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In both categories, isomorphism classes of objects are called **order types**.

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We add one element from $X \smallsetminus X_n$ to X_n and map it to one element from the corresponding part of $Y \smallsetminus Y_n$.

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We will construct an order-isomorphism between X and Y inductively. On the nth step we will build an order-isomorphism between n-element subsets $X_n \subset X$ and $Y_n \subset Y$. At the first step we take any singletons $X_1 \subset X$ and $Y_1 \subset Y$.

Inductive step. The complement $X \smallsetminus X_n$ is divided by elements of X_n into $n-1$ open segments and two rays.

We add one element from $X \smallsetminus X_n$ to X_n and map it to one element from the corresponding part of $Y \smallsetminus Y_n$. In order to obtain eventually a bijection $X \to Y$, we have to take care of including each element in X_n and Y_n for some n.

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Compare with the Cantor Theorem.

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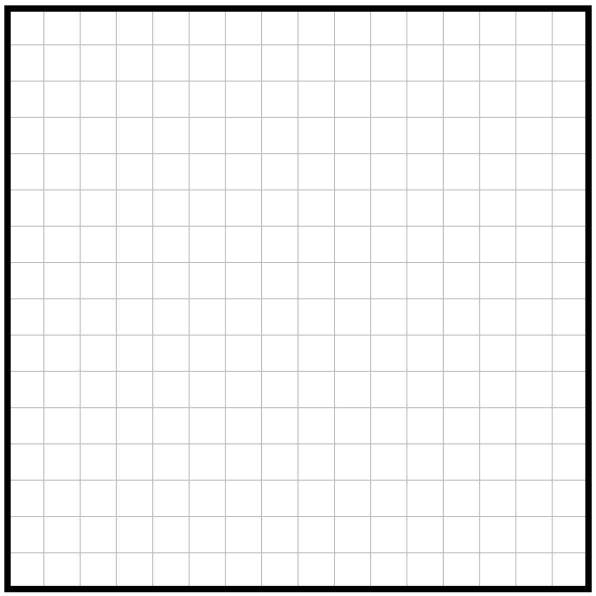

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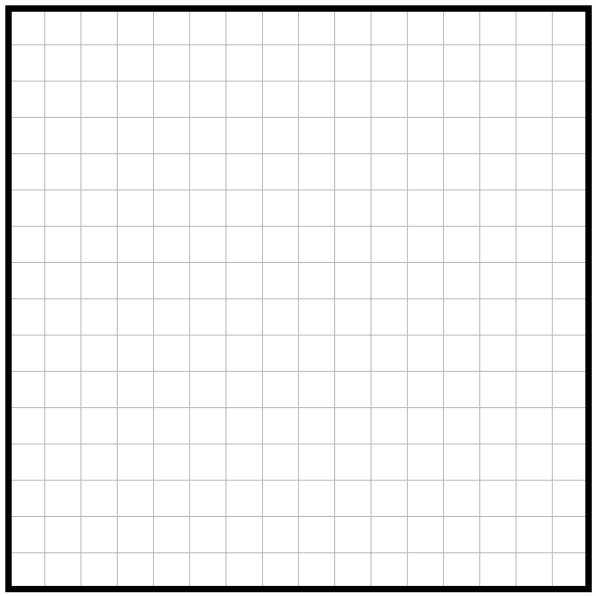

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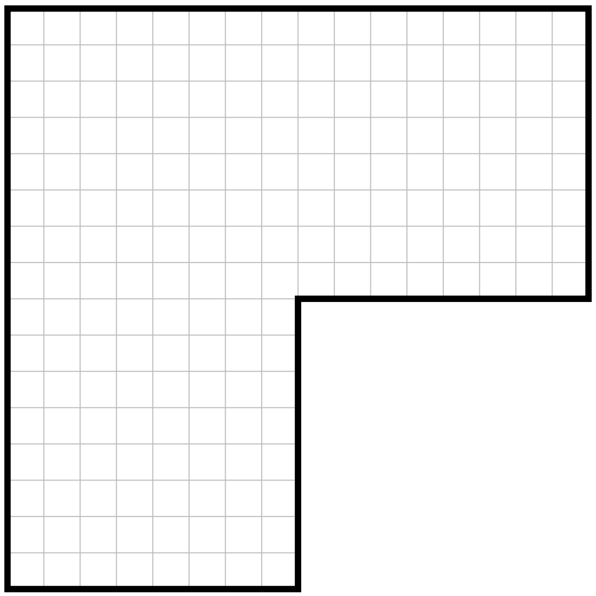

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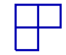
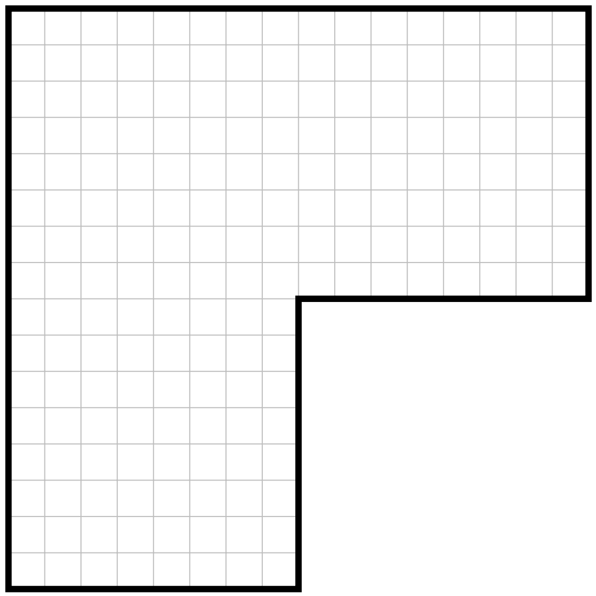
Prove that it is possible to tile the remaining shape by the corners 



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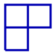
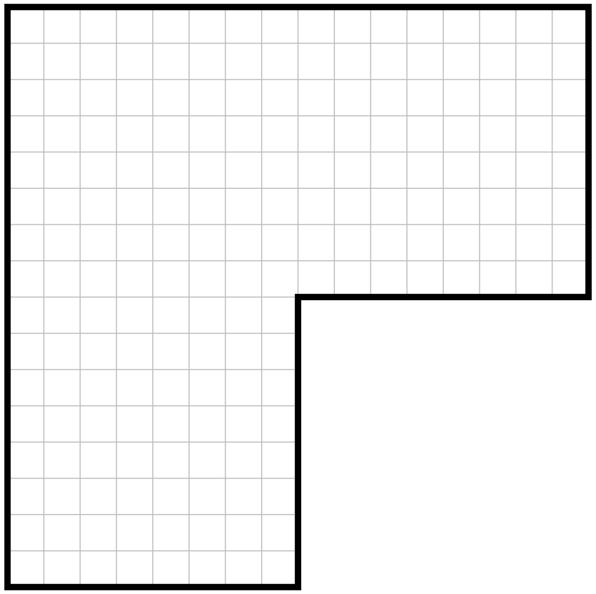
Prove that it is possible to tile the remaining shape by the corners 



Discussion.

