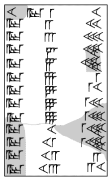
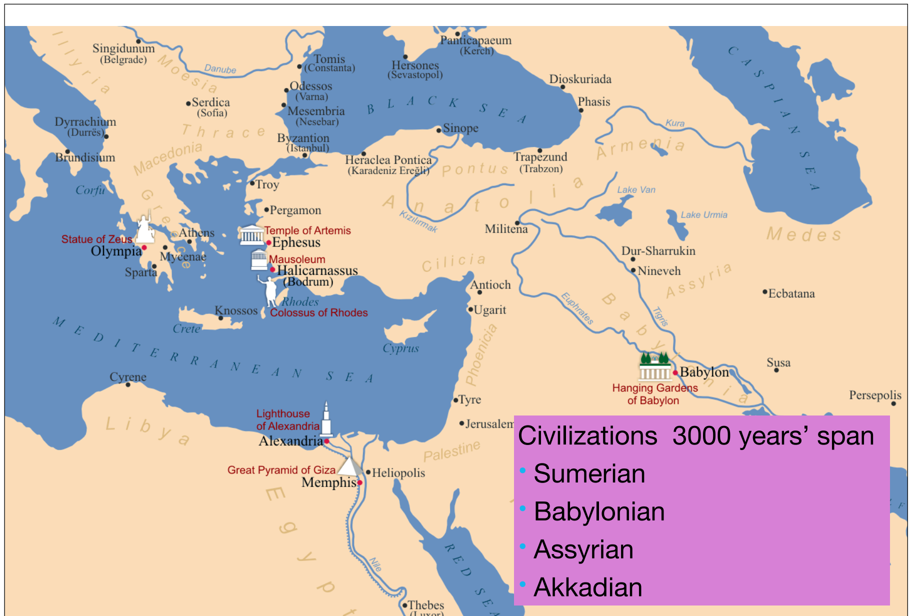
Mathematics in Mesopotamia






- Overview Mesopotamia.
- Cuneiform and the beginning of writing
- Review of numbers
- Areas
- Plimpton 322
- Number systems
- Square roots: algorithm and computations
- The “canonical“ triangle.
- Calculation of areas
- Solutions of Equations
- Reciprocals
- Multiplication tables

Overview



Civilizations 3000 years’ span
- Sumerian
- Babylonian
- Assyrian
- Akkadian

Seven wonder of the Ancient World - Kandi, CC BY-SA 3.0 <https://creativecommons.org/licenses/by-sa/3.0>, via Wikimedia Commons

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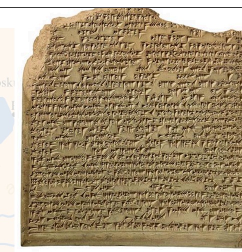


Tablet Inscribed in Babylonian with a Ritual for the Observances of Eclipses-The Morgan Library and Museum

Cuneiform Writing on Clay Tablets

- Wedge-shaped “cuneiform” script, written by trained scribes.
- Hundreds of thousands of tablets have survived.
- Most are **administrative records** (memos, receipts, wages).
- Others preserve:
  - Law codes
  - Magical rituals
  - Military campaigns and battles
  - Myths, including the Epic of Gilgamesh
  - Math!

Seven wonders of the Ancient World - Kandi, CC BY-SA 3.0 <https://creativecommons.org/licenses/by-sa/3.0>, via Wikimedia Commons

Pick one mathematical idea from the video that you use every day. Describe how you use it.

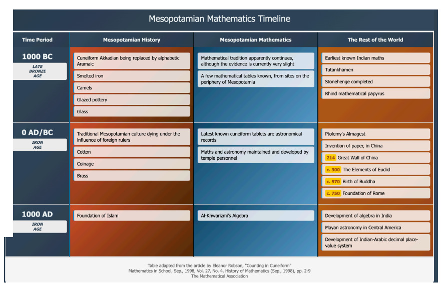
Ancient Mesopotamia 101- National Geographic
https://youtu.be/xVf5kZA0HtQ?si=PK4TY11m37V3GYK5

A praise poem of Šulgi

I am a king, offspring begotten by a king and borne by a queen.
I, Šulgi the noble, have been blessed with a favorable destiny right from the womb.
When I was small, I was at the academy, where I learned the scribal art from the tablets of Sumer and Akkad.
None of the nobles could write on clay as I could.
There where people regularly went for tutelage in the scribal art, **I qualified fully in subtraction, addition, reckoning and accounting.**
The fair Nanibgal, Nisaba, provided me amply with knowledge and comprehension.
I am an experienced scribe who does not neglect a thing.

The fair Nanibgal, Nisaba" signifies one goddess, patron of Sumerian scribes.
Nanibgal meant Lady of Great Wisdom"
King Šulgi of Ur (reigned 2000 BCE)
Electronic Text Corpus of Sumerian Literature, "A praise poem of Shulgi (Shulgi B)": etcsl.orinst.ox.ac.uk/section2/tr24202.htm

https://www.math.stonybrook.edu/~moira/courses/mat336-fall2025/zmaterial/timelineMobil.html



Mesopotamian Mathematics Timeline

Time Period	Mesopotamian History	Mesopotamian Mathematics	The Rest of the World
1000 BC LATE BRONZE AGE	Cuneiform Akkadian being replaced by alphabetic Aramaic; Smelted iron; Camels; Glazed pottery; Glass	Mathematical tradition apparently continues, although the evidence is currently very slight; A few mathematical tables known, from sites on the periphery of Mesopotamia	Earliest known Indian maths; Tutankhamen; Stonehenge completed; Rhind mathematical papyrus
0 AD/BC IRON AGE	Traditional Mesopotamian culture dying under the influence of foreign rulers; Cotton; Coinage; Brass	Latest known cuneiform tablets are astronomical records; Maths and astronomy maintained and developed by temple personnel	Ptolemy's Almagest; Invention of paper, in China; 214 Great Wall of China; c. 300 The Elements of Euclid; c. 570 Birth of Buddha; c. 750 Foundation of Rome
1000 AD IRON AGE	Foundation of Islam	Al-Khwarizmi's Algebra	Development of algebra in India; Mayan astronomy in Central America; Development of Indian-Arabic decimal place-value system

Table adapted from the article by Eleanor Robson, "Counting in Cuneiform"
Mathematics in School, Sep., 1998, Vol. 27, No. 4, History of Mathematics (Sep., 1998), pp. 2-9
The Mathematical Association

📱 **Timeline Instagram Story Activity**

https://www.math.stonybrook.edu/~moira/courses/mat336-fall2025/zmaterial/timeline-activity.html

Cuneiform Writing: Birth, Decipherment, and Surviving Texts

Mesopotamian Accounting Tokens (4000–3300 BCE)

Tokens:

- cone, spheres, flat disc → Cereal
- tetrahedron → a unit of work.

(Image by Denise Schmandt-Besserat and the University of Pennsylvania Museum of Archaeology and Anthropology, University of Pennsylvania, Philadelphia.)

Envelopes:

- Tokens were stored in clay envelopes for verification.
- Engraved with tokens content

Image Denise Schmandt-Besserat and Musée du Louvre, Département des Antiquités Orientales, Paris.)

From Accounting Tokens to Writing (3300–3200 BCE)

- 3D tokens → 2D impressions → first writing in Mesopotamia.
  - Impressions evolved into cuneiform script.
- Writing was invented independently in at least 4 regions (Mesopotamia, Egypt, China, Mesoamerica).

Image Denise Schmandt-Besserat and Musée du Louvre, Département des Antiquités Orientales, Paris.)

Each circular impression → one large measure of grain; each wedge → smaller measure of grain

(Image by by Denise Schmandt-Besserat and Musée du Louvre, Département des Antiquités Orientales, Paris.))

Behistun Inscription -the "cuneiform Rosetta stone"

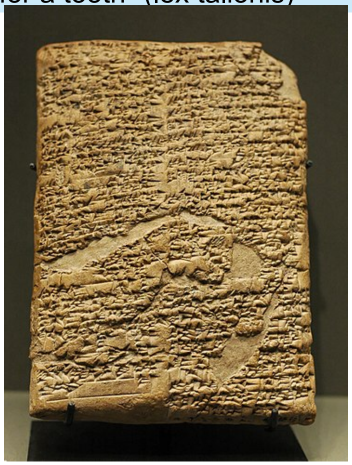
- written in three different cuneiform languages: Old Persian, Elamite, and Babylonian.
- Studied and copied by Sir Henry Rawlinson, a British army officer, beginning in 1835.

