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$|A| = |B|$, because of bijection $A \to B : x \mapsto x - 1$. Then $A \cup B = \mathbb{N}$ and
$|\mathbb{N}| = |A|$, because of bijection $\mathbb{N} \to A : x \mapsto 2x$. □

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Why? Recall that $a - b$ is a unique solution of the equation $b + x = a$.
If the subtraction could be defined, then

the equation $b + x = a$ with $b = \aleph_0$ and $a = \aleph_0$ and the equality $\aleph_0 + 1 = \aleph_0$
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In Calculus a similar phenomenon is known as "**indeterminacy** $\infty - \infty$".

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This is a bijection. The inverse map $P \times Q \to P' \times Q'$
is formed similarly out of f^{-1} and g^{-1}. $\square$

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Corollary. If $|X| = |Y| = \aleph_0$, then $|X \times Y| = \aleph_0$. For example, $|\mathbb{Z}^2 \times \mathbb{N}| = \aleph_0$.

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$\impliedby$: Let $g : Y \to X$ be a surjection. Let us construct an injection $f : X \to Y$.
For any $a \in X$ choose $b \in g^{-1}(a)$ and put $b = f(a)$. The choice is possible, as g is surjective and $g^{-1}(a) \neq \varnothing$.

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Let $f : X \to Y$ be an injection. Let us construct a surjection $g : Y \to X$.

Choose any $c \in X$ and define $g(b) = \begin{cases} a, & \text{if } b \in f(X) \wedge b = f(a) \\ c, & \text{if } b \notin f(X). \end{cases}$

$g(b)$ is well-defined, since for any $b \in f(X)$
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g is surjective, because for any $a \in X$ $\exists b \in Y : a = g(b)$, namely $b = f(a)$ $\square$.

$\Longleftarrow$: Let $g : Y \to X$ be a surjection. Let us construct an injection $f : X \to Y$.
For any $a \in X$ choose $b \in g^{-1}(a)$ and put $b = f(a)$. The choice is possible, as g is surjective and $g^{-1}(a) \neq \varnothing$. If $a_1 \neq a_2$, then $g^{-1}(a_1) \cap g^{-1}(a_2) = \varnothing$.

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Corollary. Let X, Y be sets. Then $|X| \leq |Y|$, iff $\exists$ surjection $Y \to X$.

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If nothing like this stops us, we continue and eventually come to constructing a map $X \to Y$ which is either injection or surjection. $\square$.

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Equivalence classes for grand orbit equivalence are called **grand orbits** of f.

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This is a desired bijection $X \to Y$. $\square$

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and the last digit $x \neq 9$ is replaced by the next digit $x + 1$, that is $xy9999\dots$ equals $x(y+1)0000\dots$.

For example, 33.14999...=33.15000...

Lemma. If a real number admits two different decimal presentations, then these presentations can be obtained from each other as above.

In what follows we will not use decimal presentations of real numbers which end with infinitely many 9's.

Due to Lemma, such presentation of a real number is unique.

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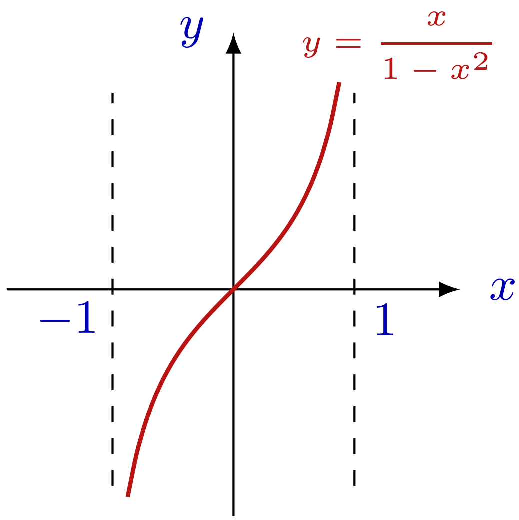

