#1. $\ell^2(\mathbb{Z})$ $\left(\sum_{n=-\infty}^{\infty}|f(n)|^2\right)^{\frac{1}{2}} < \infty$, $f(n) \in \ell^2(\mathbb{Z})$.

$T : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ $(Tf)(n) = f(n+1)$.

! $\forall \lambda \in \mathbb{C}$, $T-\lambda I$ is injective, dense range.

$(Tf - \lambda f)(n) = f(n+1) - \lambda f(n)$

Consider $\mathrm{Ker}(T-\lambda I)(n) = \{ f(n) \mid f(n+1) - \lambda f(n) = 0 \}$

Then $\sum_{n=-\infty}^{\infty} |f(n)|^2 = \sum_{n=-\infty}^{\infty} |\lambda^n f(0)|^2 < \infty$

then $f(0) = 0$ (*).

$\cdot$ $f(n+1) = \lambda f(n)$ then

$f(n) = 0$ $\forall n \in \mathbb{Z}$.

(*) If not, $\sum_{n=0}^{\infty} |\lambda^n f(0)|^2 + \sum_{n=1}^{\infty} \left|\frac{1}{\lambda^n} f(0)\right|^2$ diverges.

$\mathrm{Ker}(T-\lambda I) = \{0\}$. $\Rightarrow$ It is injective.

(**). $|f(n+1) - \lambda f(n)|^2 \le 2\left(|f(n+1)|^2 + |\lambda|^2 |f(n)|^2\right)$.

$T-\lambda I$ is well defined for all $f \in \ell^2(\mathbb{Z})$.

~~Dense~~ range

For any seq $(\cdots, a_{-n}, a_{-n+1}, \cdots, a_0, \cdots a_m, a_{m+1}, \cdots)$ in $l^2(\mathbb{Z})$, $\lim_{n \to \pm\infty} a_n = 0$ and $\sum_{n=N}^{\infty} |a_n|, \sum_{n=-N}^{-\infty} |a_n| < \varepsilon$ for any given $\varepsilon > 0$.

Then it's sufficient to show that every finite seq in $l^2(\mathbb{Z})$ is in the range.

Let $\{a_n\}_{-\infty}^{\infty}$ be a finite seq in $l^2(\mathbb{Z})$.

that is, $a_n = 0$ where $n \leq -N-1$ or $n \geq M+1$. $(N, M \geq 0)$.

$$Tf(n) = f(n+1) - \lambda f(n).$$

Solve the eq. $f(n+1) - \lambda f(n) = a_n$

$$f(-N) - \lambda f(-N-1) = a_{-N-1} = 0$$

using the recurrsive relation.

$$f(-N) - \lambda^k f(-N-k) = a_{-N-1} + \lambda a_{-N-2} + \cdots + \lambda^k a_{-N-k}$$

Take the limit $k \to \infty$.

$$f(-N) = \sum_{i=0}^{\infty} a_{-N-1-i} \lambda^i = 0.$$

Then $f(-N)=0$, moreover,

$$f(-N+j)=0 \quad \forall\, j\geq 0.$$

Similarly, $f(M+2+j)=0 \quad \forall j\geq 0$.

Then for each $-N\leq k\leq M$,

$$a_{-N} = f(-N+1) - \cancel{\lambda f(-N)}$$

$$a_{-N+1} = f(-N+2) - \lambda f(-N+1)$$

$$\vdots$$

$$a_{M-1} = f(M) - \lambda f(M+1)$$

$$a_M = f(M+1) - \cancel{\lambda f(M+2)}$$

We can solve each equation of f in terms of a_j, $-N\leq j\leq M$. (solution skipped)

Then every finite seq. is in the range of T. for each $\lambda\in\mathbb{C}$.

#2. $\|T(v)\| \leq \|T^*(v)\|$ $\forall\, v \in V$.   positive definite inner product.

$T: V \to V$. $\dim V < \infty$

! $TT^* = T^*T$. (Hint: $tr(TT^* - T^*T) = 0$)

It's sufficient to show that each eigenvalue is non-negative real #s. Then with the hint every eigenvalue of $TT^* - T^*T$ is 0.
It implies $TT^* - T^*T = 0$, namely, $TT^* = T^*T$.

By the definition of the inner product and adjoint operator, with $\|Tv\| \leq \|T^*v\|$

$$(v, T^*Tv) = (Tv, Tv) \leq (T^*v, T^*v) = (v, TT^*v)$$

Thus $(v, TT^*v) - (v, T^*Tv) = (v, (TT^* - T^*T)v) \geq 0$ for all $v \in V$.