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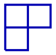
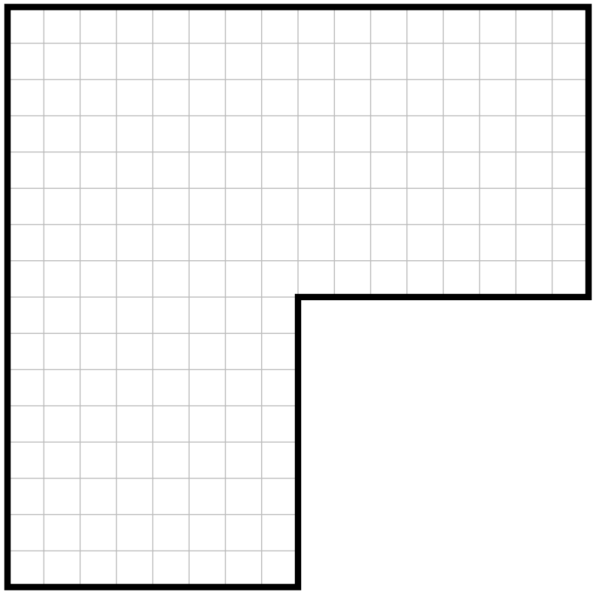
Prove that it is possible to tile the remaining shape by the corners 



Discussion. The problem seems to be difficult.

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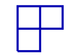
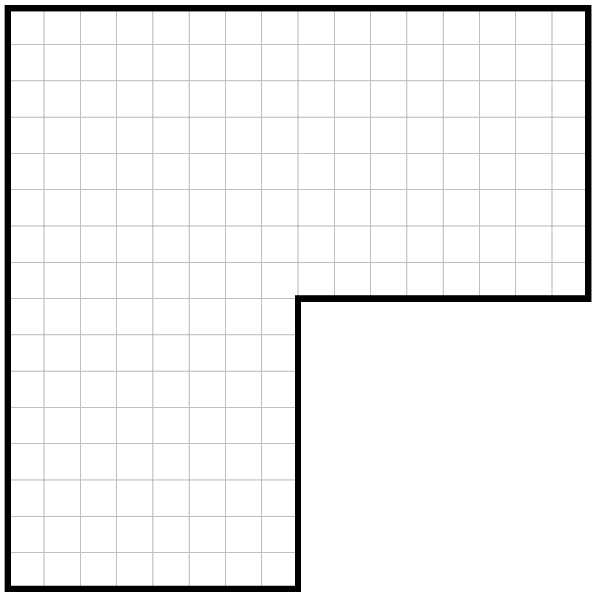
Prove that it is possible to tile the remaining shape by the corners 



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There are too many small squares...

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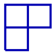
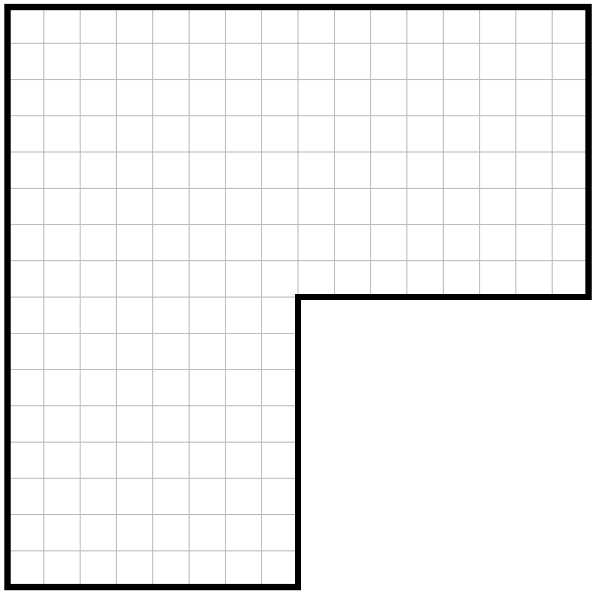
Prove that it is possible to tile the remaining shape by the corners 



Discussion. The problem seems to be difficult.

There are too many small squares...

Can we think about a simpler yet easier problem?

4×4 shape

4×4 shape

Problem. Given a 4×4 square with a quarter removed.

4×4 shape

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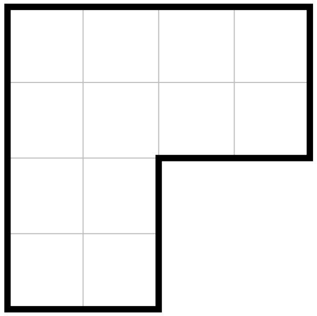

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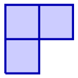
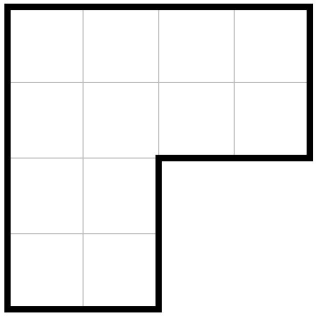
Prove that it is possible to tile this shape by the corners 



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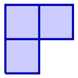
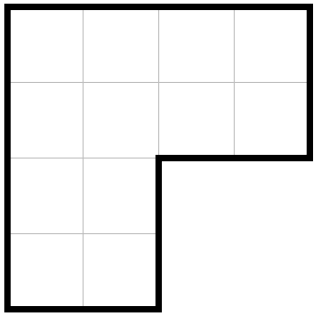
Prove that it is possible to tile this shape by the corners 



Solution.

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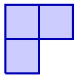
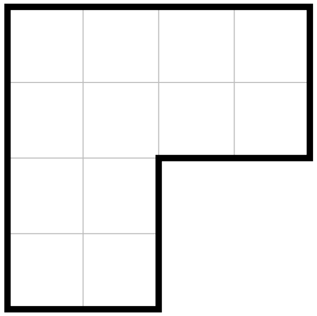
Prove that it is possible to tile this shape by the corners 



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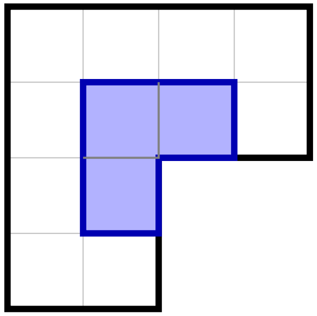