The Behistun Inscription, a large rock relief showing Darius the Great of the Achaemenid Empire punishing conspirators. Wikipedia Korosh.091 - Own work

Location of the Behistun complex

Code of Hammurabi about 1700 BC

"an eye for an eye, a tooth for a tooth" (lex talionis)

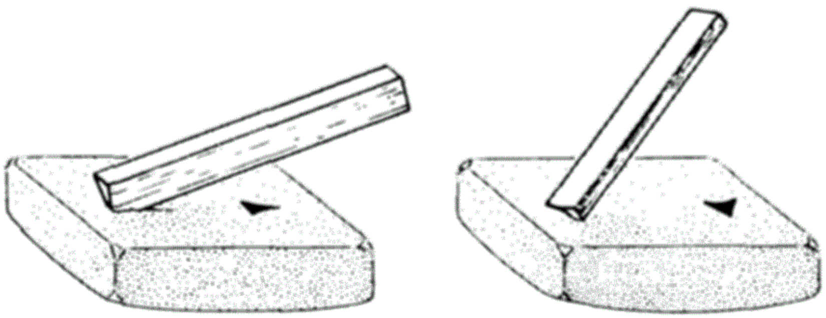


Epic of Gilgamesh

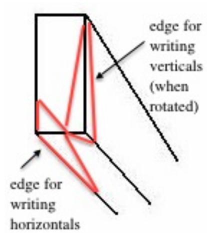
Ancient Assyrian statue currently in the Louvre, possibly representing Gilgamesh

Neo-Assyrian clay tablet. Epic of Gilgamesh, Tablet 11: Story of the Flood. Known as the "Flood Tablet" From the Library of Ashurbanipal, 7th century BC. British Museum

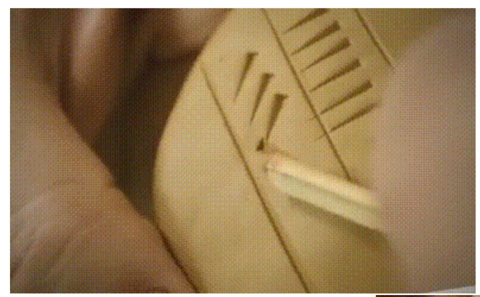
How to write in cuneiform

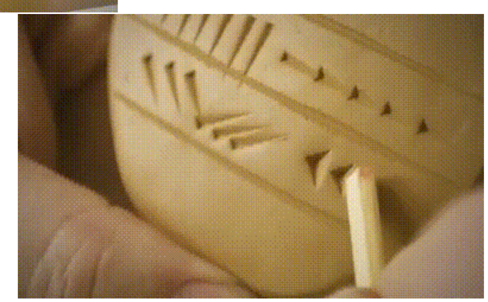


Impression of cuneiform symbols on clay tablets. (Redrawn from Neugebauer3



http://writingcuneiform.blogspot.com/2012/10/5-making-basic-wedges.html





Cuneiform writing

Clips from https://cuneiform.neocities.org/CWT/Figures/2_keilschrift_hwc.gif

Cuneiform writing: How it was done

https://cuneiform.neocities.org/CWT/Figures/2_keilschrift_hwc.gif

Mesopotamian Mathematics in a slide-nutshell

Overview of Mesopotamian Mathematics

Mathematical tablets are mainly:
Tables and Problems

Examples

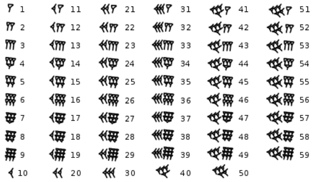
- Multiplication tables (by ***some*** numbers).
  Reciprocals tables
- Square roots tables
- **Pythagorean triples? Plimpton 322.**

- Rough work (solution of problems)
- Problems - Verbal techniques

Note: No tables of addition

Overview of Mesopotamian Mathematics

- Scribes used **positional** number system with base 60 for computations.
- Scribes also used other number systems.



positional number system with no zero!

Calendar

Algorithms

- Determine square roots.
- Solution of linear equations (with one or two unknowns)
- Solution of certain quadratic equations (often related to architecture and building)

Scribes' Shortcut: Multiply by This

- The **defining component** of an equilateral triangle was the **side** and the **coefficient** for the **height** was 7/8.
- The **defining component** of a **circle** was the **circumference**. **Coefficients:**
  - Diameter was 1/3=$(0;20)_{60}$
  - Area 1/12= $(0;5)_{60}$

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- The **defining component** of a **circle** was the **circumference**. **Coefficients:**
  - Diameter was 1/3=$(0;20)_{60}$
  - Area 1/12= $(0;5)_{60}$

Decoder key:
defining component = the given measure you start from
coefficient = the multiplier that converts the measure into something else.

Educated guess — you are now a Mesopotamian scribe.
a. Given a circle's circumference c and the coefficients for the diameter and the area, compute its diameter and its area. (Hint: Write down the area of the circle in terms of the circumference).
b. Given an equilateral triangle's side s and the coefficient for the height, compute the height and area of the triangle.

Tables in Tablets

- Amazing aspect of the Babylonian's calculating skills: **construction of tables to aid calculation.**
- Babylonians used the formula

a.b = (1/2)((a+b)^2 - a^2 - b^2)

to make multiplication easier.

- They did not have an algorithm for long division. Instead they based their method on the fact that

A % B = A x (1/B)

and used tables of reciprocals **(1/B)**.

Canals: A Major Reason to Do Mathematics

Canals = essential for irrigation and transport of goods/armies

Mathematics used to:

- Compute **dimensions of canals**.
- Plan **workers × days** needed.
- Calculate **wages and expenses**.

Old Babylonian tablets record many such **canal-digging problems**.

https://mathshistory.st-andrews.ac.uk/HistTopics/Babylonian_mathematics/
K Muroi, Small canal problems of Babylonian mathematics, Historia Sci. (2) 1 (3) (1992), 173-180.

More about base 60

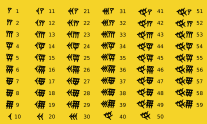
Historians have proposed several theories for the Babylonian use of base 60. Choose one and defend it.

a. Many even splits: 60 breaks evenly into many parts (2, 3, 4, 5, 6, 10, 12, 15, 20, 30), so common fractions come out neat and calculations are easier.

b. Sky counts: About 12 months × 5 "wandering stars" (Mercury, Venus, Mars, Jupiter, Saturn) = 60. (Sun and Moon were not counted as planets.)

c. Shared number for trade: Different places used different ways of counting and measuring; 60 worked as a handy common base for converting and keeping accounts.

d. Finger math: One hand can count to 12 using finger bones; the other hand tracks sets of 12 x 5 → 60.

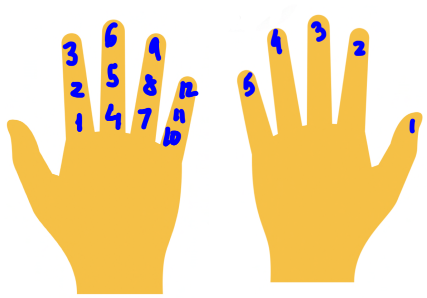


Some of the many theories explaining why base 60!

- Theon of Alexandria (~300CE): 60 is, among all the numbers the most **convenient**, because, being the smallest among all those which have the **most divisors**, it is the easiest to handle.
- More recent theory (~1950CE): **to allow for dividing weights and measures into thirds.**
- Even more recent: Sumerian civilization must have come about through the **joining of two groups**, one of whom had **base 12** for their counting and the other having base **5 or 10.**
- Common measures theory is now widely accepted.

There are many theories trying to explain base 60

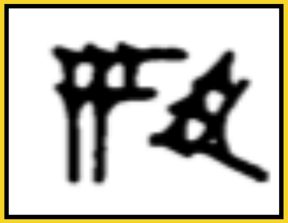
One can count up to 60 by pointing at one of the twelve parts of the fingers of the left hand with one of the five fingers of the right hand.

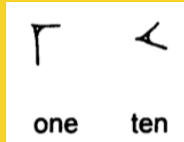


Express numbers A. and B. in Hindu-Arabic numerals

A.

B.



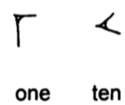


All Mesopotamian numerals

Figures from Robson, Eleanor. "Counting in Cuneiform." Mathematics in school 27.4 (1998): 2-9

5 meant
$5/60+40/60^2$ or
$5+40/60$ or
$5\times60+40$ or
$5\times60^2+40\times60$ or..

meant
$45/60(=3/4)$ or
45 or
$45\times60=2700$ or..



