

Induction vs. strong induction

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Induction is used to prove statements of type $\forall n \in \mathbb{N} P(n)$.

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Mathematical **induction**:

$P(1)$

base

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$$\boxed{P(1)} \implies P(2)$$

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$$\boxed{P(1)} \implies P(2) \implies P(3)$$

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$$\boxed{P(1)} \implies P(2) \implies P(3) \implies \dots \implies \boxed{P(n) \implies P(n+1)}$$

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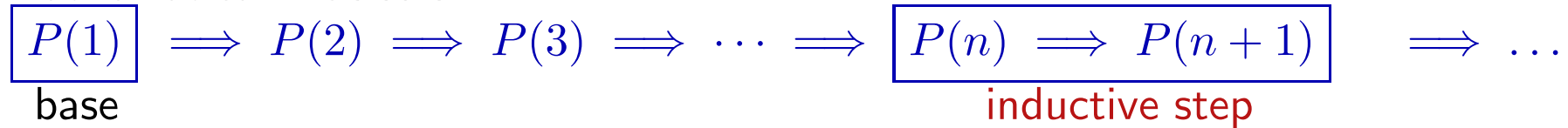
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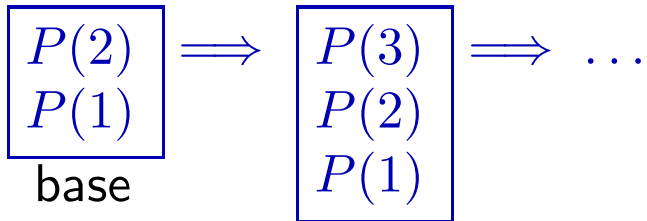
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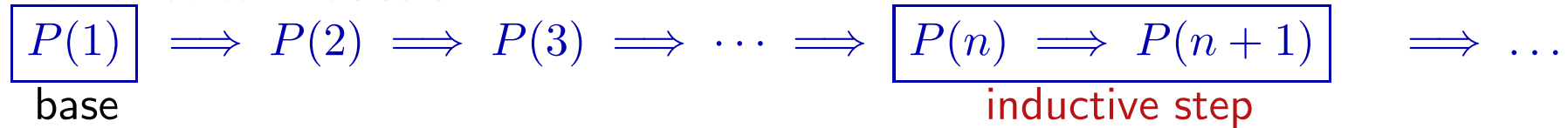
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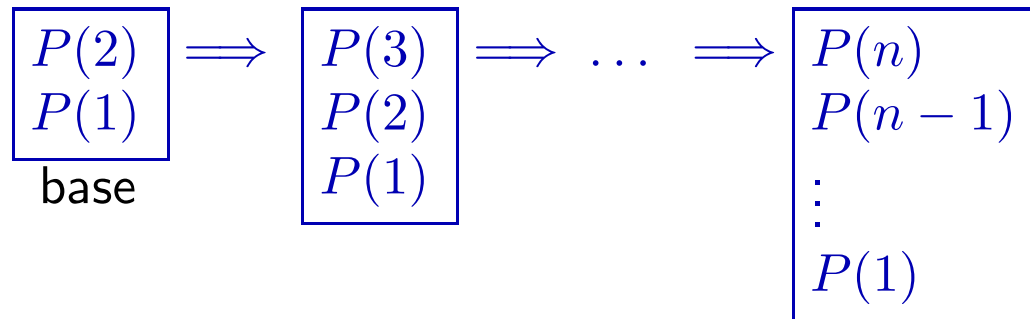
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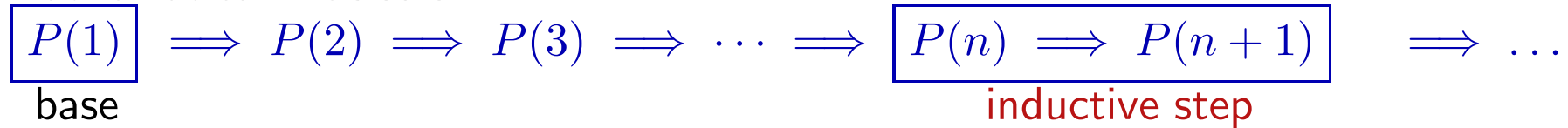
$$\boxed{\begin{matrix} P(2) \\ P(1) \end{matrix}} \implies \boxed{\begin{matrix} P(3) \\ P(2) \\ P(1) \end{matrix}} \implies \dots \implies \boxed{\begin{matrix} P(n) \\ P(n-1) \\ \vdots \\ P(1) \end{matrix}} \implies P(n+1)$$