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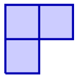
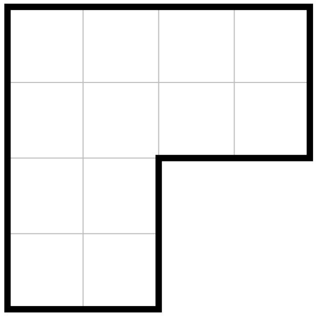
Prove that it is possible to tile this shape by the corners 



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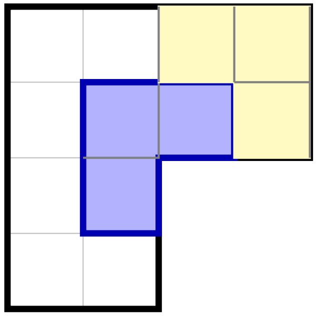

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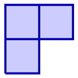
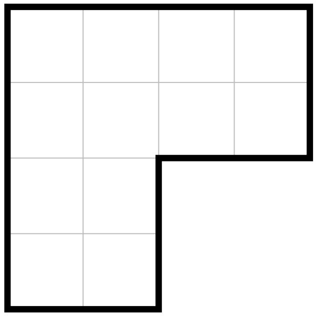
Prove that it is possible to tile this shape by the corners 



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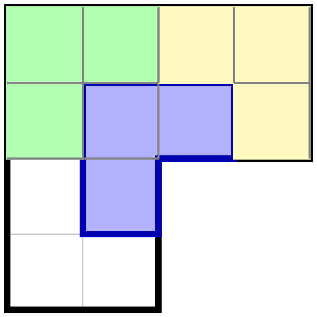

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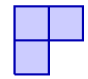
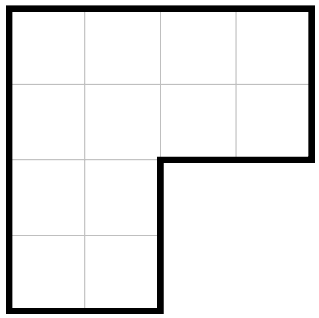
Prove that it is possible to tile this shape by the corners 



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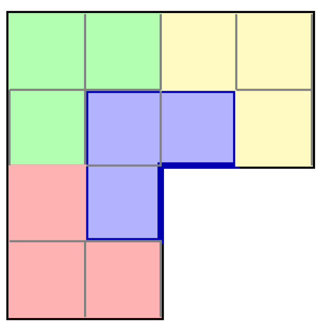

What's next?

What's next?

Can we solve now the problem about 16×16 shape?

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Can we solve now the problem about 16×16 shape? Not quite yet...

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Let's do the 8×8 shape:

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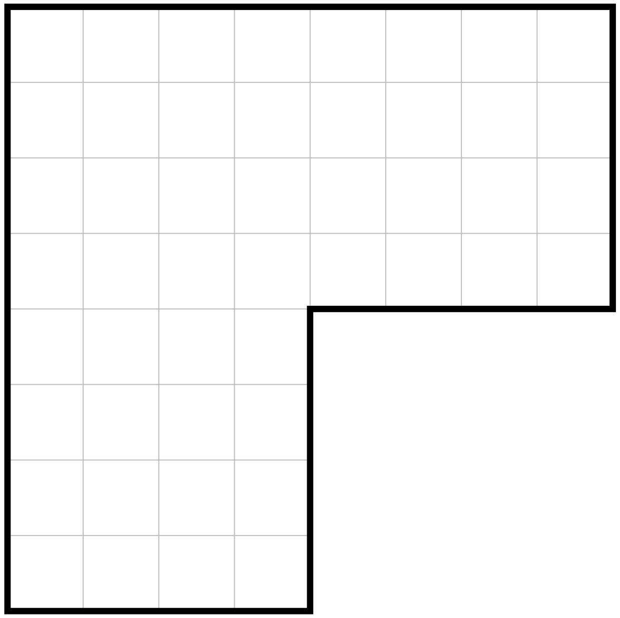

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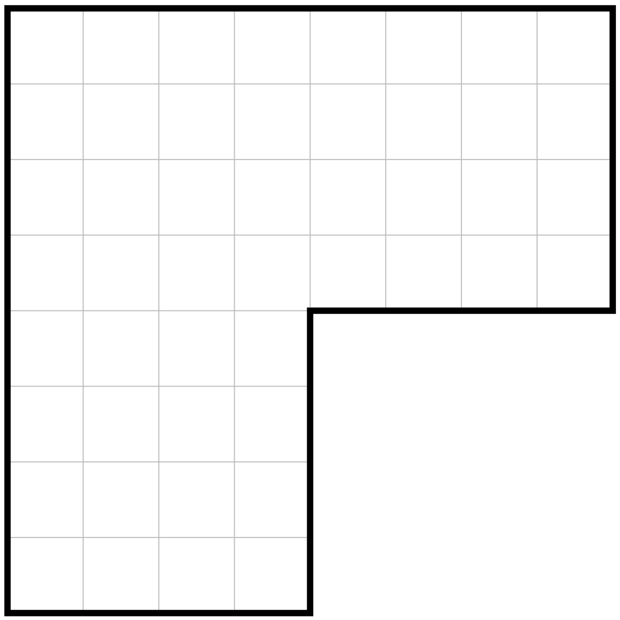

Do you see the previous 4×4 shapes here?

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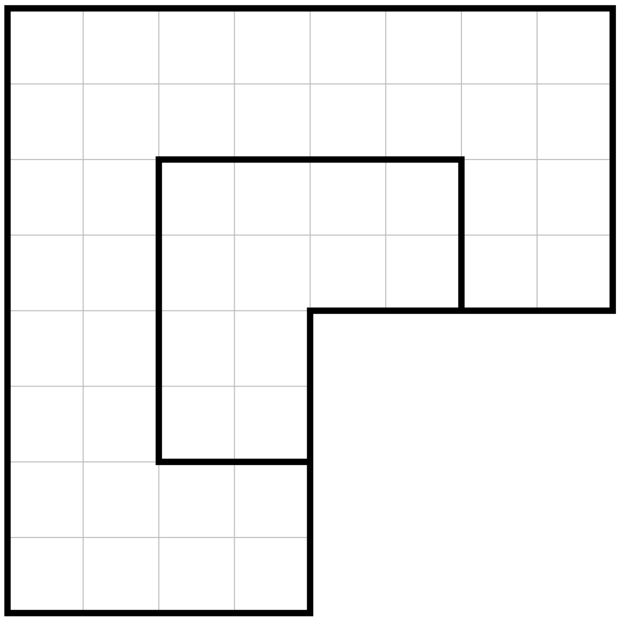

