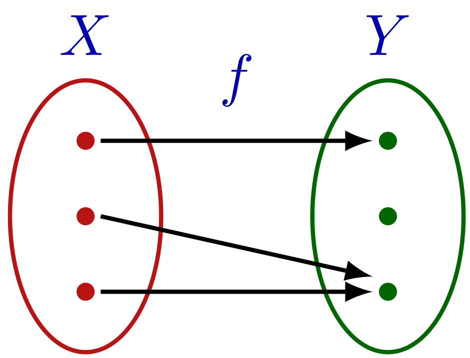


f is not bijective

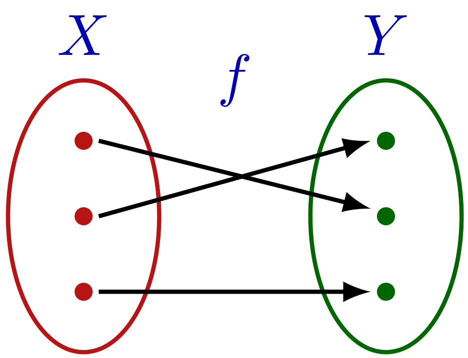
Bijective maps

Definition. Let $f : X \to Y$ be a map.

f is called **bijective** (or bijection, or one-to-one correspondence) if

f is injective and surjective: $\forall\, y \in Y \;\; \exists!\, x \in X \;\; y = f(x)$,

or, equivalently, $\forall\, y \in Y$ the equation $f(x) = y$ has a **unique** solution.



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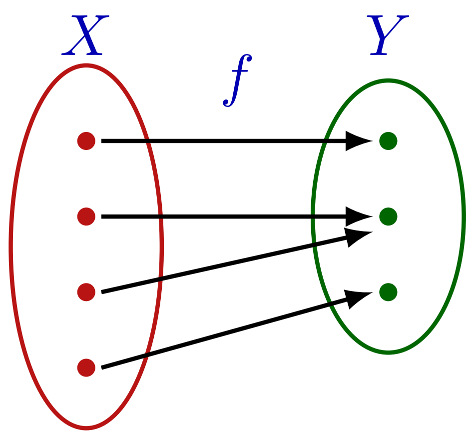

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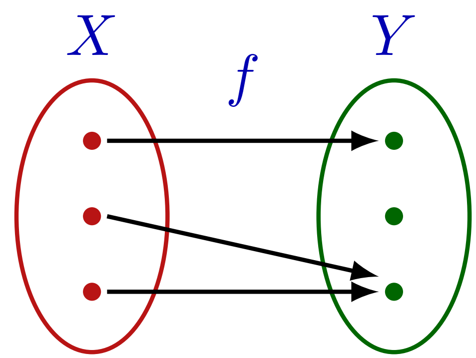

f is not bijective
(f is neither injective
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Linear function is bijective

Linear function is bijective

A linear map $f : \mathbb{R} \to \mathbb{R}$

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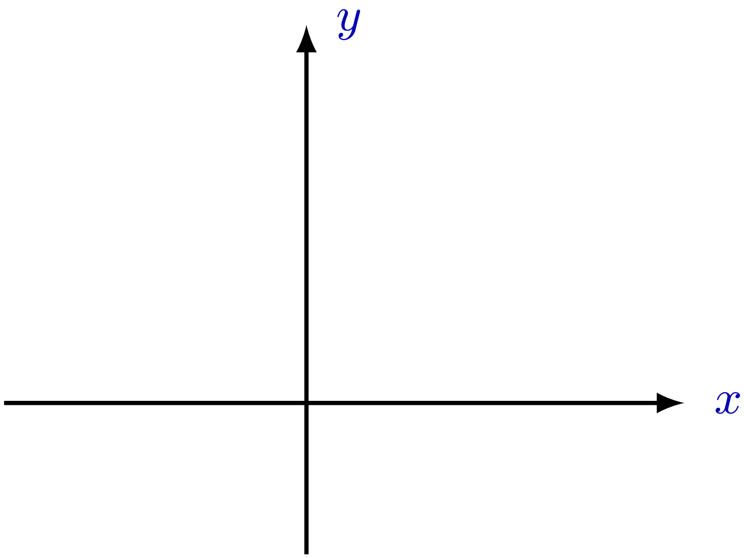