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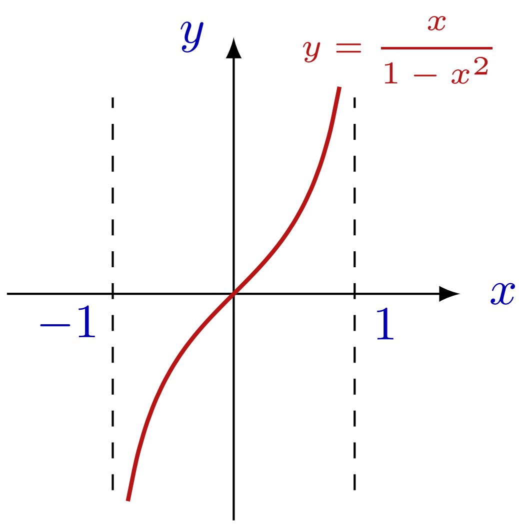

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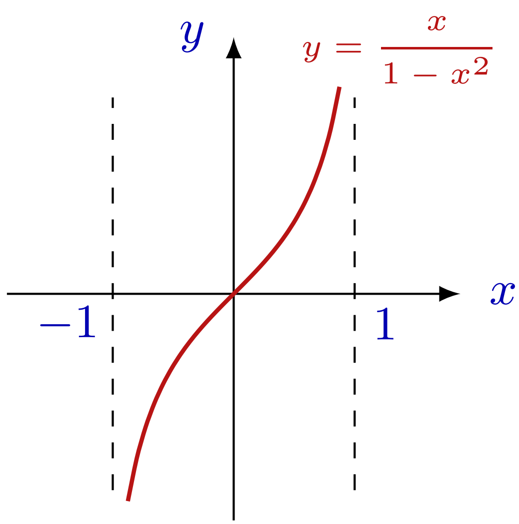

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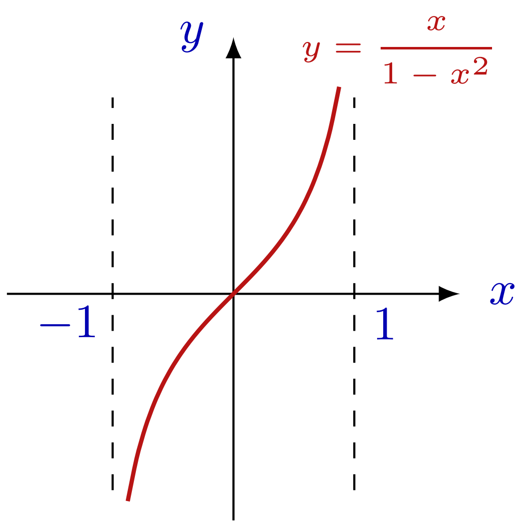

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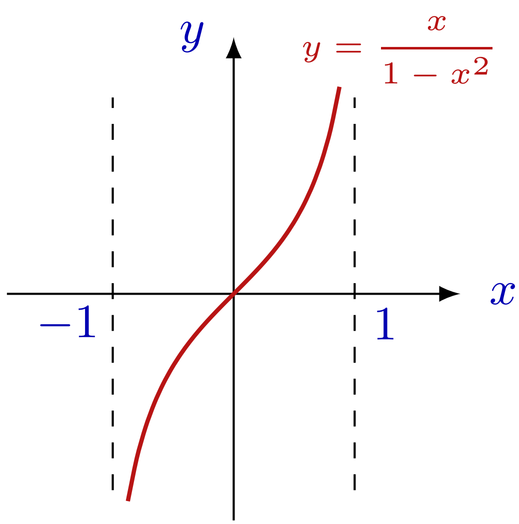