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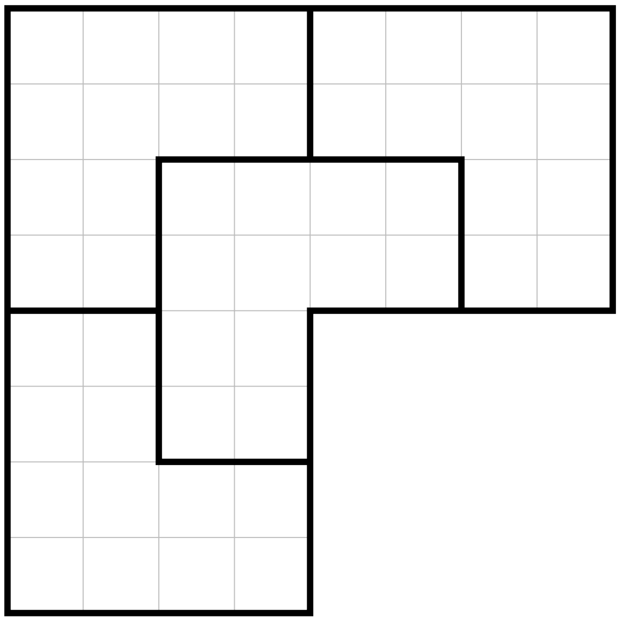

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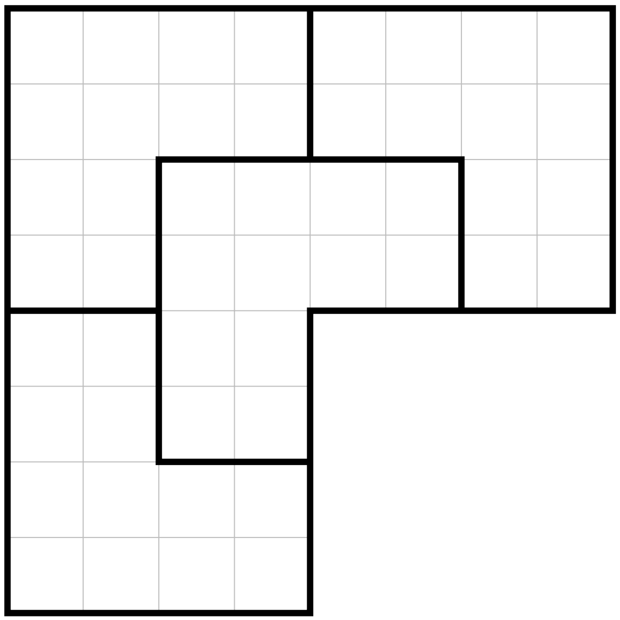

Do you see the previous 4×4 shapes here?
For each of these 4×4 shapes the problem has been already solved!

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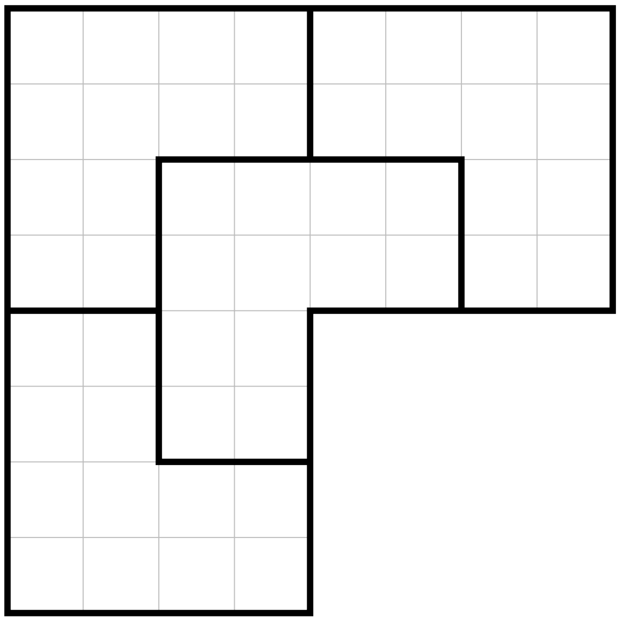

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We can tile each of 4×4 shapes by the small corners,

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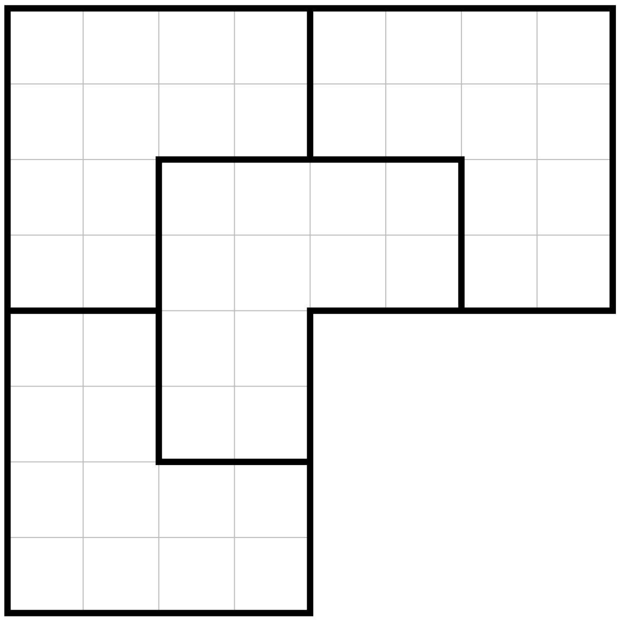

Do you see the previous 4×4 shapes here?
For each of these 4×4 shapes the problem has been already solved!
We can tile each of 4×4 shapes by the small corners,
so we can tile the 8×8 shape by the small corners.

16×16 shape

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Let us now solve the original problem about 16×16 shape.

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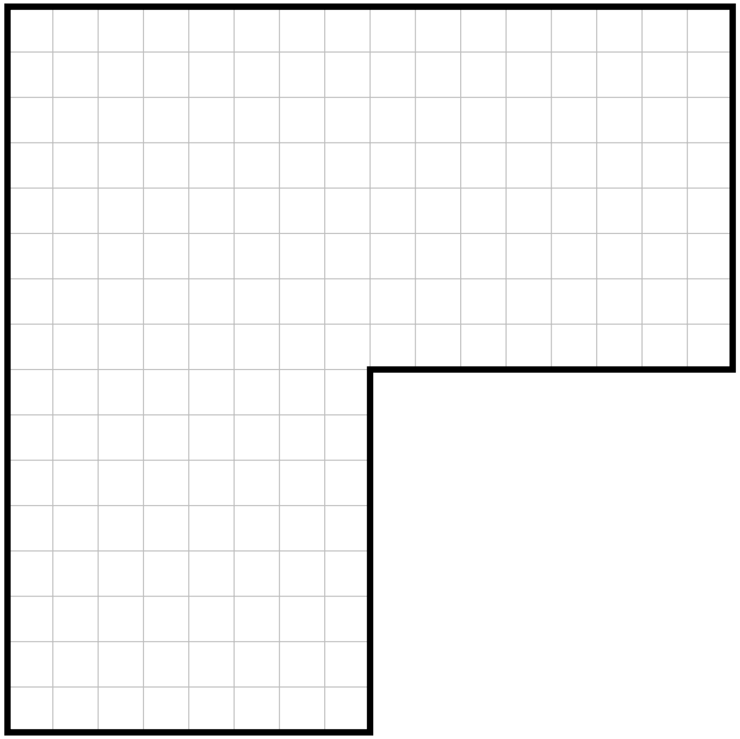