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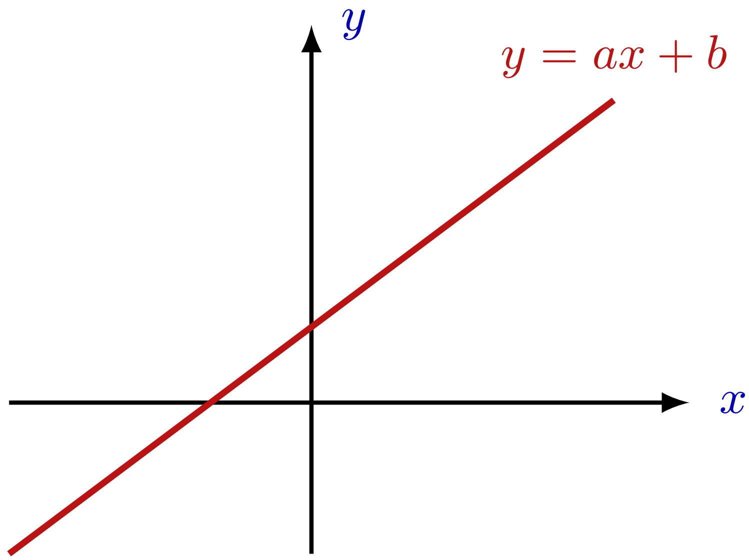

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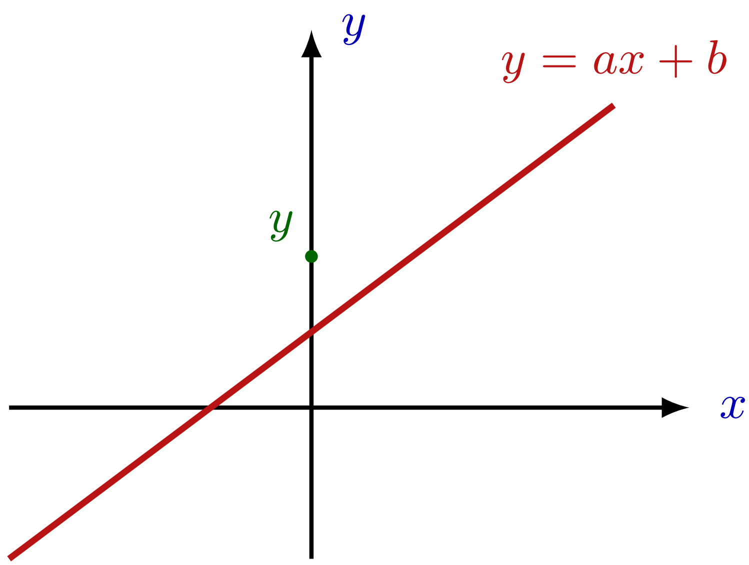

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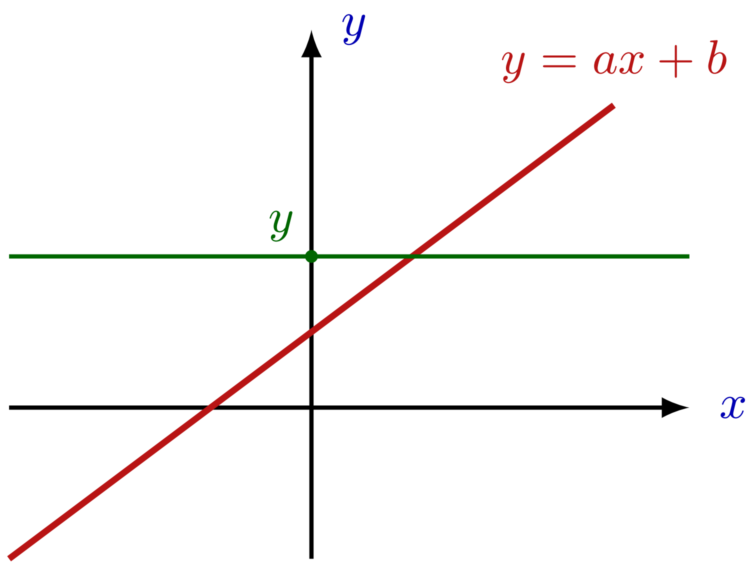