Then for every eigenvalues λ_i and corresponding eigen vectors v_i, of $TT^* - T^*T$

$$(\lambda_i v_i, v_i)$$

$$(\cancel{\bar{\lambda}_i}(v_i, (TT^* - T^*T)v_i) = \cancel{\lambda_i}\,(v_i, \lambda_i v_i) = \bar{\lambda}_i \|v_i\|^2 \geq 0$$

Then $\lambda_i = \bar{\lambda}_i$ and $\lambda_i \geq 0$.

$\Downarrow$

λ_i is real #.

//.

#3. X : vector space with positive definiteness.

! $B_r(0)$: strictly convex. (norm space case?)
$\forall r > 0$.

$\boxed{\text{Case 1}}$ $x, y \in B_r(0)$, that is, $\|x\|, \|y\| < r$.

Take the line segment between x and y.

$$tx + (1-t)y, \quad t \in [0,1].$$

$$\|tx + (1-t)y\| \le \|tx\| + \|(1-t)y\|$$
$$< tr + (1-t)r = r.$$

Hence any vector on the line segment between x and y is in $B_r(0)$.

$$\Rightarrow \boxed{B_r(0) \text{ is convex}}.$$

$\boxed{\text{Case 2}}$.

$x, y \in \overline{B_r(0)} \setminus B_r(0)$, that is, $\|x\| = \|y\| = r$.

<u>Trivial case</u>. if $x = -y$ then the line ~~seg~~ segment $\|tx + (1-t)y\|$, $t \in [0,1]$

$$= \|tx - (1-t)x\|$$
$$= \|(2t-1)x\|$$
$$= |2t-1| r \le r.$$ but equality only holds $t = 0$ and $t = 1$. $\Rightarrow$ strictly convex

Without loss of generality, we may assume that $\|x\| = \|y\| = r$ and $x \neq -y$.

$|x \cdot y| \leq \|x\| \|y\|$ Cauchy-Schwartz Inequality.
and equality holds only when $x = \lambda y$ for some $\lambda \in \mathbb{R}$. Then in this case, $|x \cdot y| < \|x\| \|y\|$.

Consider the line segment, $tx + (1-t)y$.

$$\begin{aligned}
&(tx + (1-t)y, tx + (1-t)y) \\
&= t^2 \|x\|^2 + 2t(1-t)\, x \cdot y + (1-t)^2 \|y\|^2 \\
&= t^2 r^2 + 2t(1-t)\, x \cdot y + (1-t)^2 r^2 \\
!\ &< (t^2 + 2t(1-t) + (1-t)^2) r^2 \quad \text{except } t=0 \text{ or } t=1. \\
&= (\cancel{t^2} + \cancel{2t} - \cancel{2t^2} + 1 - \cancel{2t} + \cancel{t^2}) r^2 \\
&= r^2.
\end{aligned}$$

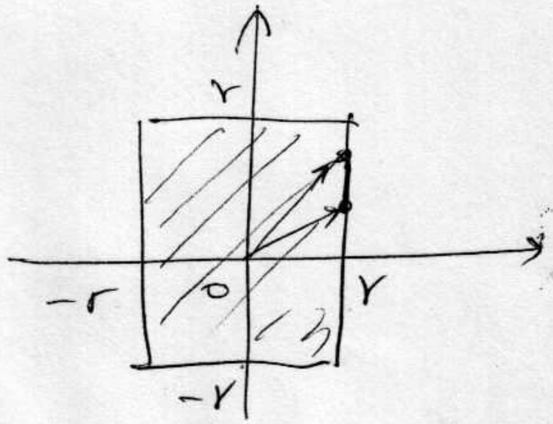
Hence, any vector in the line segment between x and y (with $\|x\|, \|y\| = r$) are in $B_r(0)$.

$\therefore$ Strictly convex.

Example) normed space.

In $\mathbb{R}^2$, $\|\cdot\|$, Define $\|(a,b)\| = \sup\{|a|, |b|\}$.

Then the ball $B_r(0)$ in the plane is the following.



It cannot be strictly convex.

#4. $L : X \longrightarrow Y$ and $\exists k,\ \|L(v)\| \geq k\|v\|$.

Conti. Banach space. ! $\boxed{\text{range of } L \text{ is closed}}$

Take a sequence $\{x_n\}$ in X and let

$L(x_n) = y_n$ for each $n \in \mathbb{N}$. with $y_n \xrightarrow{n\to\infty} y \in Y$??$\in \operatorname{Im}(L)$

$$\|y_n - y_m\| = \|L(x_n) - L(x_m)\| = \|L(x_n - x_m)\|$$

$$\geq k\|x_n - x_m\| \quad \forall m, n \in \mathbb{N}.$$

X is complete (it's a Banach space), Thus

Cauchy seq. $\{x_n\}$ in X

$$x_n \xrightarrow{n\to\infty} x \text{ in } X.$$

By the continuity of L,

$$L(x) = \lim_{n\to\infty} L(x_n) = \lim_{n\to\infty} y_n = y.$$

Hence $y \in \operatorname{Im}(L)$

$\hookrightarrow$ closed in Y.