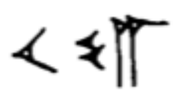
How do they know which one? Context!

Figures from Robson, Eleanor. "Counting in Cuneiform." Mathematics in school 27.4 (1998): 2-9

Notations for intermediate zero across time

12

10 02
or 602
before
1600 BC



10 02
or 602
after
1600 BC

Figures from Robson, Eleanor. "Counting in Cuneiform." Mathematics in school 27.4 (1998): 2-9

Goal: Write a number **a=0.205** between 0 and 1 in base 60. We want numbers $c_1, c_2, c_3 \ldots$ such that

$$\mathbf{0.205} = c_1/60 + c_2/60^2 + c_3/60^3 + \ldots$$

1) Each c_i is an integer
2) $0 \leq c_i < 60$

Step 1: $c_1 = \lfloor 60\cdot 0.205 \rfloor = \lfloor 12.3 \rfloor = 12$

$a_1 = 60a - c_1 = 12.3 - 12 = 0.3$

Step 2: $c_2 = \lfloor 60a_1 \rfloor = \lfloor 60\cdot 0.3 \rfloor = \lfloor 18 \rfloor = 18$

$a_2 = 60a_1 - c_2 = 60\cdot 0.3 - 18 = 0$

Stop: If we reach i such that $a_i = 0$ the process ends.

Note: $\lfloor x \rfloor$ denotes the largest integer smaller or equal than x.

Goal: Write a number **a** between 0 and 1 in base 60. We want numbers c_1, c_2, c_3... such that

$$a = c_1/60 + c_2/60^2 + c_3/60^3 + \ldots$$

10 Each c_i is an integer
2) $0 \le c_i < 60$

Step 1: $c_1 = \lfloor 60 \cdot a \rfloor$

$a_1 = 60a - c_1$

Step 2: $c_2 = \lfloor 60 a_1 \rfloor$

$a_2 = 60 a_1 - c_2$

Step n: $c_n = \lfloor 60 a_{n-1} \rfloor$,

$a_n = 60 a_{n-1} - c_n$)

Stop: If we reach i such that $a_i = 0$ the process ends.

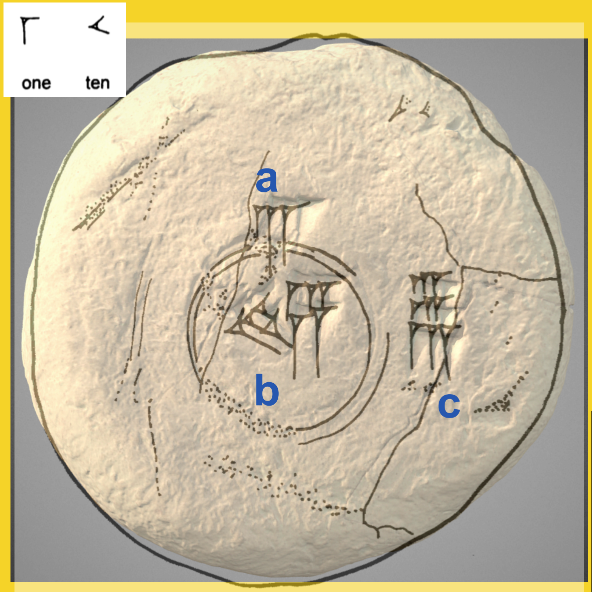
Note: $\lfloor x \rfloor$ denotes the largest integer smaller or equal than x.

Convert these fractions into base 60 (sexagesimal). Then, show what the result would look like in cuneiform.
Answer in Wooclap using base-60 notation.
Upload a screenshot of your cuneiform work.

A. 1/20
B. 21/80
C. (Optional) 1/7

Calculation of Areas

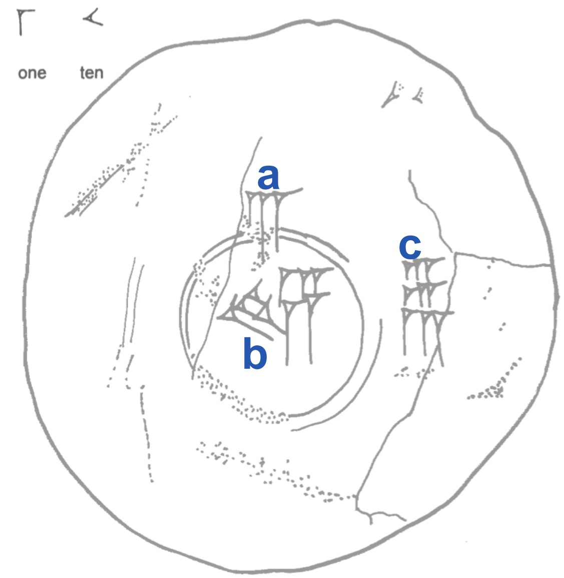
https://sketchfab.com/3d-models/the-area-of-a-circle-tablet-16f3e288b7be485297ddd0945fd10ece



1. Write a, b, c in the Hindu-Arabic number system.
2. What do these numbers represent?
3.(Optional) What is the purpose of this tablet?

Hint: To find the purpose, consider the arrangement of the numbers with respect to the circle.

Old Babylonian tables from Yale Collection
Drawing by Eleanor Robson The American Mathematical Monthly , Feb., 2002, Vol. 109, No. 2 (Feb., 2002), pp. 105-120)



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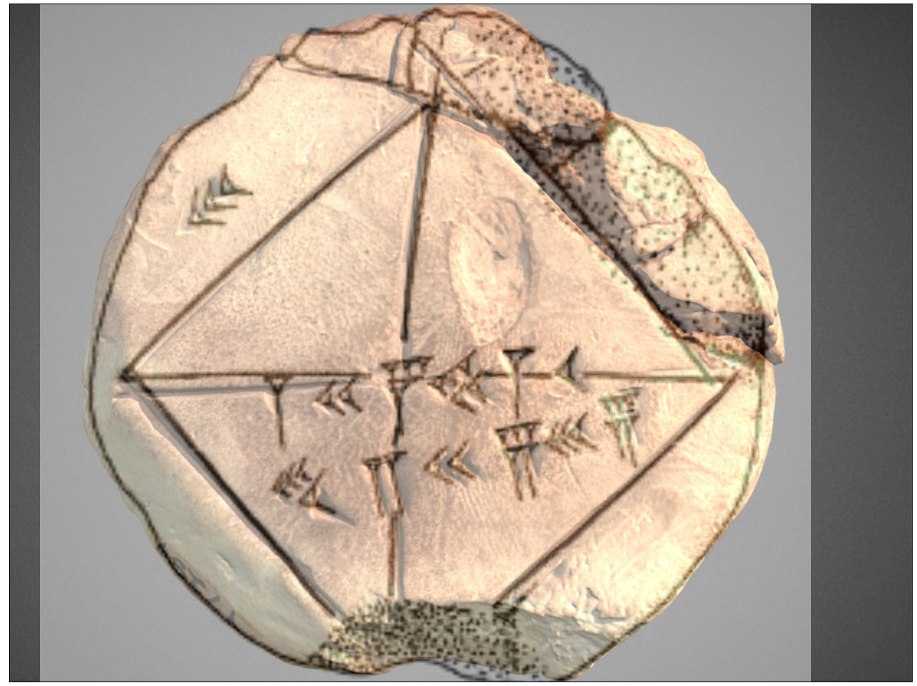
Old Babylonian tables from Yale Collection
Drawing by Eleanor Robson The American Mathematical Monthly , Feb., 2002, Vol. 109, No. 2 (Feb., 2002), pp. 105-120)

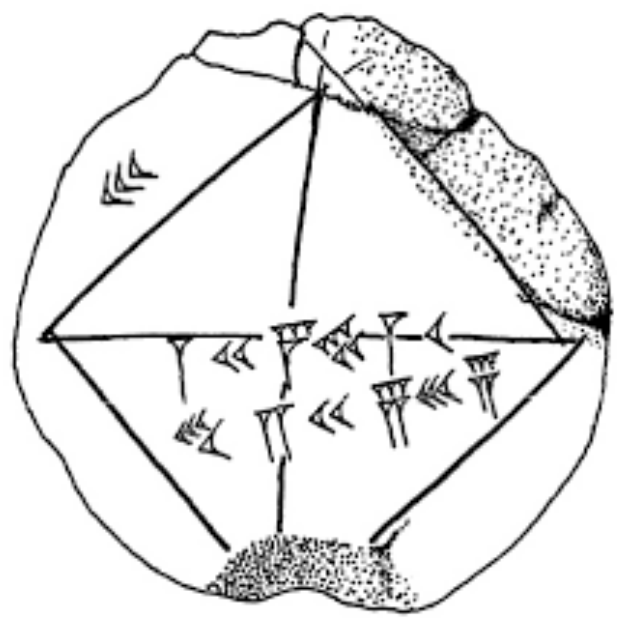
A trapezoid

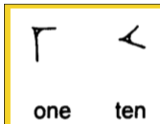
An amazing computation

A digital replica to hold in your digital hands

https://sketchfab.com/3d-models/the-diagonal-of-a-square-







Picture from Yale Collection
Drawing by Eleanor Robson

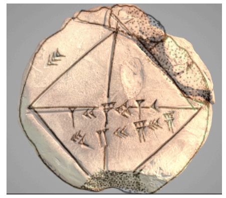
1. What are the values of a, b and c?
2. Educated guess: What do you think is the purpose of this tablet?