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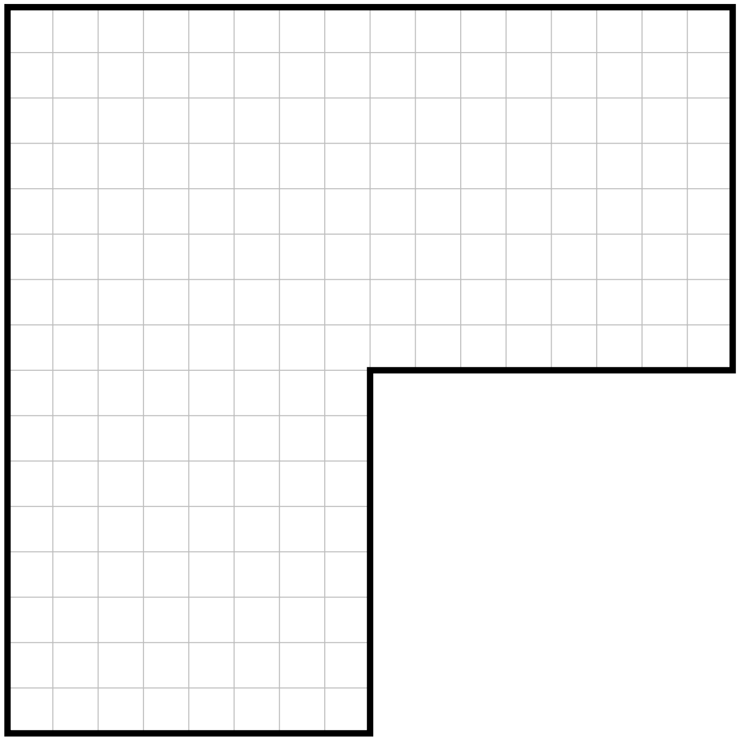

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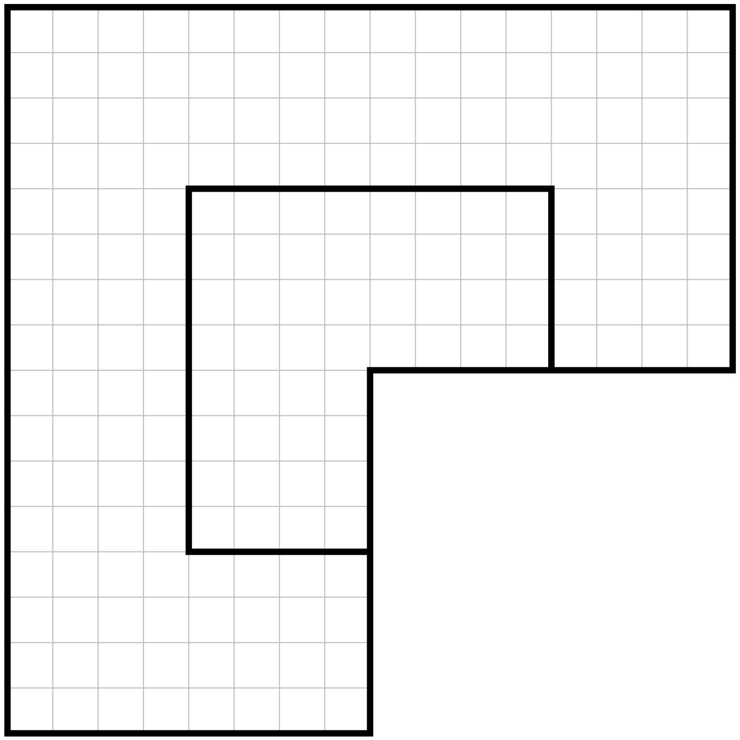

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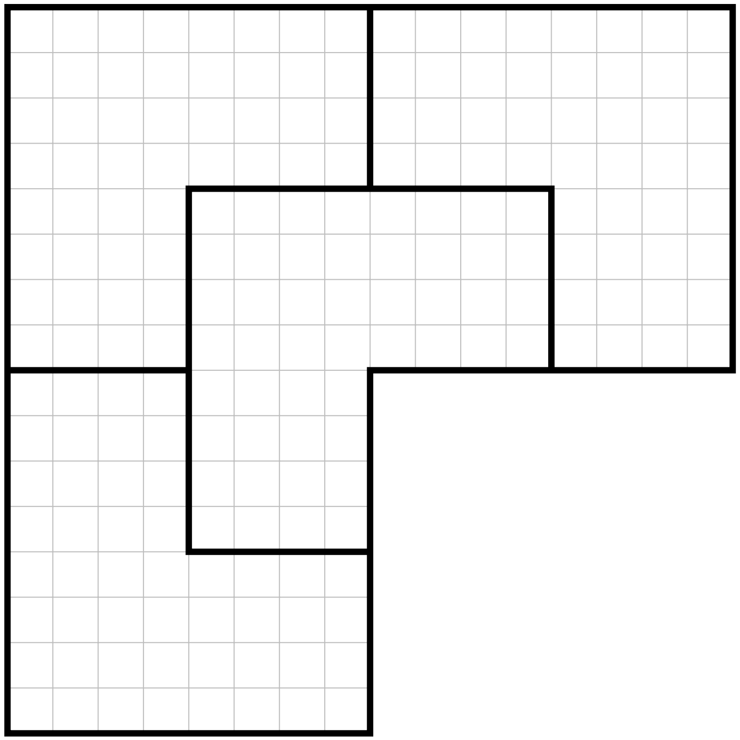