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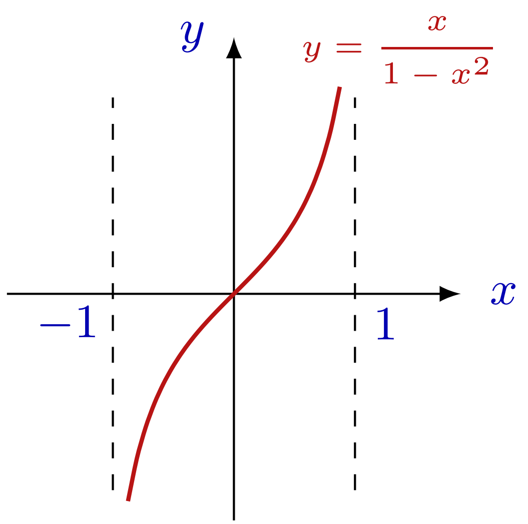

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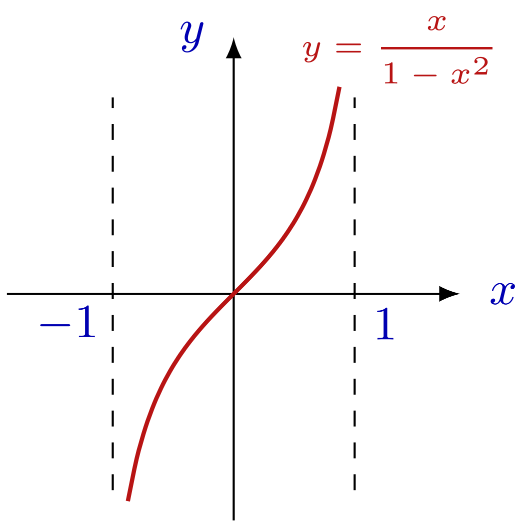

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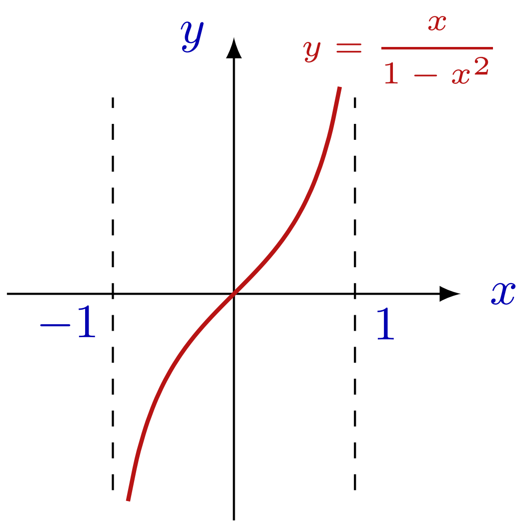

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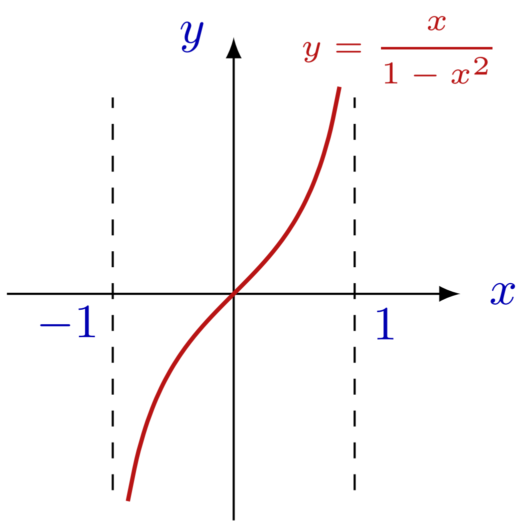

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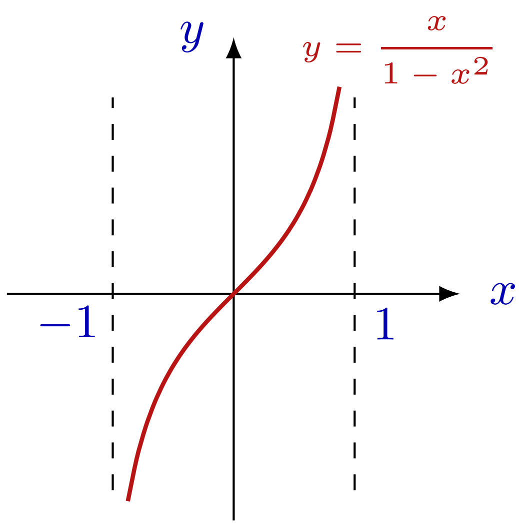

