Section 10

Categories of maps

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Families Γ and Δ of sets are equivalent
if there exists a bijection $\Gamma \to \Delta$ formed of bijections.

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If the orbit of a does not contain a periodic element, then $\mathbb{N} \to X : n \mapsto f^n(a)$ is injective

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X is considered as the set of states of the system, f describes an evolution.

A map $f : X \to X$ can be iterated (composed with itself): $X \xrightarrow{f} X \xrightarrow{f} X \dots$

This gives iterates $f^1 = f$, $f^2 = f \circ f$, $f^3 = f \circ f \circ f$, . . . of f .

The images of $a \in X$ under iterations of f form the **orbit**

$\{a, f(a), f^2(a), f^3(a), \dots\}$ of a .

A map $f : X \to X$ defines an equivalence relation on X :
elements $a, b \in X$ are **grand orbit equivalent** if their orbits intersect,

i.e., $f^p(a) = f^q(b)$ for some $p, q \in \mathbb{N}$.

Check if this is really an equivalence relation on X .

The equivalence classes are called **grand orbits** of f .

The set of all grand orbits of f is called the **orbit set** of f and denoted by $X/_f$.

If $f^p(a) = a$ for some $p \in \mathbb{N}$, then a is said to be **periodic** of period p .

If the orbit of a does not contain a periodic element, then $\mathbb{N} \to X : n \mapsto f^n(a)$ is injective
and then X is **infinite**, because $|\mathbb{N}| \le |X|$.

Conjugate maps

Maps $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are called **conjugate** if
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Similarly, if $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are conjugate via $h : X \to Y$, then grand orbits of f are mapped by $h : X \to Y$ to grand orbits of g.

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Hence, we have proved

Theorem. *In a finite set an injection may have only periodic grand orbits.* □

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Corollary 3. *For a map $X \to X$ of a finite set X, injectivity, surjectivity and bijectivity are equivalent.* □

Maps between finite sets

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Control question: Is this true for infinite sets?

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Proof. Exercise.

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How many classes of self-maps are out there?

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Then $|X| \geq |U| = |\mathbb{N}|$, which contradicts to the assumption that X is finite. □

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Arithmetic operations match: $|\mathbb{N}_p \times \mathbb{N}_q| = |\mathbb{N}_{pq}|$, etc.

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Example 6. Five points are placed randomly inside an equilateral triangle with a side of length 1 inch. Prove that among these five points there are two points which are at most $1/2$ inches apart.

Example 7. Prove that there exists a power of 3 that ends up with digits 01.

Example 8. Prove that if 51 points are placed randomly in a unit square, then one can always find three points that can be covered by a circle of radius $1/7$.

Example 9. Given 6 points inside a unit circle. Prove that some two of them are within 1 unit from each other.