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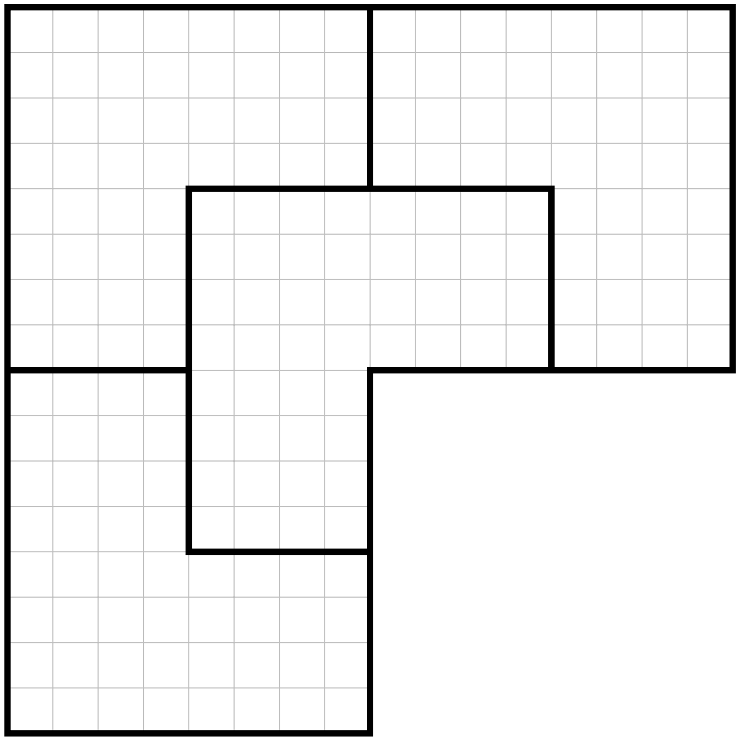

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We can tile each of the 8×8 shapes by the small corners,

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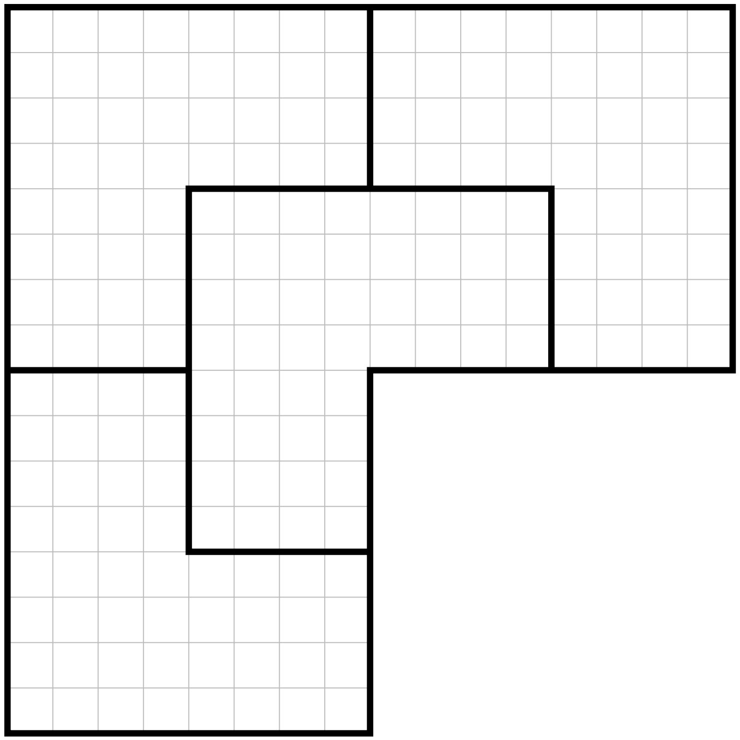

Do you see 8×8 shapes here?
We can tile each of the 8×8 shapes by the small corners,
so we can tile the 16×16 shape by the small corners.

Go further!

Go further!

The 16×16 problem has been solved,

Go further!

The 16×16 problem has been solved, but we may go further!

Go further!

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Let us **generalize** the problem.

Go further!

The 16×16 problem has been solved, but we may go further!

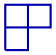
Let us **generalize** the problem.

Problem. Given a $2^n \times 2^n$ square with a lower right quarter removed.

Go further!

The 16×16 problem has been solved, but we may go further!

Let us **generalize** the problem.

Problem. Given a $2^n \times 2^n$ square with a lower right quarter removed. Prove that this shape can be tiled by small corners .

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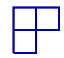
Prove that this shape can be tiled by small corners .

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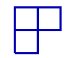
Problem. Given a $2^n \times 2^n$ square with a lower right quarter removed. Prove that this shape can be tiled by small corners .

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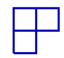
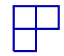
Prove that this shape can be tiled by small corners .

Solution. We have shown how to solve 4×4 problem,
and how to pass to the solution for 8×8 problem,

Go further!

The 16×16 problem has been solved, but we may go further!

Let us **generalize** the problem.

Problem. Given a $2^n \times 2^n$ square with a lower right quarter removed.
Prove that this shape can be tiled by small corners .

Solution. We have shown how to solve 4×4 problem,
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The 16×16 problem has been solved, but we may go further!

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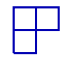
Prove that this shape can be tiled by small corners .

Solution. We have shown how to solve 4×4 problem,
and how to pass to the solution for 8×8 problem,
and how then to pass to the solution for 16×16 problem:

$$4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16\,.$$

Go further!

The 16×16 problem has been solved, but we may go further!

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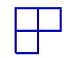
Prove that this shape can be tiled by small corners .

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We can go further to 32×32 problem

Go further!

The 16×16 problem has been solved, but we may go further!

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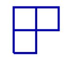
Prove that this shape can be tiled by small corners .

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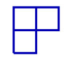
$$4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16\,.$$

We can go further to 32×32 problem
by tiling the 32×32 shape by four 16×16 shapes:

Go further!

The 16×16 problem has been solved, but we may go further!

Let us **generalize** the problem.

Problem. Given a $2^n \times 2^n$ square with a lower right quarter removed. Prove that this shape can be tiled by small corners .

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We can go further to 32×32 problem
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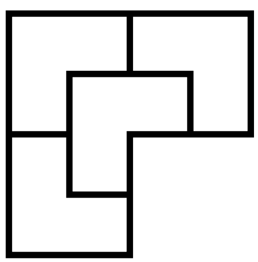

Generalized problem

By continuing the procedure,

Generalized problem

By continuing the procedure, we will reach $2^n \times 2^n$ shape:

Generalized problem

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$$4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16 \longrightarrow 32 \times 32 \longrightarrow \cdots \longrightarrow 2^n \times 2^n$$

Generalized problem

By continuing the procedure, we will reach $2^n \times 2^n$ shape:

$$4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16 \longrightarrow 32 \times 32 \longrightarrow \cdots \longrightarrow 2^n \times 2^n \longrightarrow \ldots$$

Generalized problem

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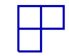
It's natural to start with 2×2 problem, representing the small corner 

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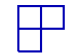
It's natural to start with 2×2 problem, representing the small corner 

$$2 \times 2 \longrightarrow 4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16 \longrightarrow 32 \times 32 \longrightarrow \cdots \longrightarrow 2^n \times 2^n \longrightarrow \dots$$

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By continuing the procedure, we will reach $2^n \times 2^n$ shape:

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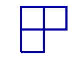
It's natural to start with 2×2 problem, representing the small corner 

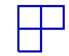
$$2 \times 2 \longrightarrow 4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16 \longrightarrow 32 \times 32 \longrightarrow \cdots \longrightarrow 2^n \times 2^n \longrightarrow \ldots$$

This step-by-step process of deriving each statement from the previous one

Generalized problem

By continuing the procedure, we will reach $2^n \times 2^n$ shape:

$$4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16 \longrightarrow 32 \times 32 \longrightarrow \cdots \longrightarrow 2^n \times 2^n \longrightarrow \ldots$$

It's natural to start with 2×2 problem, representing the small corner 

$$2 \times 2 \longrightarrow 4 \times 4 \longrightarrow 8 \times 8 \longrightarrow 16 \times 16 \longrightarrow 32 \times 32 \longrightarrow \cdots \longrightarrow 2^n \times 2^n \longrightarrow \ldots$$

This step-by-step process of deriving each statement from the previous one is called *induction*.