√2 Babylonian clay tablet (circa 1800–1600 BCE)

- Likely “written” **by a student.**
- Evidence suggests its **author copied** some of the results from an existing table of values and did not compute them himself.

ChatGPT said “avoid gendered assumption. What do you think?

- The value demonstrates one of the **greatest known computational accuracy** obtained in the ancient world.



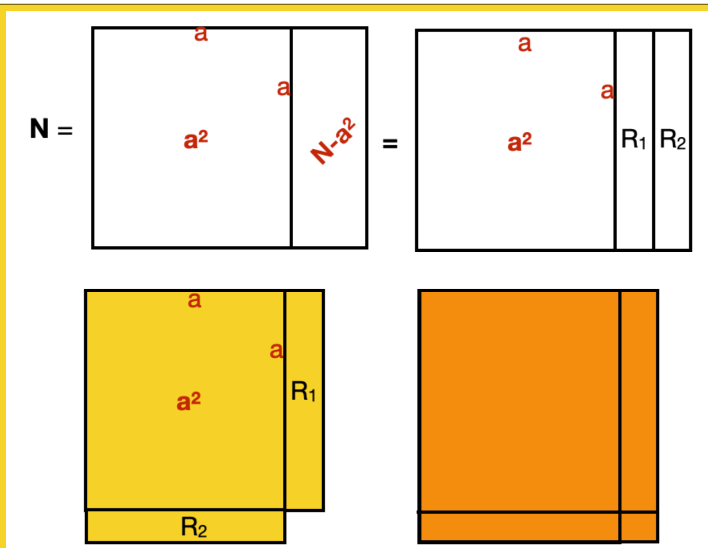
Picture from Yale Collection
Drawing by Eleanor Robson

A digital replica to hold in your digital hands

https://sketchfab.com/3d-models/the-diagonal-of-a-square-tablet-605c9d9573d14b52b8880e14c826e133?utm_medium=embed&utm_campaign=share-popup&utm_content=605c9d9573d14b52b8880e14c826e133

Some examples of notable computational accuracy in the Ancient World

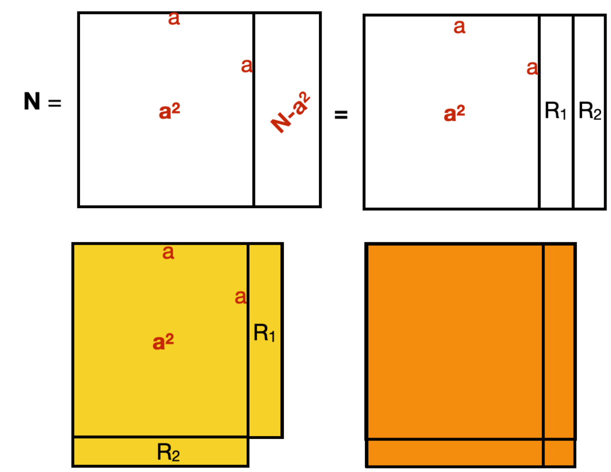
	Who and Where	When	Value	Accuracy
√2	Babylonian	1800–1600 BCE	$(1;24,51,10)_{60}$	error ≈ 6.0×10^{-7} → **6 decimals**
√2	Śulba Sūtras - India	800–500 BCE	$1 + 1/3 + 1/(3\cdot4) - 1/(3\cdot4\cdot34)$	error ≈ 2.1×10^{-6} → **5 decimals**
π	Archimedes - Greece	3rd c. BCE	$223/71 < \pi < 22/7$ (Greek ciphered)	**≈2–3 decimals** bounds
π	Ptolemy - Alexandria	2nd c. CE	$(3;8,30)_{60}$	error ≈ 7.4×10^{-5} → **3 decimals**
π	Zu Chongzhi - China	5th c. CE	355/113	error ≈ 2.7×10^{-7} → **6 decimals**



What is the area of the yellow figure? What is the area of the orange figure? (Express both areas in terms of N and a)

Problem: Given a square of area N, find its side

- **Start with a good guess**: Pick **a** where a^2 is close to N (but $a^2 < N$)



The side of the new (orange) square is a’ = a + $(N-a^2)/(2a)$

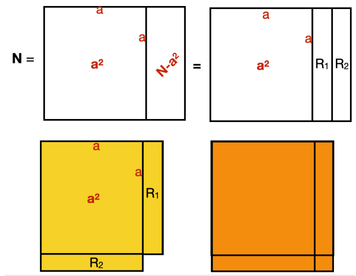
Problem: Given a square of area N, find its side

Babylonian approximation of the square root of a number N.

- **Start with a good guess**: Pick **a** where **a^2** is close to **N** (but **$a^2 < N$**)
- **Form a rectangle**: Create a rectangle with area **N** and one side of length **a.** (The other side must have length **N/a**)
- **Key insight**: Since **$a^2 < N$**, we have **$a < \sqrt{N} < N/a$** (Our rectangle has one side too short, one side too long)
- **Goal**: Cut up the rectangle and rearrange the pieces to form "almost" **a** square closer to area **N.** This gives a better approximation to **$\sqrt{N}$**

The side of the new (orange) square is $a' = a + (N-a^2)/(2a)$

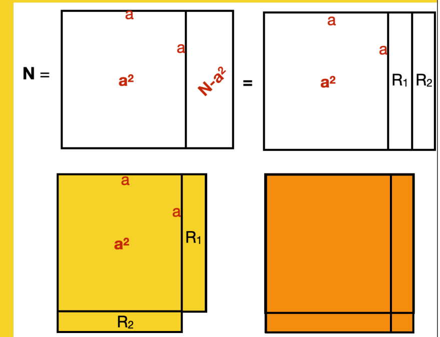
Does this algorithm rings a bell?



Using the algorithm, compute an approximation of $\sqrt{2}$ starting with a=1

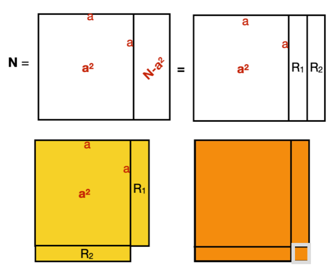
- **Start with a good guess**: Pick **a** where **a^2** is close to **N** (but **$a^2 < N$**)
- **Form a rectangle**: Create a rectangle with area **N** and one side of length **a.** (The other side must have length **N/a**)
- **Goal**: Cut up the rectangle and rearrange the pieces to form "almost" **a** square closer to area **N.** This gives a better approximation to **$\sqrt{N}$**

The side of the new (orange) square is $a' = a + (N-a^2)/(2a)$



Babylonian approximation of the square root of a number N when $a^2 < N$

1. Find a such that a^2 is close to N and $a^2 < N$.
2. The yellow shape has area N $= a^2 + (N - a^2)$.
3. The "leftover area $(N - a^2)$ is split into two equal rectangles, each of area $(N - a^2)/2$. So their width is $(N - a^2)/(2a)$
4. Since a^2 is close to N, the area of the yellow shape (N) is close to the area of the orange square."They differ only by the small corner square of side
5. Therefore, $\sqrt{N}$ is close to the side length of the orange square.
6. The orange square's side length is $a + (N - a^2)/(2a)$. Thus, $a + (N - a^2)/(2a)$ is the new approximation to $\sqrt{N}$.



Babylonian approximation of the square root of a number N.

Why the error using a′ is smaller than using a.

Recall $a^2 < N$ and $a' = a + (N - a^2)/(2a)$.

Old error (using a to estimate $N^{1/2}$) is $N^{1/2} - a$.