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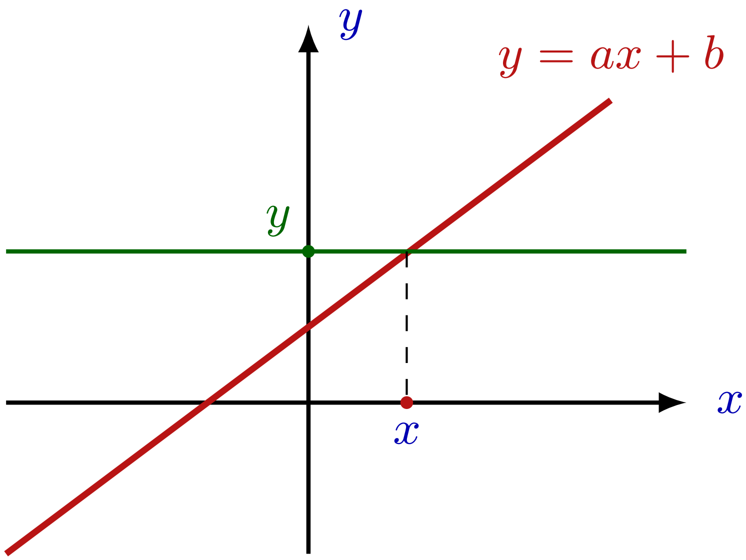

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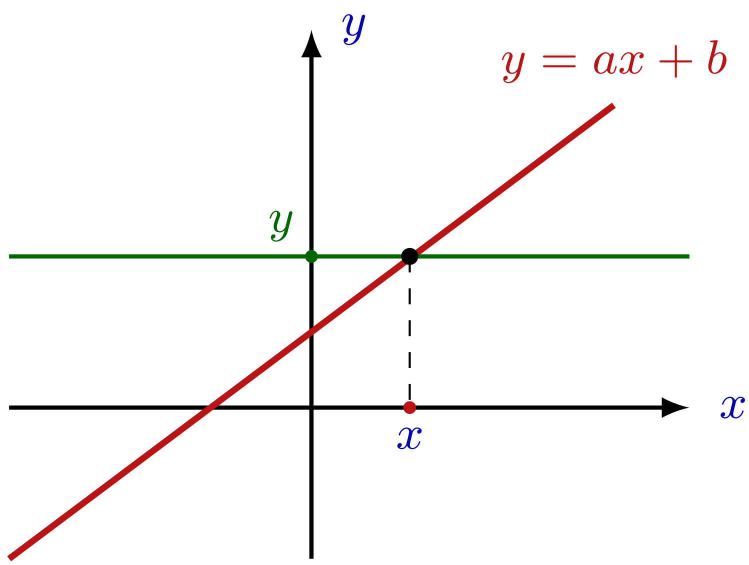

$$\forall\, y \in \mathbb{R} \quad \exists\,!\, x \in \mathbb{R} \quad y = ax + b$$

Inverse map

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Definition.

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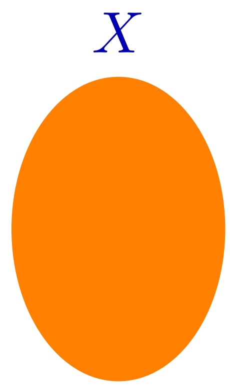

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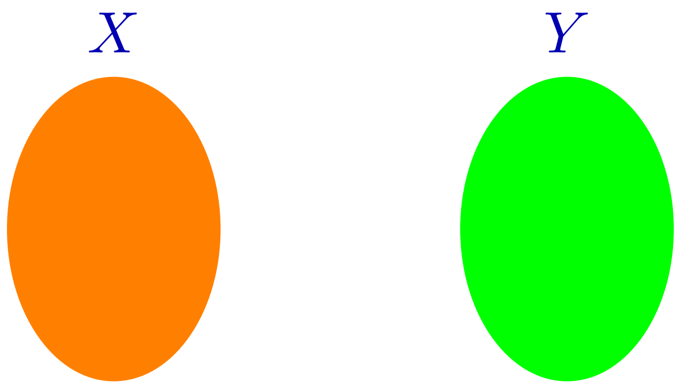

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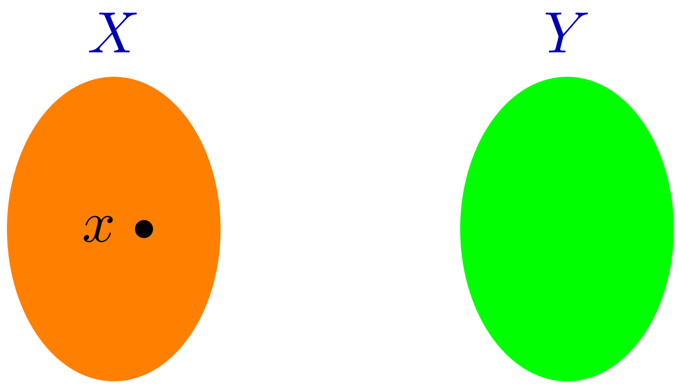