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Principle of Mathematical Induction (to prove $\forall n \in \mathbb{N} \ \ P(n)$):

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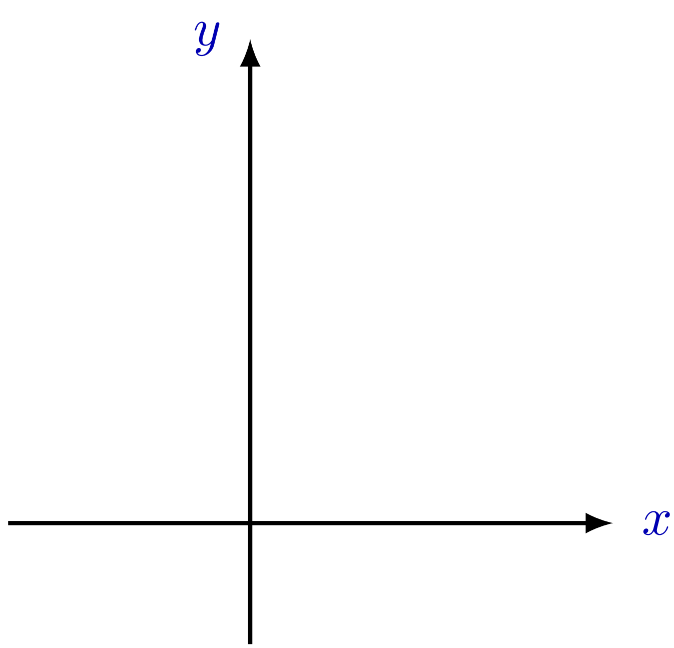

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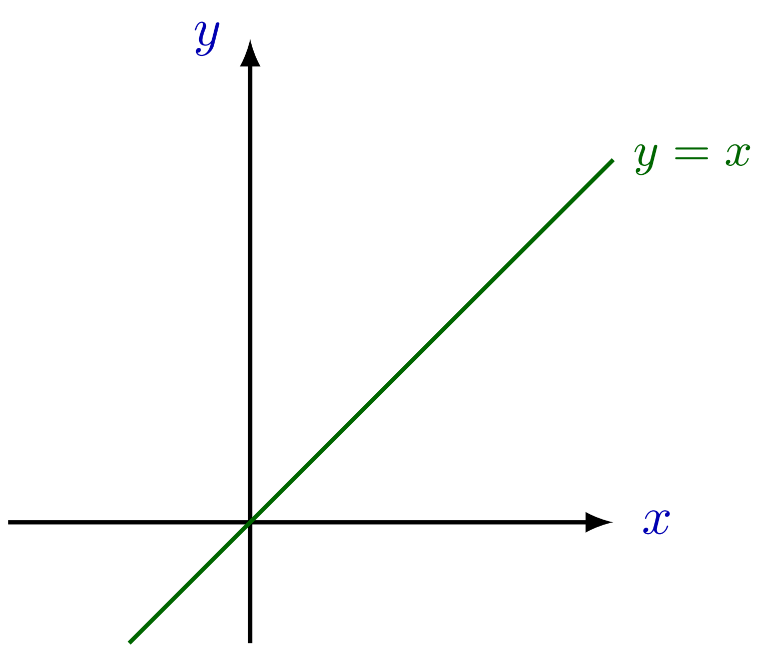

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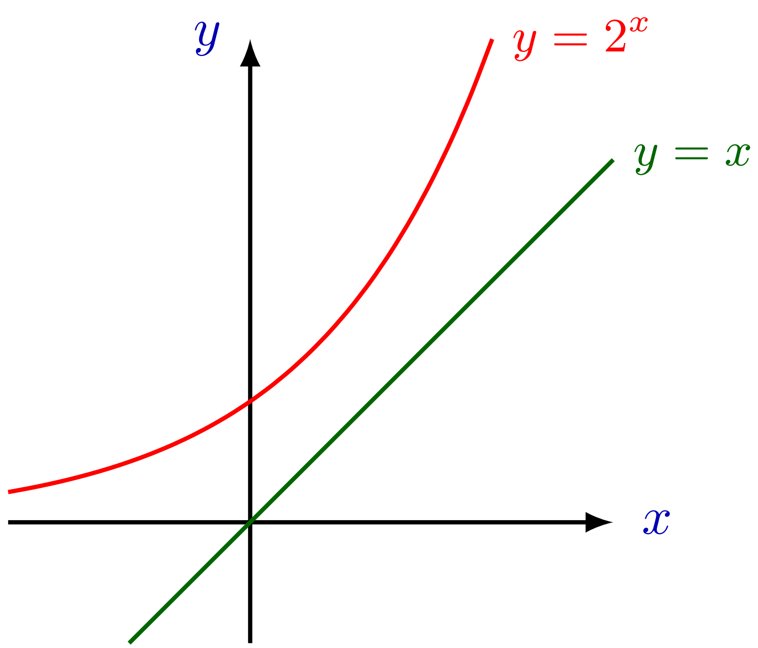

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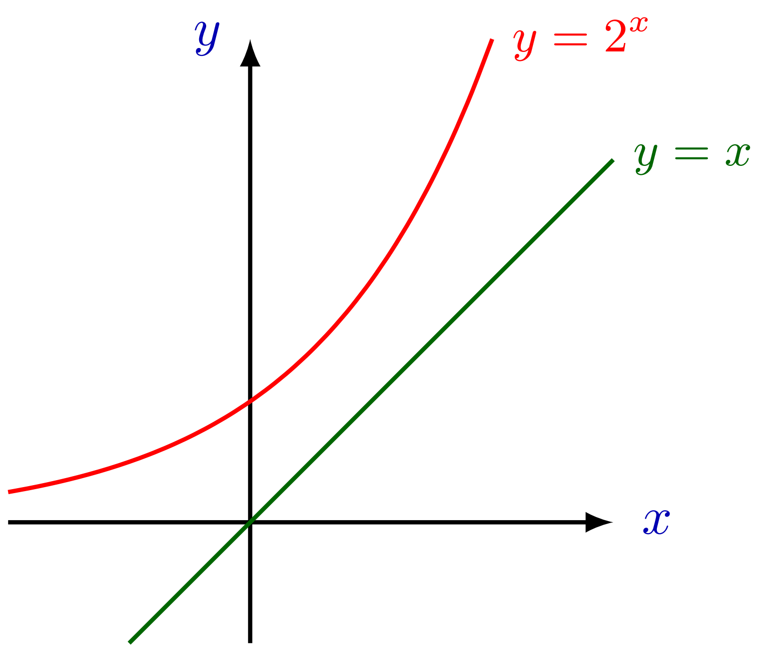