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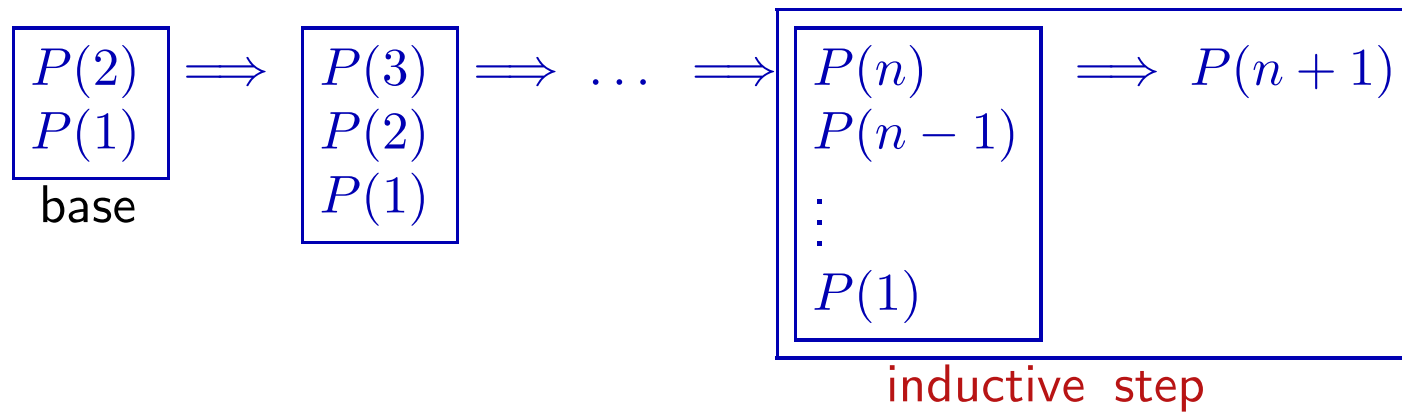
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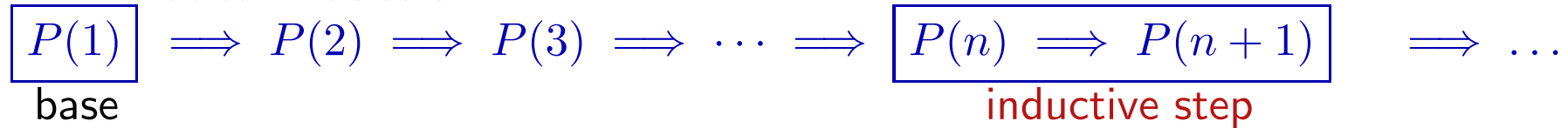
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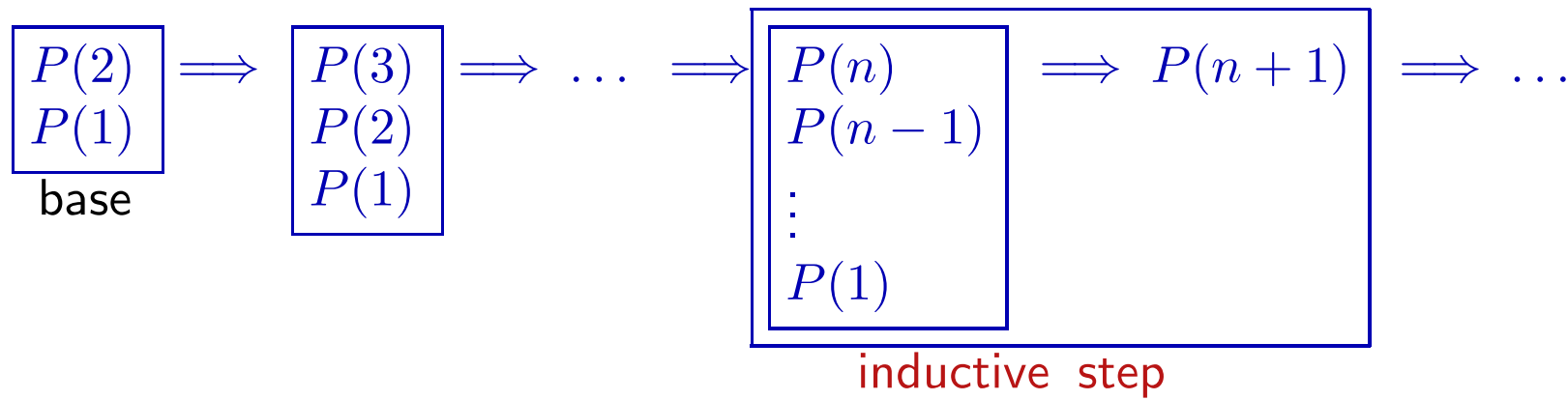
