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- Mathematical induction and strong induction are **equivalent**.

General term of a sequence

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Example.

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Example. A numerical sequence $\{a_n\}_{n=1}^{\infty}$

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Show that $\underbrace{a_n = 2^n + 1}_{\text{gerenal term of sequence}}$ for all $n \in \mathbb{N}$.

Solution. Given: $a_1 = 3,\ a_2 = 5,\ \ a_{n+1} = 3a_n - 2a_{n-1}$ for $n \geq 2$.

Prove: $\forall\, n \in \mathbb{N}$ $\boxed{a_n = 2^n + 1}$ $P(n)$

Proof (by strong induction).

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3. Conclusion: Therefore, by the principle of strong induction,

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Fibonacci numbers

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A morphism $(X \xrightarrow{f} X) \to (Y \xrightarrow{g} Y)$ is a map $X \xrightarrow{h} Y$ such that $h \circ f = g \circ h$.

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Similarly, if $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are conjugate via $h : X \to Y$, then grand orbits of f are mapped by $h : X \to Y$ to grand orbits of g.

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Theorem. *In a finite set an injection may have only periodic grand orbits.* □

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Corollary 3. *For a map $X \to X$ of a finite set X, injectivity, surjectivity and bijectivity are equivalent.* □

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Control question: Is this true for infinite sets?

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Proof. Exercise.

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Proof. Let X be a finite set. Recall that this means that $|X| < \aleph_0 = |\mathbb{N}|$.
There exists a subset $A \subset X$ such that there exists a bijection $f_A : A \to \mathbb{N}_n$.
For example $A = \varnothing$ and $n = 0$.
If $A \neq X$, choose $b \in X \smallsetminus A$,
and extend f_A to a map $A \cup \{b\} \to \mathbb{N}_{n+1}$ by mapping b to $n + 1$.
This cannot be done only if $X \smallsetminus A = \varnothing$, then $A = X$
and then the required bijection $X \to \mathbb{N}_n$ has been found.
If this would never happen, then the union U of the subsets of X
which have been chosen will be mapped bijectively onto $\mathbb{N} = \cup_{n=0}^{\infty} \mathbb{N}_n$.
Then $|X| \geq |U| = |\mathbb{N}|$, which contradicts to the assumption that X is finite. $\square$

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Arithmetic operations match: $|\mathbb{N}_p \times \mathbb{N}_q| = |\mathbb{N}_{pq}|$, etc.

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Example 5. Does there exists a number written by the digits 1 only
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Pigeonhole principle. Examples 6-9

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Example 6. Five points are placed randomly inside an equilateral triangle with a side of length 1 inch. Prove that among these five points there are two points which are at most $1/2$ inches apart.

Example 7. Prove that there exists a power of 3 that ends up with digits 01.

Example 8. Prove that if 51 points are placed randomly in a unit square, then one can always find three points that can be covered by a circle of radius $1/7$.

Example 9. Given 6 points inside a unit circle. Prove that some two of them are within 1 unit from each other.