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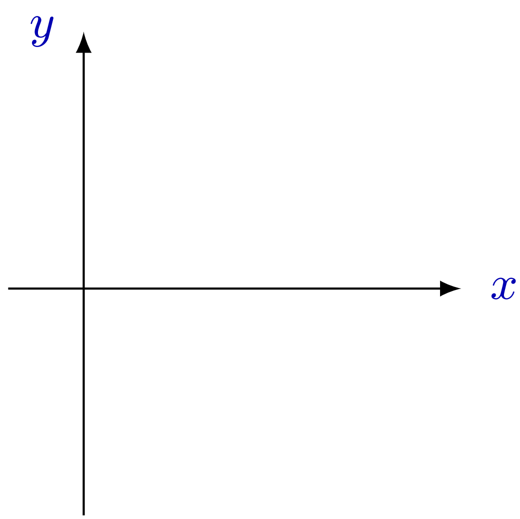

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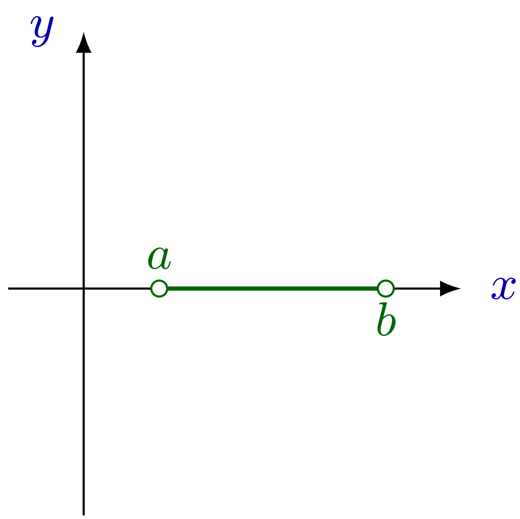

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Proof. A linear function $f : (a, b) \to (-1, 1)$ defined by

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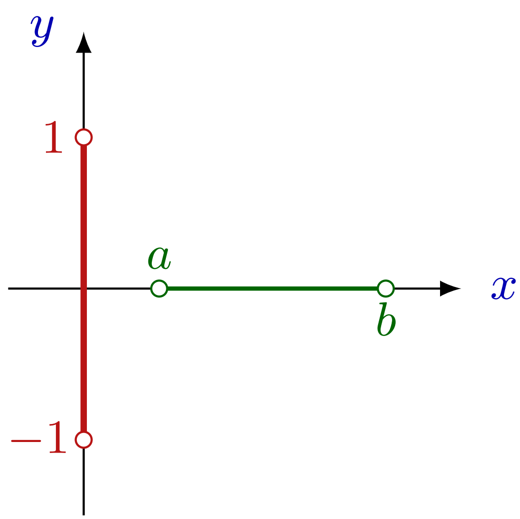

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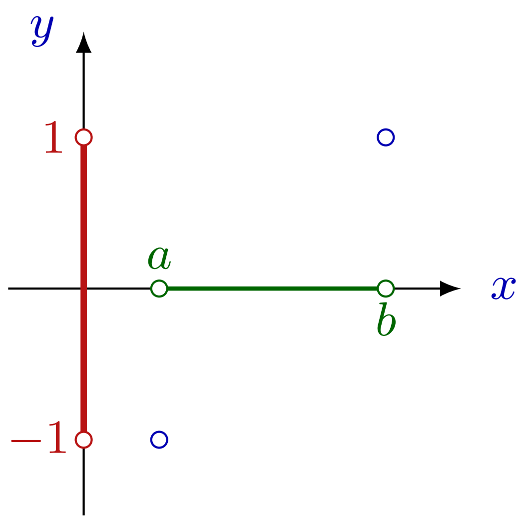