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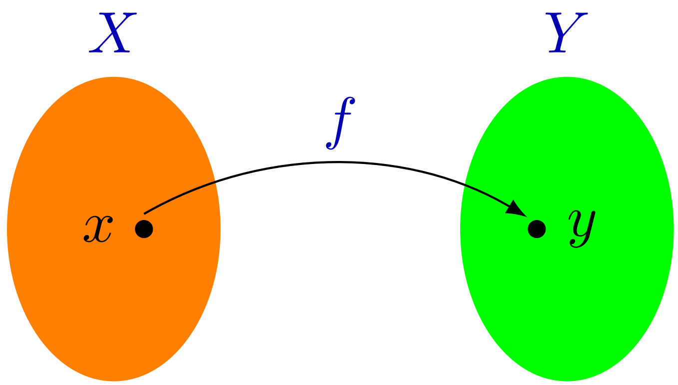

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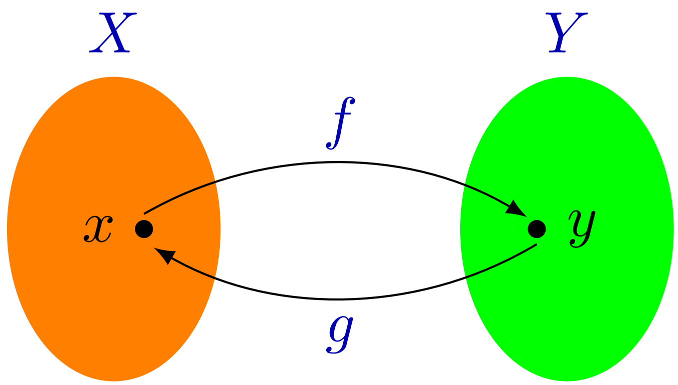

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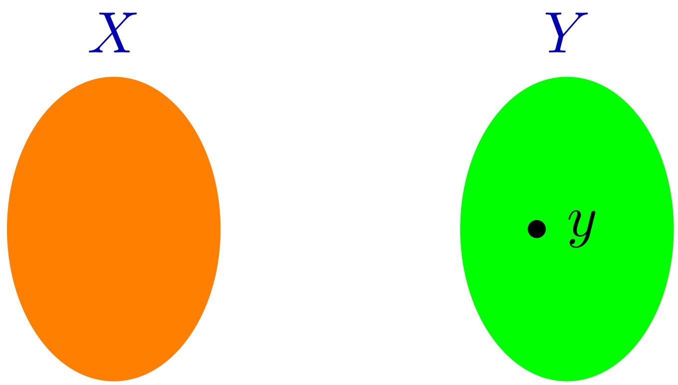

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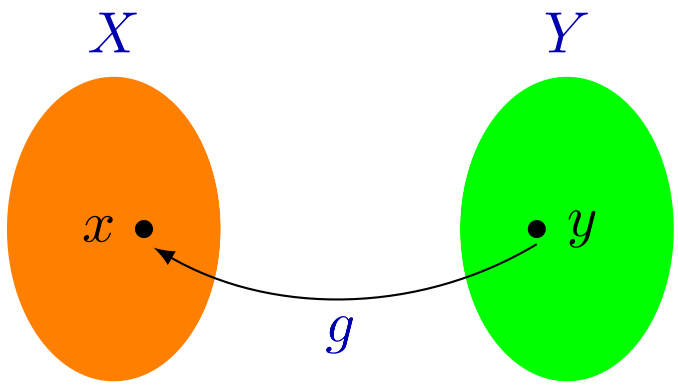