New error (using a′ to estimate $N^{1/2}$):

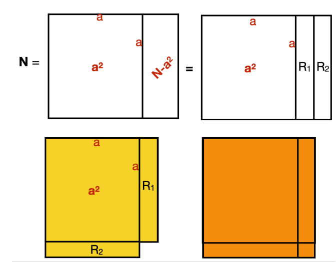
$$a' - N^{1/2}$$

$$= a + [(N - a^2)/(2a)] - N^{1/2}$$

$$= a + [((N^{1/2})^2 - a^2)/(2a)] - N^{1/2}$$

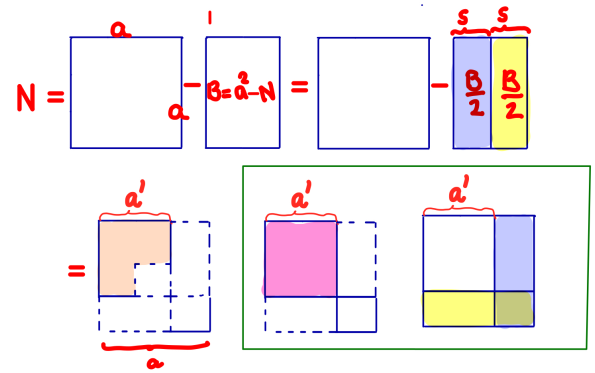
$$= (N^{1/2} - a) \ ((N^{1/2} + a)/(2a) - 1)$$

$$= (N^{1/2} - a)^2 / (2a).$$



Conclusion: new-error = $(\text{old\_error})^2 / (2a)$.

If old_error is <1 and a>1, we get a better estimate.



Babylonian method to find square roots, using an approximation a with $a^2>N$.

Given **N** and **a** such that:

- $N>0$, $a>0$
- $a^2>N$
- a^2 is close to N (thus a is close to $\sqrt{N}$)
- Hint: Write **s** in terms of **N** and **a**

Following the figures, find a' such that (in terms of a and N) such that $(a')^2$ is even closer N than a^2 (so a' is even closer than a to $\sqrt{N}$)

Using the formula given by the previous algorithm, compute an approximation of $\sqrt{2}$ starting with a=1.5

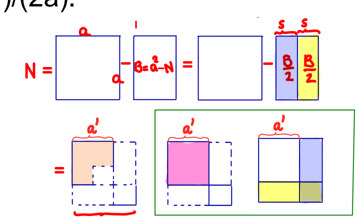
- **Start with a good guess**: Pick **a** where $\mathbf{a^2}$ is close to **N** (but $\mathbf{a^2 > N}$)

The side of the new (orange) square is $a' = a + (N-a^2)/(2a)$

Babylonian approximation of the square root of a number N when $a^2 > N$

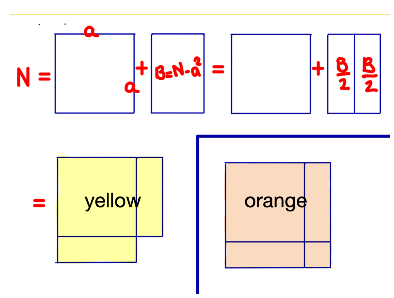
1. Pick a so that a^2 is close to N and $a^2 > N$.
2. The yellow shape has area $N = a^2 - (a^2 - N)$.
3. Split the excess area $(a^2 - N)$ into two equal rectangles, each of area $(a^2 - N)/2$, so their width is $(a^2 - N)/(2a)$.
4. Remove one strip from the right and one from the bottom of the a × a square to get the orange shape. Since the two stripes share a small square, we remove it too.
5. The small missing corner has side $(a^2 - N)/(2a)$, so the orange area is close to N.
6. Therefore $\sqrt{N}$ is close to the side of the orange square.
7. The orange square's side is $a - (a^2 - N)/(2a)$.

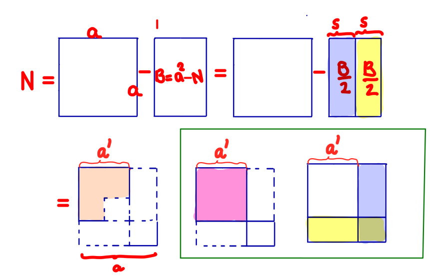
Thus the new approximation is $a' = a - (a^2 - N)/(2a) = a + (N - a^2)/(2a)$.



Babylonian approximation of the square root of a number N

Given a guess a, one finds a new guess $a' = a + (N-a^2)/(2a)$. In both cases (whether we start with $a^2>N$ or $a^2<N$.





- **What were some main features of Mesopotamian mathematics?** *(Think: number system, √2 approximation, reciprocal tables…)*
- **Why was base 60 practical for calculations?** *(Hint: divisors, fractions, and reciprocals.)*
- **How did Mesopotamians perform division?** *(Remember how they used tables…)*
- **What do we learn from the √2 tablet? (***Consider the value written, its accuracy, and how it was used.)*
- **How did Mesopotamians achieve accuracy in their work?** *(Think about pre-calculated values and reference lists.)*
- **How was mathematical knowledge preserved and passed on?***(Hint: scribal schools, coefficient lists, training.)*

Draft Slide Review – Instructions

- Divide into groups of **3**.
- **Divide 20 minutes by the number of students with slides in your group → that's each student's time.**
- Each student shows their slides. Teammates give helpful feedback by answering:
  1. Are there math concepts?
  2. Is there history?
  3. Are the slides clear and readable? Is the number of words appropriate?
- The **person whose first name comes after the presenter's alphabetically** writes 1–2 sentences for each question (more than yes/no), includes the **presenter's name**, and submits it in **Wooclap**.
- Remember: you're helping each other improve, not judging

Draft Slide Review – Instructions

- Divide into groups of **3 or 4**, with at least **3 members who brought slides**.
- Divide **20 minutes** by the number of students with slides in your group → that's each student's time.
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1. Are there math concepts?
2. Is there history?
3. Are the slides clear and readable? Is the number of words appropriate?

Example Outline – Plimpton 322

- **Slide 1:** Description of the tablet and when and where it was found.
- **Slide 2:** Background on Mesopotamian mathematics (base 60, use of tables).
- **Slide 3:** Transcription of Plimpton 322 into base 60 and Hindu–Arabic numerals, , discussion of the patterns in the entries..
- **Slide 4:** Example of a problem posed using one of the rows.
- **Slide 5:** Interpretation 1 – Trigonometric table.
- **Slide 6:** Interpretation 2 – Pythagorean triples.
- **Slide 7:** Interpretation 3 – Teacher's aid or school exercise.
- **Slide 8:** Conclusion – What Plimpton 322 tells us about ancient mathematics and how historians study it.

Tables of Reciprocals

n	1/n
2	0.5
3	0.333333333333333
4	0.25
5	0.2
6	0.166666666666667
7	0.142857142857143
8	0.125
9	0.111111111111111
10	0.1
11	0.0909090909090909
12	0.0833333333333333
13	0.0769230769230769
14	0.0714285714285714
15	0.0666666666666667
16	0.0625
17	0.0588235294117647
18	0.0555555555555556
19	0.0526315789473684
20	0.05
21	0.0476190476190476

n	1/n
22	0.0454545454545455
23	0.0434782608695652
24	0.0416666666666667
25	0.04
26	0.0384615384615385
27	0.037037037037037
28	0.0357142857142857
29	0.0344827586206897
30	0.0333333333333333
31	0.032258064516129
32	0.03125
33	0.0303030303030303
34	0.0294117647058824
35	0.0285714285714286
36	0.0277777777777778
37	0.027027027027027
38	0.0263157894736842
39	0.0256410256410256
40	0.025
41	0.024390243902439

Describe the numbers n such the 1/n has a terminating decimal. (here you have some approximations that might help.)

n	1/n
2	0.5
3	0.333333333333333
4	0.25
5	0.2
6	0.166666666666667
7	0.142857142857143
8	0.125
9	0.111111111111111
10	0.1
11	0.0909090909090909
12	0.0833333333333333
13	0.0769230769230769
14	0.0714285714285714
15	0.0666666666666667
16	0.0625
17	0.0588235294117647
18	0.0555555555555556
19	0.0526315789473684
20	0.05
21	0.0476190476190476

n	1/n
22	0.0454545454545455
23	0.0434782608695652
24	0.0416666666666667
25	0.04
26	0.0384615384615385
27	0.037037037037037
28	0.0357142857142857
29	0.0344827586206897
30	0.0333333333333333
31	0.032258064516129
32	0.03125
33	0.0303030303030303
34	0.0294117647058824
35	0.0285714285714286
36	0.0277777777777778
37	0.027027027027027
38	0.0263157894736842
39	0.0256410256410256
40	0.025
41	0.024390243902439

The first numbers n such the 1/n has a terminating decimal.

