Lecture 11

Modular Arithmetic

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$$\mathbb{Z}/_m = \{[0],\ [1],\ [2],\ \ldots,\ [m-1]\}$$

What can we do with congruence classes? Add, multiply, and raise to a power.

Define addition in $\mathbb{Z}/_m$ as follows: $[a] + [b] = [a + b]$

We have to prove that the addition is **well defined**, that is,
the result of addition doesn't depend on the choice of class representatives.

Theorem 1. If $a \equiv a_1 \mod m$ and
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By this, $a + b \equiv a_1 + b_1 \mod m$. □

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Example. Find the remainder when

$2018 + 123456789 + 876543$ is divided by 5.

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Problem. Find the remainder when $235 \cdot 9823 \cdot 757$ is divided by 3.

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Solution. $235 \equiv 1 \mod 3$ since $235 = 3 \cdot 78 + 1$

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Answer: the reminder is 2.

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Answer: the reminder is 2.

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Powers in $\mathbb{Z}/_m$

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Therefore, $a^n \equiv b^n \mod m$. □

Ring $\mathbb{Z}/m$

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It is called the **ring of congruence classes modulo** m or **ring of integers modulo** m.

Proof is by checking the ring axioms. For any $a, b, c \in \mathbb{Z}/_m$

1. $a+b \in \mathbb{Z}/_m$
2. $a \cdot b \in \mathbb{Z}/_m$
3. $(a+b)+c = a+(b+c)$
4. $a+b = b+a$
5. $\exists 0 \in \mathbb{Z}/_m \qquad a+0=a$
6. $\exists -a \in \mathbb{Z}/_m \quad a+(-a)=0$
7. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
8. $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$

- $a \cdot b = b \cdot a$
- $\exists 1 \in \mathbb{Z}/_m \qquad 1 \cdot a = a \cdot 1 = a$

Modular arithmetic for divisibility

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Problem 1.

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Therefore, $7^{2n+1} + 17^n \equiv -1 + 1 \equiv 0 \mod 8$, as required. □

Last digit

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Problem 3.

Last digit

Problem 3. Find the last digit of 2022^{2023}.

Last digit

Problem 3. Find the last digit of 2022^{2023}.

Solution.

Last digit

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Solution. $2022^{2023} \equiv ? \mod 10$

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$$2022 \equiv 2 \mod 10$$

Last digit

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$$2022 \equiv 2 \mod 10 \implies 2022^{2023} \equiv 2^{2023} \mod 10$$

Last digit

Problem 3. Find the last digit of 2022^{2023}.

Solution. $2022^{2023} \equiv ? \mod 10$

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Last digit

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$2^1 \equiv 2 \mod 10$

Last digit

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$2^1 \equiv 2 \mod 10$

$2^2 \equiv 4 \mod 10$

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. . . . . .

Last digit

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$$2^{2023} \equiv ? \mod 10$$

$$\left.\begin{aligned} 2^1 &\equiv 2 \mod 10 \\ 2^2 &\equiv 4 \mod 10 \\ 2^3 &\equiv 8 \mod 10 \\ 2^4 &\equiv 6 \mod 10 \end{aligned}\right]$$

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cycle of length 4

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cycle of length 4

$$2023 \equiv ? \mod 4$$
$$2023 \equiv 3 \mod 4$$

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$2^1 \equiv 2 \mod 10$

$2^2 \equiv 4 \mod 10$

$2^3 \equiv 8 \mod 10$

$2^4 \equiv 6 \mod 10$

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. . . . . .

cycle of length 4

$2023 \equiv ? \mod 4$

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Last digit

Problem 3. Find the last digit of 2022^{2023}.

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Answer:

the last digit of 2022^{2023} is 8.

Diophantine equations

Diophantine equations

Problem 3.

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Problem 3. Prove that the equation $x^2 - 3y^2 = 17$

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Therefore, the original equation has **no** integer solutions.

Reduction modulo m

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Definition. Let R, S be rings.

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Therefore, f is a ring homomorphism. $\square$

Control of calculations

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Problem.

Control of calculations

Problem. Is it true that

Control of calculations

Problem. Is it true that $74,218 \cdot 21,363 - 81,835 = 1,585,447,299$?