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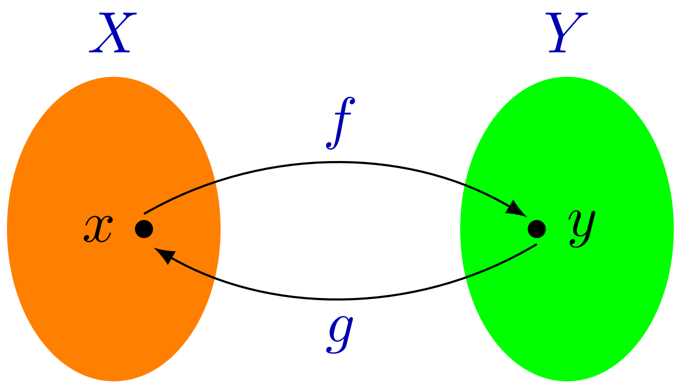

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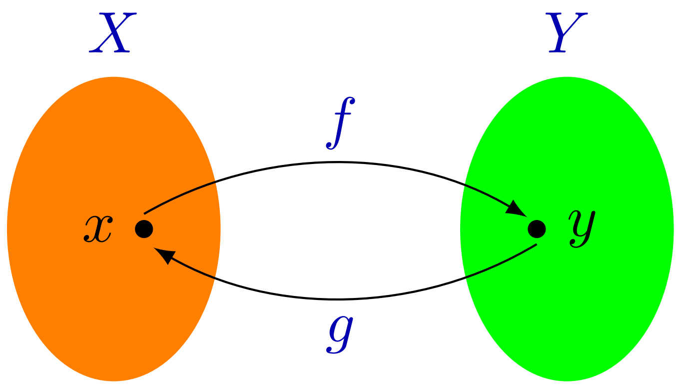

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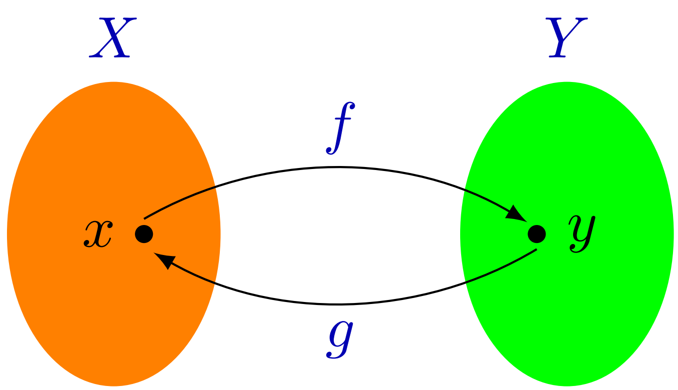

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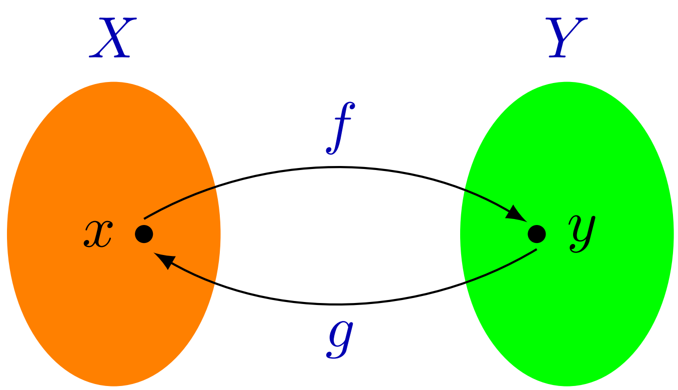

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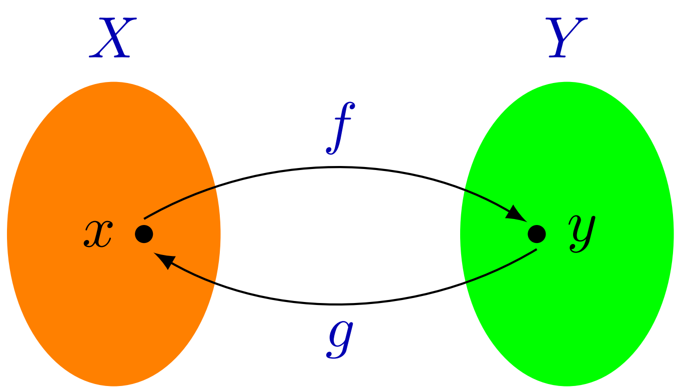

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Warning. Not all maps are invertible!

Inverse is unique

Inverse is unique

Theorem.

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Theorem. If an inverse map exists,

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Proof.

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Since the inverse map is **unique**, it deserves a **functional notation**.

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Therefore, $g = h$ and the inverse map is unique. $\square$

Since the inverse map is **unique**, it deserves a **functional notation**.