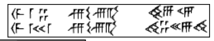
n	1/n	C= 1/n with integer fraction separator removed	C.n
2	0.5	5	10
4	0.25	25	100
5	0.2	2	10
8	0.125	125	1000
10	0.1	1	10
16	0.0625	625	10000
20	0.05	5	100
25	0.04	4	100
32	0.03125	3125	100000
40	0.025	25	1000
50	0.02	2	100

Integer fraction separator is the decimal point

2	30
3	20
4	15
5	12
6	10
8	7,30
9	6,40
10	6
12	5
15	4
16	3,45
18	3,20
20	3
24	2,30
25	2,24

27	2,13,20
30	2
32	1,52,30
36	1,40
40	1,30
45	1,20
48	1,15
50	1,12
54	1,6,40
1	1
1,4	56,15
1,12	50
1,15	48
1,20	45
1,21	44,26,40

What do you think this table is about?
Can you find a pattern?
(Numbers are in base 60)



Photographs courtesy of Jöran Friberg. https://www.math.stonybrook.edu/~tony/whatsnew/may12/tablets/bab-mult2.html

2	30
3	20
4	15
5	12
6	10
8	7,30
9	6,40
10	6
12	5
15	4
16	3,45
18	3,20
20	3
24	2,30
25	2,24

27	2,13,20
30	2
32	1,52,30
36	1,40
40	1,30
45	1,20
48	1,15
50	1,12
54	1,6,40
1	1
1,4	56,15
1,12	50
1,15	48
1,20	45
1,21	44,26,40

What do you think this table is about?
Can you find a pattern?
(Numbers are in base 60)

n	B=1/n in sexagesimal with integer-fraction separator removed	n	B decimal	n.B
2	30	2	30	60
3	20	3	20	60
4	15	4	15	60
5	12	5	12	60
6	10	6	10	60
8	7,30	8	450	3600
9	6,40	9	400	3600
10	6	10	6	60
12	5	12	5	60
15	4	15	4	60
16	3,45	16	225	3600
18	3,20	18	200	3600
20	3	20	3	60
24	2,30	24	150	3600
25	2,24	25	144	3600
27	2,13,20	27	8000	216000

Tables of Reciprocals and Regular numbers

Base-10: Reciprocals have finite decimal expansions only when prime factors are 2 and 5

Base-60: Reciprocals have finite sexagesimal expansions only when prime factors are 2, 3, and 5

Regular numbers: Numbers with only prime factors 2, 3, or 5

Why regular numbers mattered: Mesopotamians performed division by multiplying with reciprocals—but only regular numbers have finite sexagesimal representations

Reciprocal tables: Scribes created reference tables containing **only regular numbers** and their reciprocals.

Recall: Tables in Tablets

- Amazing aspect of the Babylonian's calculating skills: **construction of tables to aid calculation.**
- Babylonians used the formula

$$a.b = (1/2)((a+b)^2 - a^2 - b^2)$$

to make multiplication easier.

- They did not have an algorithm for long division. Instead they based their method on the fact that

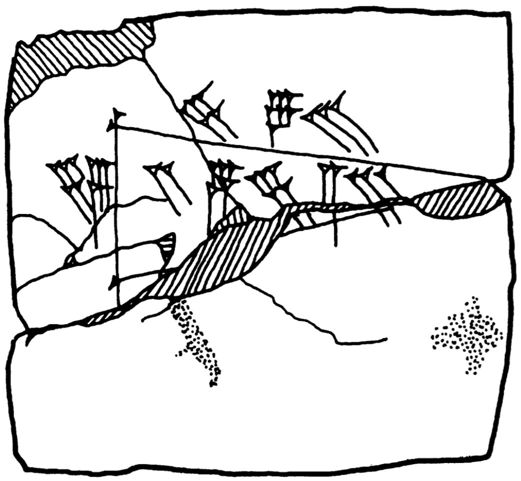
$$A \% B = A \times (1/B)$$

and used tables of reciprocals **(1/B)**.

Plimpton 322

Without overthinking, quickly draw a triangle (paper or tablet) and upload it.

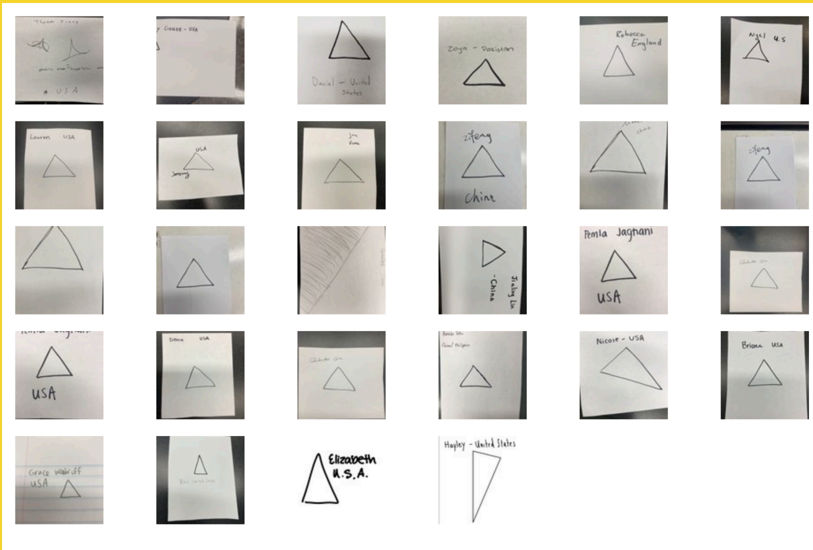
A Babylonian typical triangle



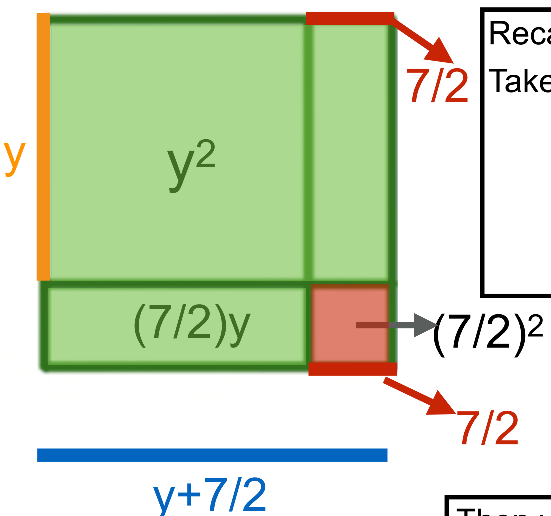
M 29-15-709 (obverse). Drawing by the Eleanor Robson - Words and Pictures: New Light on Plimpton 322 - The American Mathematical Monthly , Feb., 2002, Vol. 109, No. 2 (Feb., 2002), pp. 105-120

List features common to most drawings (besides the obvious fact that they are triangles)

List features common to most drawings (besides the obvious fact that they are triangles)



A number and its reciprocal differ by 7. Your task: form a quadratic equation whose solution gives that number.



Recall $x^2 - 7x - 1=0$.

Take $y=1/x \rightarrow y^2+7y-1=0$

$(y + 7/2)^2 =$

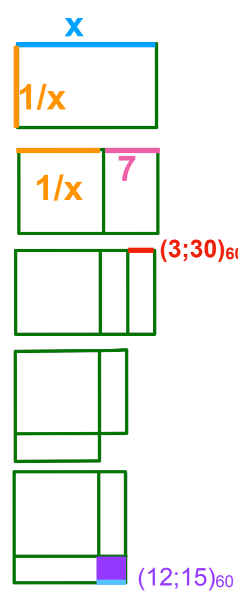
$(y^2+7y) + (7/2)^2 =$

$1 + (7/2)^2.$

Then $y + 7/2 = (1 + (7/2)^2)^{1/2}$

Completing the square

Find a number and its reciprocal whose difference is 7 (Solve the Babylonian way)



1. You: break in half the 7 by which the reciprocal exceeds its reciprocal, and **$(3;30)_{60}$** (will come up).
2. Multiply **$(3;30)_{60}$** by **$(3;30)_{60}$** and **$(12;15)_{60}$** (will come up).
3. Append [1 00, the area,] to the **$(12;15)_{60}$** which came up for you and **$(1\ 12;15)_{60}$** (will come up).
4. What is [the square-side of 1] 12;15? **$(8;30)_{60}$**.
5. Put down [**$(8;30)_{60}$** and] **$(8;30)_{60}$**, its equivalent, and subtract **$(3;30)_{60}$**, the takiltum-square, from one (of them); append **$(3;30)_{60}$** to one (of them).
6. One is **12**, the other is **5**.
7. The reciprocal is **12**, its reciprocal **5**.

