Numbers

How to create other numbers

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Let us do this human work.

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Notice that $a + d$, $b + c$ are natural numbers.

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Indeed, $x \cdot (1/x) = [(a, b)][(b, a)] = [(ab, ba)] = [(1, 1)] = 1$.

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Which sets do have the same supremum?

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The map $\mathbb{Q} \to \mathcal{B}^+(\mathbb{Q})/_\sim, \;\; x \mapsto [\{x\}]$ is injective,
so we may identify a rational number x with the class $[\{x\}] \in \mathbb{R}$.

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Exercise. Prove that the relation $\leq$ above is a non-strict total order.
Show also that it agrees with $\leq$ on $\mathbb{Q}$.

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One can easily prove that $\equiv \mod (x^2+1)$ is an equivalence relation on $\mathbb{R}[x]$.

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