The inverse for f is denoted by f^{-1}.

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Therefore, $g = h$ and the inverse map is unique. □

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Theorem. If an inverse map exists, then it is **unique**.

Proof. Let $f : X \to Y$ has two inverse maps, g and h. $g, h : Y \to X$

Since g is an inverse for f, we have

$g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

Since h is an inverse for f, we have

$h \circ f = \text{id}_X$ and $f \circ h = \text{id}_Y$.

$g = g \circ \text{id}_Y = g \circ (f \circ h) = g \circ f \circ h = (g \circ f) \circ h = \text{id}_X \circ h = h$

Therefore, $g = h$ and the inverse map is unique. □

Since the inverse map is **unique**, it deserves a **functional notation**.

The inverse for f is denoted by f^{-1}. By the definition of the inverse,

$$\boxed{f^{-1} \circ f = \text{id}_X \quad \text{and} \quad f \circ f^{-1} = \text{id}_Y}$$

Bijection = invertible map

Bijection = invertible map

Theorem.

Bijection = invertible map

Theorem. A map is invertible iff

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Theorem. A map is invertible iff it is a bijection.

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Theorem. A map is invertible iff it is a bijection.

Proof.

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Proof. **Assume** that $f : X \to Y$ is **invertible** and prove that f is a bijection.

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Proof. Assume that $f : X \to Y$ is **invertible** and prove that f is a bijection.

To show **injectivity**,

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To show **injectivity**, assume that $f(x_1) = f(x_2)$

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Apply f^{-1}

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Apply f^{-1} (it exists since f is invertible)

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Apply f^{-1} (it exists since f is invertible) to this identity:

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By this, f is injective.

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Apply f^{-1} (it exists since f is invertible) to this identity:

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To show **surjectivity**, take any $y \in Y$

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To show **injectivity**, assume that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$.

Apply f^{-1} (it exists since f is invertible) to this identity:

$f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \implies x_1 = x_2$.

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To show **surjectivity**, take any $y \in Y$ and apply f^{-1} .

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$$f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \implies x_1 = x_2 \, .$$

By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1} . Let $x = f^{-1}(y)$.

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So for any $y \in Y$

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

such that $f(x)$

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1} . Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

such that $f(x) = f(f^{-1}(y))$

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

such that $f(x) = f(f^{-1}(y)) = y$.

Bijection = invertible map

Theorem. A map is invertible iff it is a bijection.

Proof. Assume that $f : X \to Y$ is **invertible** and prove that f is a bijection.

To show **injectivity**, assume that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$.

Apply f^{-1} (it exists since f is invertible) to this identity:

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

such that $f(x) = f(f^{-1}(y)) = y$.

By this, f is surjective.

Bijection = invertible map

Theorem. A map is invertible iff it is a bijection.

Proof. Assume that $f : X \to Y$ is **invertible** and prove that f is a bijection.

To show **injectivity**, assume that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$.

Apply f^{-1} (it exists since f is invertible) to this identity:

$f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \implies x_1 = x_2$.

By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

such that $f(x) = f(f^{-1}(y)) = y$.

By this, f is surjective.

We have proved that f is injective and surjective,

Bijection = invertible map

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To show **injectivity**, assume that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$.

Apply f^{-1} (it exists since f is invertible) to this identity:

$f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \implies x_1 = x_2$.

By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1} . Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

such that $f(x) = f(f^{-1}(y)) = y$.

By this, f is surjective.

We have proved that f is injective and surjective, therefore, f is **bijective**.

Bijection = invertible map

Theorem. A map is invertible iff it is a bijection.

Proof. Assume that $f : X \to Y$ is **invertible** and prove that f is a bijection.

To show **injectivity**, assume that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$.

Apply f^{-1} (it exists since f is invertible) to this identity:

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By this, f is injective.

To show **surjectivity**, take any $y \in Y$ and apply f^{-1}. Let $x = f^{-1}(y)$.

So for any $y \in Y$ there exists $x \in X$, namely $x = f^{-1}(y)$,

such that $f(x) = f(f^{-1}(y)) = y$.

By this, f is surjective.

We have proved that f is injective and surjective, therefore, f is **bijective**.

The half of the proof is done!

Bijection = invertible map

Assume now that f is a **bijection**,

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Assume now that f is a **bijection**, and prove that f is **invertible**.

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By definition of bijectivity,

Bijection = invertible map

Assume now that f is a **bijection**, and prove that f is **invertible**.