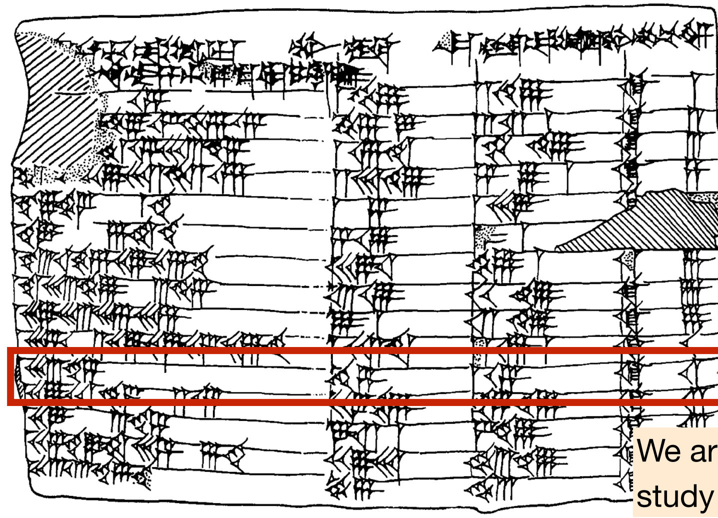
https://www.nytimes.com/2010/11/23/science/23babylon.html?smid=url-share

Plimpton 322

An Exhibition That Gets to the (Square) Root of Sumerian Math - NYTimes - Nov 22, 2010

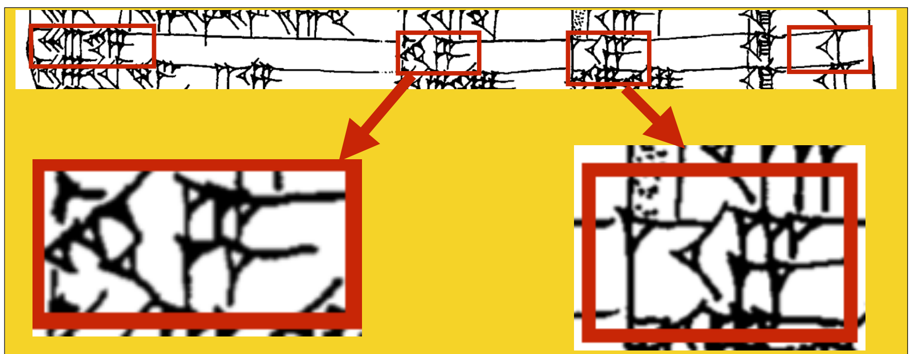
- First Western owner, George A. Plimpton bequeathed to Columbia University in the mid-1930s.
- Surviving correspondence shows that he bought the tablet for $10 from a dealer called Edgar J. Banks.
- Banks told him it came from an archaeological site called Senkereh in southern Iraq, whose ancient name was Larsa
- Approximate date of the tablet: 1800 BCE.

Plimpton 322 - Drawing by Eleanor Robson



We are going to study this row

Plimpton 322 (Robson, Eleanor. "Neither Sherlock Holmes nor Babylon: A Reassessment of Plimpton 322." Historia Mathematica 28.3 (2001): 167-206.)



Express one or two of these numbers in Hindu Arabic numerals

Plimpton 322 Transcription (and Minor Corrections)

$(d/l)^2$ or $(s/l)^2$	Short side s	Diagonal d	Row
$(59;0;15)_{60}$	$(1;59)_{60}$	$(2;49)_{60}$	1
$(56;56;58;14;50;6;15)_{60}$	$(56;7)_{60}$	$(3;12;1)_{60}$	2
$(55;7;41;15;33;45)_{60}$	$(1;16;41)_{60}$	$(1;50;49)_{60}$	3
$(53;10;29;32;52;16)_{60}$	$(3;31;49)_{60}$	$(5;9;1)_{60}$	4
$(48;54;1;40)_{60}$	$(1;5)_{60}$	$(1;37)_{60}$	5
$(47;6;41;40)_{60}$	$(5;19)_{60}$	$(8;1)_{60}$	6
$(43;11;56;28;26;40)_{60}$	$(38;11)_{60}$	$(59;1)_{60}$	7
$(41;33;45;14;3;45)_{60}$	$(13;19)_{60}$	$(20;49)_{60}$	8
$(38;33;36;36)_{60}$	$(9;1)_{60}$	$(12;49)_{60}$	9
$(35;10;2;28;27;24;26;40)_{60}$	$(1;22;41)_{60}$	$(2;16;1)_{60}$	10
$(33;45)_{60}$	$(45)_{60}$	$(1;15)_{60}$	11
$(29;21;54;2;15)_{60}$	$(27;59)_{60}$	$(48;49)_{60}$	12
$(27;0;3;45)_{60}$	$(7;12;1)_{60}$	$(4;49)_{60}$	13
$(25;48;51;35;6;40)_{60}$	$(29;31)_{60}$	$(53;49)_{60}$	14
$(23;13;46;40)_{60}$	$(56;56)_{60}$	$(53)_{60}$	15

$(1;\ 33\ 45)_{60}$	$(0;45)_{60}$	$(1;15)_{60}$	11
	45	75	

Row 11

a = $(1;33,45)_{60}$ → "d"
s = $(0;45)_{60}$ → "short side"
d = $(1;15)_{60}$ → "diagonal"

The scribe sets up the igi–igibi problem

- From **s**, I compute 2·**s**= $(1;30)_{60}$.
- I speak the igi-igibi problem: *"Find a number and its reciprocal; their difference is $(1;30)_{60}$."*

This is a simplified version of a problem that can be set up with Row 11.

The student's solution

- Halve the difference: $(1;30)_{60}$ ÷ 2 = **s** = $(0;45)_{60}$.
- Make a square and append one: 1 + s^2 = **a** = $(1;33,45)_{60}$.
- Take the square-side of a: $\sqrt{a}$ = **d** = $(1;15)_{60}$.
- Take the two numbers:
  - x = **d** + **s** = $(2;00)_{60}$, and
  - 1/x = **d** − **s** = $(0;30)_{60}$.
- Check: x × 1/x = 1.

Short side s	Diagonal d	Row	d^2–s^2
119	169	1	14,400
3367	4825	2	11,943,936
4601	6649	3	23,040,000
12709	18541	4	182,250,000
65	97	5	5,184
319	481	6	129,600
2291	3541	7	7,290,000
799	1249	8	921,600
481	769	9	360,000
4961	8161	10	41,990,400
45	75	11	3,600
1679	2929	12	5,760,000
161	289	13	57,600
1771	3229	14	7,290,000
56	106	15	8,100

The columns labeled by *Short side s, Diagonal d and Row* are three of the four columns of Plimpton 322.

The orange column ($d^2 - s^2$) wasn't on the original Plimpton 322 tablet — it was added to help you spot a pattern.
What kind of numbers do you see in this column?
(Hint: Square root)

Plimpton 322 in Hindu Arabic number system

Short side s	Diagonal d	Row	d^2–s^2	$(d^2–s^2)^{1/2}$
119	169	1	14,400	120
3367	4825	2	11,943,936	3,456
4601	6649	3	23,040,000	4,800
12709	18541	4	182,250,000	13,500
65	97	5	5,184	72
319	481	6	129,600	360
2291	3541	7	7,290,000	2,700
799	1249	8	921,600	960
481	769	9	360,000	600
4961	8161	10	41,990,400	6,480
45	75	11	3,600	60
1679	2929	12	5,760,000	2,400
161	289	13	57,600	240
1771	3229	14	7,290,000	2,700
56	106	15	8,100	90

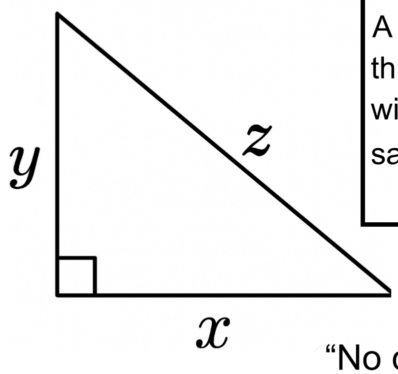
The numbers in the column (d^2-s^2) are all perfect squares!

Recall: a **perfect square** is an integer whose square root is also an integer.