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Then $[a]_m$ is a zero-divisor. Indeed, $[a]_m \cdot [b]_m = [ab]_m = [m]_m = 0$.

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Theorem, If a set X is finite, then any injection $f : X \to X$ is surjective.

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My mathematical ancestors

Mathematics Genealogy Project https://www.genealogy.math.ndsu.nodak.edu/

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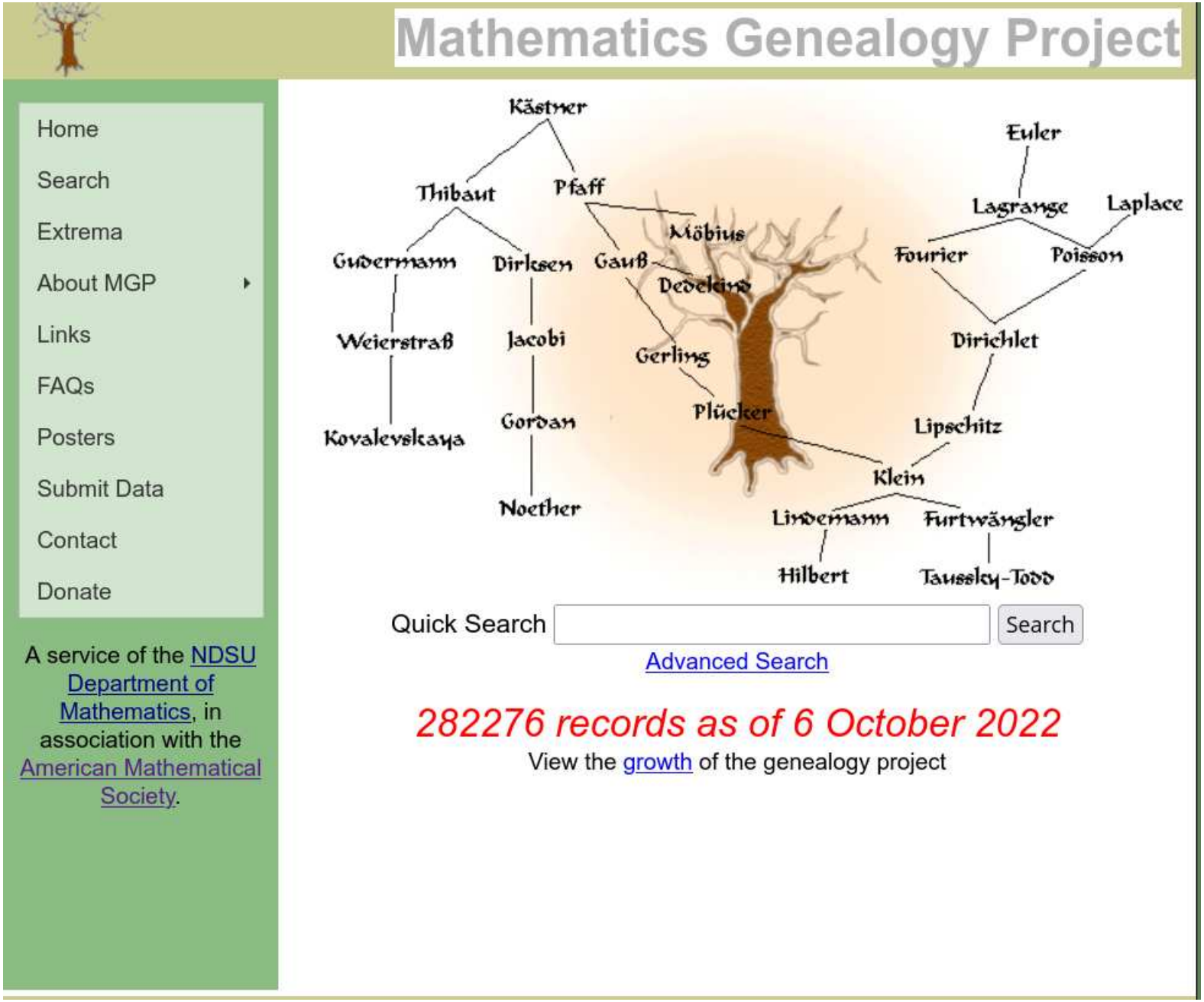

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