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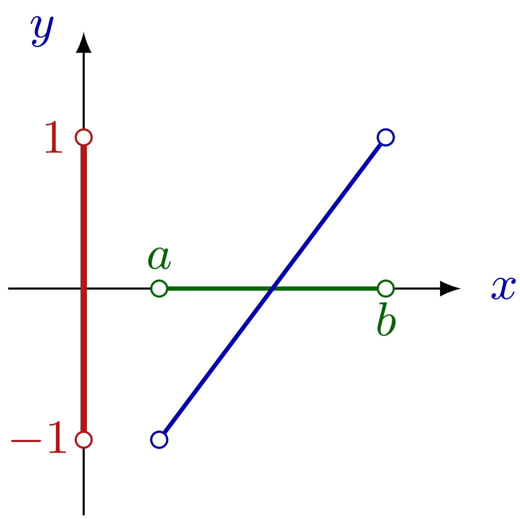

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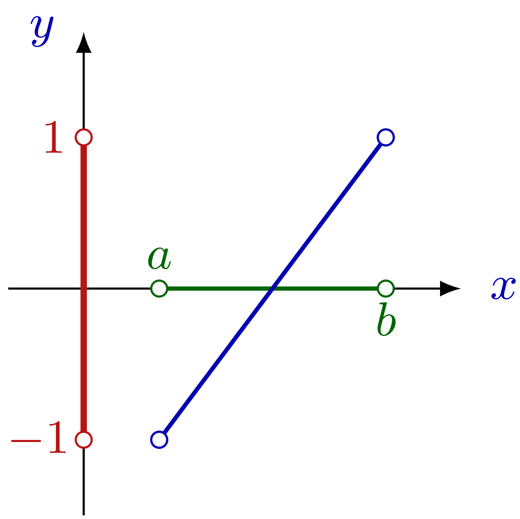

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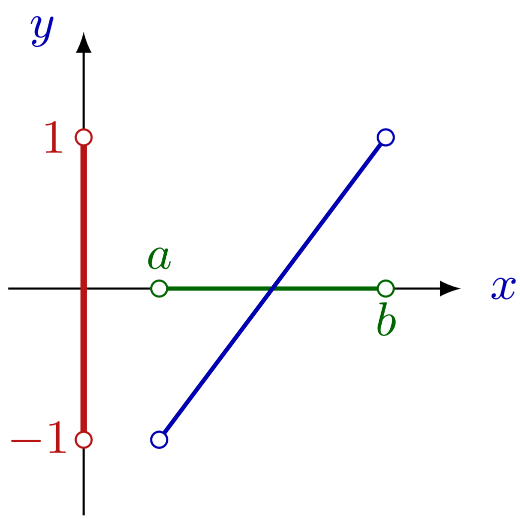

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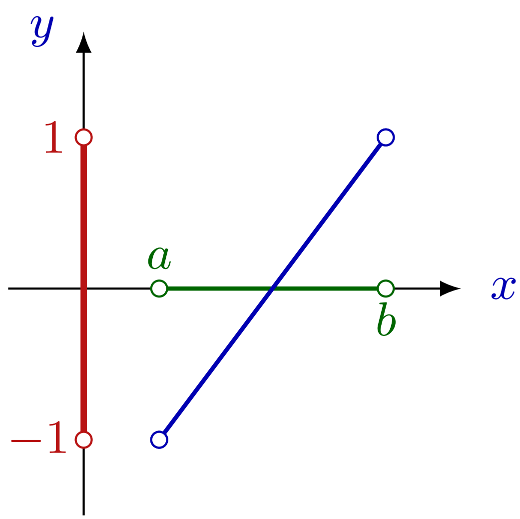