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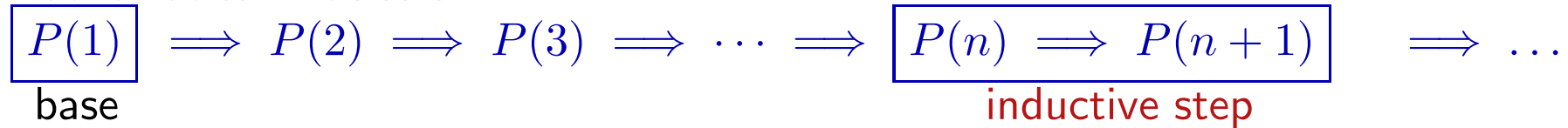
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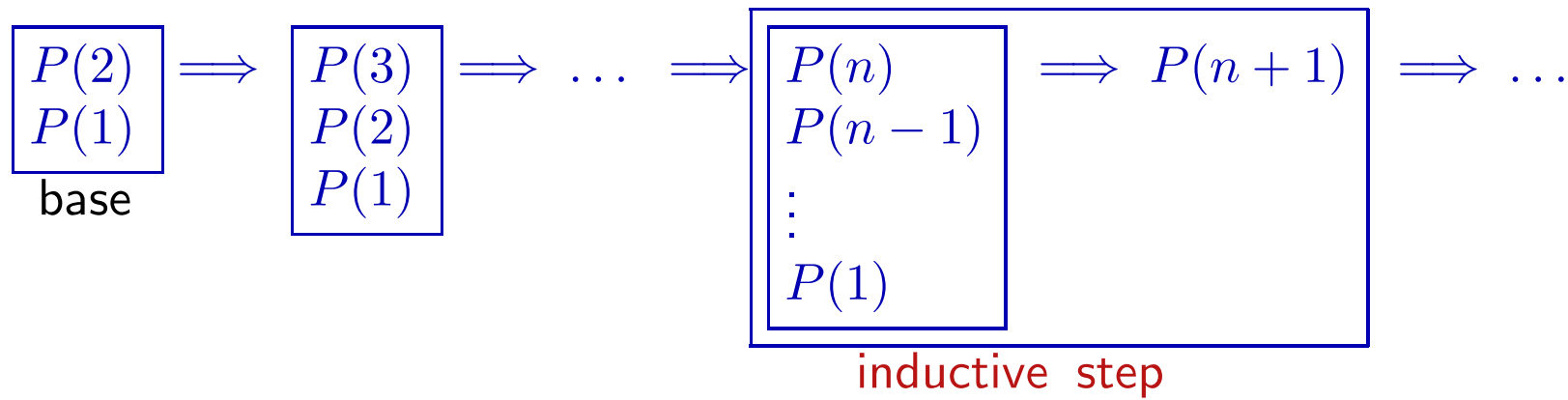
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- Mathematical induction and strong induction are **equivalent**.