By definition of bijectivity, $\forall\, y \in Y \quad \exists!\, x \in X \quad y = f(x)$.

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Assume now that f is a **bijection**, and prove that f is **invertible**.

By definition of bijectivity, $\forall\, y \in Y \quad \exists !\, x \in X \quad y = f(x)$.

Define a map $g : Y \to X$

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By definition of bijectivity, $\forall\, y \in Y \quad \exists!\, x \in X \quad y = f(x)$.

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Let us prove that g is the inverse for f.

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$\forall\, x \in X \quad (g \circ f)(x)$

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$\forall\, x \in X \quad (g \circ f)(x) = g(f(x))$

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Let us prove that g is the inverse for f.

$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathrm{id}_X$.

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$\forall\, y \in Y \quad (f \circ g)(y)$

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Let us prove that g is the inverse for f.

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$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathrm{id}_X$.

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Let us prove that g is the inverse for f.

$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathrm{id}_X$.

$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathrm{id}_Y$.

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By definition of bijectivity, $\forall\, y \in Y \quad \exists!\, x \in X \quad y = f(x)$.

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Therefore, by the definition of the inverse,

Bijection = invertible map

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$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathsf{id}_X$.

$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathsf{id}_Y$.

Therefore, by the definition of the inverse, g is the **inverse** for f ,

Bijection = invertible map

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$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathsf{id}_Y$.

Therefore, by the definition of the inverse, g is the **inverse** for f, $g = f^{-1}$.

Bijection = invertible map

Assume now that f is a **bijection**, and prove that f is **invertible**.

By definition of bijectivity, $\forall\, y \in Y \quad \exists!\, x \in X \quad y = f(x)$.

Define a map $g : Y \to X$ by the formula $g(y) = x$, where $y = f(x)$.

Let us prove that g is the inverse for f.

$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathrm{id}_X$.

$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathrm{id}_Y$.

Therefore, by the definition of the inverse, g is the **inverse** for f, $g = f^{-1}$.

Thus, f is **invertible**.

Bijection = invertible map

Assume now that f is a **bijection**, and prove that f is **invertible**.

By definition of bijectivity, $\forall\, y \in Y \quad \exists!\, x \in X \quad y = f(x)$.

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Let us prove that g is the inverse for f.

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$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathrm{id}_Y$.

Therefore, by the definition of the inverse, g is the **inverse** for f, $g = f^{-1}$.

Thus, f is **invertible**. And the other half of the proof is done!

Bijection = invertible map

Assume now that f is a **bijection**, and prove that f is **invertible**.

By definition of bijectivity, $\forall\, y \in Y \quad \exists !\, x \in X \quad y = f(x)$.

Define a map $g : Y \to X$ by the formula $g(y) = x$, where $y = f(x)$.

Let us prove that g is the inverse for f.

$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathrm{id}_X$.

$\forall\, y \in Y \quad (f \circ g)(y) = f(g(y)) = f(x) = y$. So $f \circ g = \mathrm{id}_Y$.

Therefore, by the definition of the inverse, g is the **inverse** for f, $g = f^{-1}$.

Thus, f is **invertible**. And the other half of the proof is done! $\square$

Bijection = invertible map

Assume now that f is a **bijection**, and prove that f is **invertible**.

By definition of bijectivity, $\forall\, y \in Y \quad \exists!\, x \in X \quad y = f(x)$.

Define a map $g : Y \to X$ by the formula $g(y) = x$, where $y = f(x)$.

Let us prove that g is the inverse for f.

$\forall\, x \in X \quad (g \circ f)(x) = g(f(x)) = g(y) = x$. So $g \circ f = \mathrm{id}_X$.

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In these identities, what is f from the point of view of f^{-1}?

f is the inverse for f^{-1}!

Therefore, f^{-1} is invertible (and by this, is a bijection) and $(f^{-1})^{-1} = f$.

□

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□

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If f is strictly increasing or strictly decreasing on X ,
then it is called (strictly) **monotonic**.

Theorem. A monotonic function is injective.

Proof. Let $X, Y \subset \mathbb{R}$ and $f : X \to Y$, $x \mapsto f(x)$ be a function.

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Therefore, f is injective. $\square$

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the **definition** of logarithmic function as the inverse for exponential function,
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To avoid this ambiguity, always use $\arctan x$ as a notation for the inverse function for $\tan x$.

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