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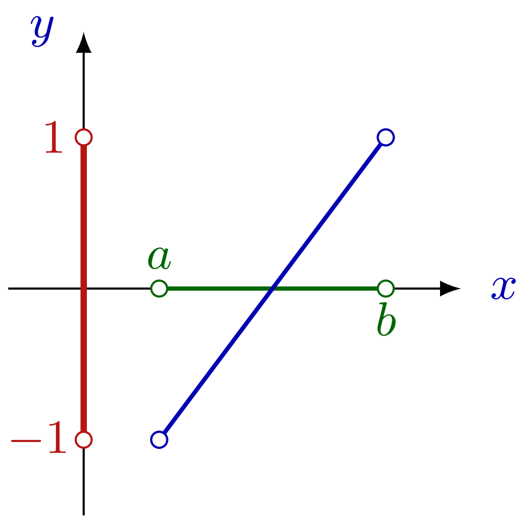

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Does $a \in A$? If $a \in A$, then $a \notin F(a)$ by definition of A . But $A = F(a)$, so $a \notin A$. Contradiction!

If $a \notin A$, then it is not true that $a \notin F(a)$, that is, $a \in F(a) = A$.
It's a contradiction.

Cardinality of power set

Theorem (Cantor). *For any set* X, $|X| < |\mathcal{P}(X)|$.

Proof. $|X| \le |\mathcal{P}(X)|$, since there is an injection $X \to \mathcal{P}(X)$, $x \mapsto \{x\}$.
We have to prove that $|X| \neq |\mathcal{P}(X)|$, that is, there is **no** bijection $X \to \mathcal{P}(X)$.

Assume that there exists a bijection $F : X \to \mathcal{P}(X)$. $F(x) \subset X$ for $\forall x \in X$.
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Therefore, the assumption about existing a bijection $X \to \mathcal{P}(X)$ was wrong, and there is no such a bijection. $\square$

Huge sets

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f is **injective** since the binary representation of a number is unique.

$c \leq 2^{\aleph_0}$

Theorem. $\mathcal{P}(\mathbb{N}) \approx (0,1)$.

Proof. We will prove that $|(0,1)| \leq |\mathcal{P}(\mathbb{N})|$ and $|\mathcal{P}(\mathbb{N})| \leq |(0,1)|$.

Any number in the interval $(0,1)$ can be written in **binary** representation $0.a_1a_2a_3\ldots$, where each a_i is either 0 or 1 .

We exclude the sequences that terminate with infinitely many 1's to guarantee the uniqueness of the representation (so $0.0111\ldots$ should be written as $0.100\ldots$).

For example, the binary presentation 0.011 stands for the number $0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 1 \cdot 2^{-3} = 3/8$.

Define a map $f : (0,1) \to \mathcal{P}(\mathbb{N})$ by $f(0.a_1a_2a_3\ldots) = \{n \in \mathbb{N} \mid a_n = 1\}$.

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f is **injective** since the binary representation of a number is unique.
Therefore, $|(0,1)| \leq |\mathcal{P}(\mathbb{N})|$, that is $\mathbf{c} \leq \mathbf{2}^{\aleph_0}$.

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Similarly, if $X \xrightarrow{f} X$ and $Y \xrightarrow{g} Y$ are conjugate via $h : X \to Y$, then grand orbits of f are mapped by $h : X \to Y$ to grand orbits of g .

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Grand orbits of injection

In the orbit of a, each element $f^q(a)$ has a unique **successor** $f^{q+1}(a)$.
If $f: X \to X$ is an **injection**,
then each element of a grand orbit may have at most one **predecessor**.

Theorem. Grand orbits of an **injection** can be one of the following **three types**:

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Hence, we have proved

Theorem. *In a finite set an injection may have only periodic grand orbits.*

□

Maps of a finite set to itself

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Corollary 3. *For a map $X \to X$ of a finite set X, injectivity, surjectivity and bijectivity are equivalent.* □

Maps between finite sets

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Control question: Is this true for infinite sets?

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Proof. Exercise.

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Then $|X| \geq |U| = |\mathbb{N}|$, which contradicts to the assumption that X is finite. $\square$

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