General term of a sequence

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$$\underbrace{a_1 = 3, a_2 = 5}_{\text{initial values}} \text{ and } \underbrace{a_{n+1} = 3a_n - 2a_{n-1}}_{\text{recursive formula}} \text{ for } n \geq 2.$$

Show that $\underbrace{a_n = 2^n + 1}_{\text{general term of sequence}}$ for all $n \in \mathbb{N}$.

Solution. Given: $a_1 = 3, a_2 = 5, a_{n+1} = 3a_n - 2a_{n-1}$ for $n \geq 2$.

Prove: $\forall n \in \mathbb{N}$ $a_n = 2^n + 1$ $P(n)$

Proof (by strong induction).

- Base case:** For $n = 1$, $P(1)$ is $a_1 = 2^1 + 1$. Is it true? Yes, since $a_1 = 3$.
For $n = 2$, $P(2)$ is $a_2 = 2^2 + 1$. Is it true? Yes, since $a_2 = 5$.
For $n = 3$, $P(3)$ is $a_3 = 2^3 + 1 = 9$.

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3. Conclusion: Therefore, by the principle of strong induction,

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Fibonacci numbers

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Category of self-maps

Objects are maps $f : X \rightarrow X$, where X is a set.

A morphism $(X \xrightarrow{f} X) \rightarrow (Y \xrightarrow{g} Y)$ is a map $X \xrightarrow{h} Y$ such that $h \circ f = g \circ h$.

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Maps $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are called **conjugate** if they are isomorphic in the category of self-maps, i.e., there exists a (conjugating) bijection $h : X \rightarrow Y$ such that $g = h^{-1} \circ f \circ h$.

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Similarly, if $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are conjugate via $h : X \rightarrow Y$, then grand orbits of f are mapped by $h : X \rightarrow Y$ to grand orbits of g .

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Hence, we have proved

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Since f is assumed to be surjective, this means that f is not injective.

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Theorem. *In a finite set an injection may have only periodic grand orbits.* □

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Proof. Let $f : X \rightarrow X$ be a surjection. For each $a \in X$ choose $b \in f^{-1}(a)$ and put $b = g(a)$. The choice is possible, since f is surjective and $f^{-1}(a) \neq \emptyset$.

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Corollary 3. *For a map $X \rightarrow X$ of a finite set X ,
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Control question: Is this true for infinite sets?

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Proof. Exercise.

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Then $|X| \geq |U| = |\mathbb{N}|$, which contradicts to the assumption that X is finite. \square

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Arithmetic operations match: $|\mathbb{N}_p \times \mathbb{N}_q| = |\mathbb{N}_{pq}|$, etc.

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Pigeonhole Principle

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Example 5. Does there exist a number written by the digits 1 only
which is divisible by 2023?

Pigeonhole principle. Examples 6-9

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Example 6. Five points are placed randomly
inside an equilateral triangle with a side of length 1 inch.
Prove that among these five points
there are two points which are at most $1/2$ inches apart.

Example 7. Prove that there exists a power of 3 that ends up with digits 01.

Example 8. Prove that if 51 points are placed randomly in a unit square,
then one can always find three points
that can be covered by a circle of radius $1/7$.

Example 9. Given 6 points inside a unit circle. Prove that some two of them
are within 1 unit from each other.