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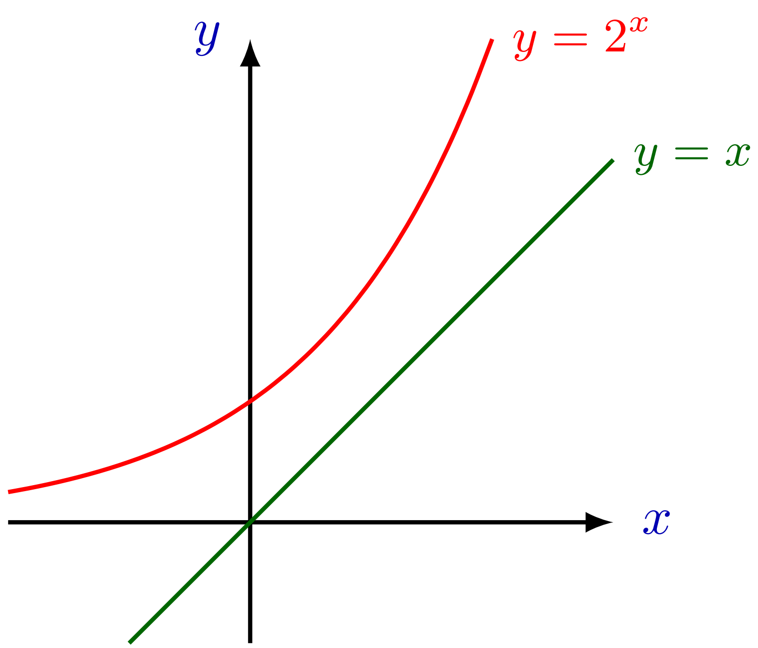

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The implication

$\forall x \in \mathbb{R} \;\; x < 2^x \implies \forall n \in \mathbb{N} \;\; n < 2^n$

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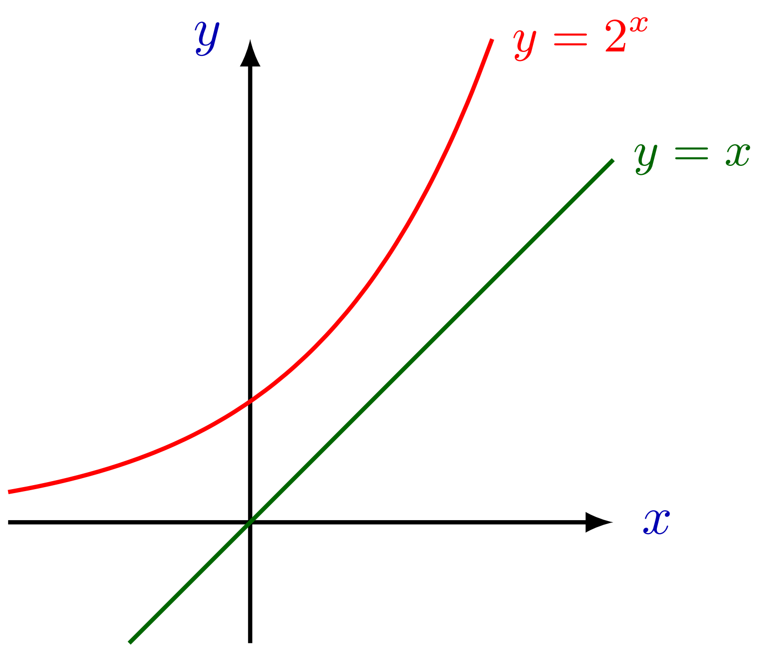

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is called **specialization**.

Proving summation formula

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Problem 2.

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Problem 2. Prove that $\displaystyle\sum_{j=1}^{n}(2j-1) = n^2$

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Discussion.

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and prove that $\sum_{j=1}^{k+1}(2j-1) = (k+1)^2$.

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3. Conclusion: Therefore, by the principle of mathematical induction,

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$\square$

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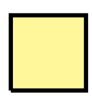

1

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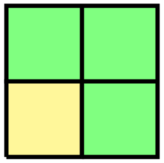

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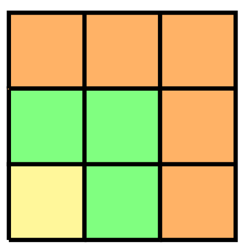

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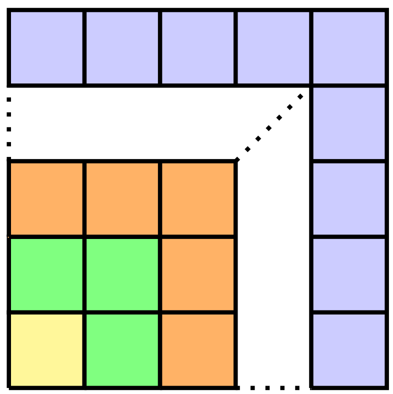

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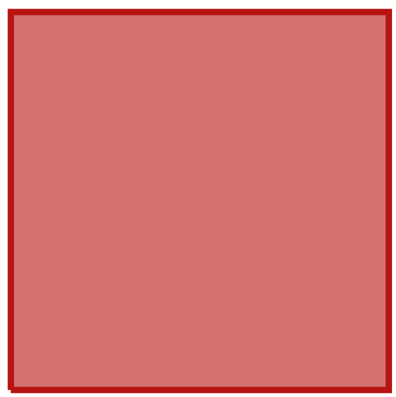

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

Formula for product

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Problem 3.

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Problem 3. Prove that $\displaystyle\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}$

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Discussion. $\prod$ is a sign for a **product**:

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3. Conclusion: Therefore, by the principle of mathematical induction,

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n} \quad \text{for all } n \geq 2.$$

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Problem 4.

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Conclusion. Therefore, $4^{2n-1} + 3^{n+1}$ is divisible by 13 for any $n \in \mathbb{N}$.

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• Mathematical induction and strong induction are **equivalent**.

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Example.

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2. **Inductive step:**

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3. Conclusion: Therefore, by the principle of strong induction,

$$a_n = 2^n + 1 \text{ for all } n \in \mathbb{N}.$$

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