Plimpton 322 in Hindu Arabic number system

$(d/l)2$ or $(s/l)2$	Short side s	Diagonal d	Row	d^2–s^2	$(d^2–s^2)^{1/2}$
(1).9834028	119	169	1	14400	120
(1).9491586	3367	4825	2	11943936	3456
(1).9188021	4601	6649	3	23040000	4800
(1).8862479	12709	18541	4	182250000	13500
(1).8150077	65	97	5	5184	72
(1).7851929	319	481	6	129600	360
(1).7199837	2291	3541	7	7290000	2700
(1).6927094	799	1249	8	921600	960
(1).6426694	481	769	9	360000	600
(1).5861226	4961	8161	10	41990400	6480
(1).5625	45	75	11	3600	60
(1).4894168	1679	2929	12	5760000	2400
(1).4500174	161	289	13	57600	240
(1).4302388	1771	3229	14	7290000	2700
(1).3871605	56	106	15	8100	90

Pythagorean triples



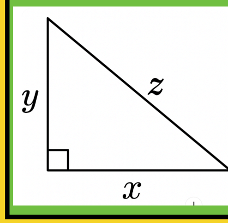
A **Pythagorean triple** is a set of three positive integers, ***x, y* and *z*** with no common factors that satisfy the equation $x^2 + y^2 = z^2$.

"No common factors" means that the largest positive integer that divides ***x, y,*** and ***z*** is 1.

These three interpretations of Plimpton 322 were proposed in different eras, each reflecting the point of view of its time. Sort them from the oldest to the most recent.

Trigonometric table: The first column lists a ratio related to a triangle's angle (like a tangent), making the tablet an early form of trigonometric table

Pythagorean triples: positive integers x, y, and z such that**:**



Teaching tool:The tablet gives examples (or starting values) for generating exercises, possibly used by teachers to create math problems for students.

Pythagorean triples (that is integer numbers A, L, D such that $A^2+L^2=D^2$)
In this case, entries are generated by pairs (p, q), with no common divisor, not both odd and such that p>q.
$L = 2pq$,
$D=p^2+q^2$
The remaining leg is p^2-q^2

Trigonometric table and pythagorean triples interpretations

line	*α*	*p*	*q*
1	44.76°	12	5
2	44.25°	1 04	27
3	43.79°	1 15	32
4	43.27°	2 05	54
5	42.08°	9	4
6	41.54°	20	9
7	40.32°	54	25
8	39.77°	32	15
9	38.72°	25	12
10	37.44°	1 21	40
11	36.87°	2	1
12	34.98°	48	25
13	33.86°	15	8
14	33.26°	50	27
15	31.89°	9	5

Eleanor Robson, Words and Pictures: New Light on Plimpton 322, The American Mathematical Monthly , Feb., 2002, Vol. 109, No. 2 (Feb., 2002), pp. 105-120

Trigonometric table: if Columns L and D contain the Legs and Diagonals of right-triangles, then the values in the first column are $\tan^2$ or $1/\cos^2$. The acute angles of the triangles decrease by approximately 1°

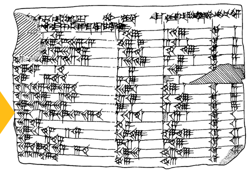
Robson's Reinterpretation: Educational Tool, Not Pythagorean Nor Trigonometry Table - Key Arguments

- **No trigonometry evidence**: Old Babylonian geometry lacked measurable angles - undermines trigonometric table theory
- **Consistent with Robson's reconstruction**: Rows can be derived from (x, 1/x) via half-sum/half-difference, then scaled by a regular factor.
- **Teacher's aid (interpretation):** A set of clean parameters for school problems (igi–igibi problems), not a function/angle table
- **Number choices:** Regular values (built from 2,3,5) keep sexagesimal exact and short—typical school practice..
- **Error pattern:** Slips match copying/scaling with reciprocals, not trig computation.
- **Context:** Found within scribal-school traditions (reciprocal/square tables), supporting a pedagogical use.

Note: **Historical context**: Ancient mathematics must be interpreted within its original cultural framework

Detour about (des)information online

I found this drawing of Plimpton 322 online, and looked for the source to include the credit in the slides.

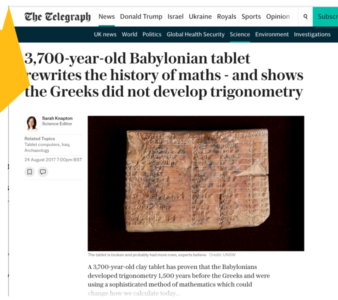


Plimpton 322 (Robson, Eleanor. "Neither Sherlock Holmes nor Babylon: A Reassessment of Plimpton 322." Historia Mathematica 28.3 (2001): 167-206.)

An image search lead me to this article

https://interestingengineering.com/culture/this-ancient-babylonian-tablet-proves-the-greeks-did-not-invent-trigonometry

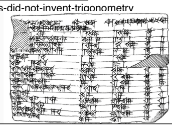
Which quoted this article



https://www.telegraph.co.uk/science/2017/08/24/3700-year-old-babylonian-tablet-rewrites-history-maths-could/

Which made me 😡

with wrong source!

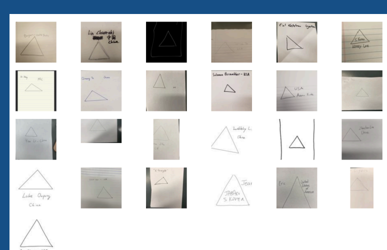


"Further, if we believe that **Plimpton 322 was intended to be a list of parameters to aid the setting of school mathematics problems** (and the typological evidence suggests that it was), the question '**how was the tablet calculated?**' does not have to have the same answer as the question 'what problems does the tablet set?' The first can be answered most satisfactorily **by reciprocal pairs,** as first suggested half a century ago, and the second by some sort of right-triangle problems."

E. Robson

Robson, Eleanor (August 2001), "Neither Sherlock Holmes nor Babylon: a reassessment of Plimpton 322", Historia Math., 28 (3): 167–206, doi:10.1006/hmat.2001.2317,

Ancient mathematical texts and artefacts, if we are to understand them fully, must be viewed in the light of their mathematico-historical context, and not treated as artificial, self-contained creations in the style of detective stories.
Eleanor Robson



Answer two or more of the following questions

1. Why were tables of reciprocals important in Mesopotamian mathematics?
Hint: Think about how they made division possible in base 60.

2. Looking at Plimpton 322, what mathematical pattern connects the short side and diagonal in each row?
Hint: Recall the "orange column" we added in class.

3. Different scholars have proposed different interpretations of Plimpton 322. What does the debate (trigonometric table, Pythagorean triples, teaching tool) show us about the way historians study ancient texts?
Hint: Each interpretation reflects the concerns of its own era.

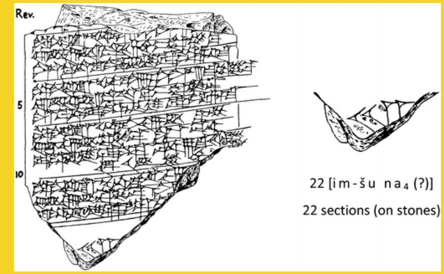
4. Think about the triangle drawing activity at the beginning of class. How did this experience connect to what we learned about Mesopotamian mathematics today?

Hint: Consider what this taught you about assumptions we make without realizing it.

Solutions of equations

Write an equation whose solution will be the answer to the problem below

Do you think this is an actual practical problem? Have you seen aa problem like this before? Can you suggest what the tablet might have been for?



Colophon of Tablet C4: end of the reverse (Neugebauer/Sachs 1945, Plate 13)

- I found a stone, (but) did not weigh it;
- (after) I subtracted one-seventh,
- added one-eleventh,
- (and) subtracted one-thir[teenth],
- I weighed (it): 1 ma-na.
- What was the origin(al weight) of the stone?

The origin(al weight)] of the stone was 1 ma–na, 9½ gin, (and) 2½ se.

- 60 gin = 1 ma-na
- 180 se = 1 gin

Educated guess: This problem is giving the instructions to solve a certain kind of equations. Which kind? (Linear, quadratic cubic… one unknown, two three…)

Numbers are in base 60

1. I summed the area and my square-side so that it was 0;45.
2. You put down 1, the projection.
3. You break off half of 1.
4. You combine 0;30 and 0;30.
5. You add 0;15 to 0;45.
6. 1 squares 1.
7. You take away 0;30 which you combined from inside 1 so that the square-side is 0;30.

Translation by Eleanor Robson