#5. $\ell^1(\mathbb{N})$. $x = \{x_n\}$, $\|x\| = \sup |x_n|$ [illegible]

separable.

Take any countably many seq. in $\ell^1(\mathbb{N})$

$\{x_{1n}\} = \{x_{11}, x_{12}, x_{13}, x_{14}, \cdots, x_{1n}, \cdots\}$

$\{x_{2n}\} = \{x_{21}, x_{22}, x_{23}, x_{24}, \cdots, x_{2n}, \cdots\}$

$\vdots$

$\{x_{nn}\} = \{x_{n1}, x_{n2}, x_{n3}, x_{n4}, \cdots, x_{nn}, \cdots\}$

$\vdots$

Then $\sup\limits_{m \in \mathbb{N}} |\{x_{km}\}| \leq C(k) < \infty$.

Take another seq. $\{y_n\}$, $y_n = x_{nn} + 1$ $\forall n \in \mathbb{N}$.

using diagonal process, $\sup |\{y_n\}| \leq C(k) + 1 < \infty$

Then $\sup\limits_{n \in \mathbb{N}} \| \{x_{mn}\} - \{y_n\} \| \geq 1$ for every $m \in \mathbb{N}$.

Hence $\ell^1(\mathbb{N})$ is not separable.

#6. $L: X \longrightarrow Y$.

1. L is bounded $\Rightarrow$ $\operatorname{Ker} L$ is closed.

L is bounded $\Leftrightarrow$ L is continuous (skip the proof).

$\operatorname{Ker} L \overset{\text{def}}{=\!=} L^{-1}(\{0\})$

$\{0\}$ is closed and L is continuous.

$L^{-1}(\{0\}) = \operatorname{Ker} L$ is also closed.

2. $\operatorname{Ker} L$ is closed. $\Rightarrow$ L is bounded.

Let P is the linear subspace of $C[0,1]$ which contains all polynomials. with sup norm

$$T: P \longrightarrow P$$
$$f \longmapsto f'$$

is not continuous (unbounded operator)

$p_n \in P$ $\quad \|p_n\| = \sup\{p_n(t),\ t \in [0,1]\}$.

n: degree.

$T(t^n) = n t^{n-1}$, and $\|n t^{n-1}\| = n$.

unbounded.

but $\operatorname{Ker} T = \{\text{constants}\}$ is closed in P.

#7. $A : H \longrightarrow H$

bounded operator on a separable Hilbert space.

1. $\{e_i\}$ orthogonal basis on H. Set $a_{ij} = (Ae_i, e_j)$

$$! \sum_{i,j} |a_{ij}| = \sum_{i \geq 0} \|Ae_i\|^2 = \sum_{i \geq 0} \|A^* e_i\|^2$$

We can let $Ae_i = \sum_j a_{ij} e_j$.

Then $\|Ae_i\|^2 = \|\sum_j a_{ij} e_j\|^2 = \left(\sum_j a_{ij} e_j, \sum_j a_{ij} e_j\right)$

$$= \sum_j |a_{ij}|^2 \quad \text{becaus } (e_i, e_j) = 0 \quad i \neq j$$

$$(Ae_i, e_j) = \overline{(e_j, Ae_i)} = \overline{(A^* e_j, e_i)} = \sum_j a_{ij} e_i$$

Then $A^* e_i = \sum_j \overline{a_{ji} e_i}$

$$\|A^* e_i\|^2 = \|\sum_j a_{ji} e_i\|^2 = \left(\sum_j \overline{a_{ji}} e_j, \sum_j \overline{a_{ji}} e_j\right)$$

$$= \sum_j |a_{ji}|^2 \quad \text{because } (e_i, e_j) = 0 \quad i \neq j$$

Hence. $\sum_{i,j} |a_{ij}|^2 = \sum_i \|Ae_i\|^2 = \sum_i \|A^* e_i\|^2$.

(skip the unbounded case) _

#7

2. For some orthonormal basis

$$\sum_{i,j} |a_{ij}| < \infty$$

! The sum is independent of the basis.

Let $U : H \longrightarrow H$ be the operator on $\mathcal{H}$

$$e_i \longmapsto \tilde{e}_i$$

from orthonormal basis to itself.

$x \in H$

$$x = \sum_i (x, e_i) e_i \qquad \{e_i\} \text{ orthonormal basis}$$

$$= \sum_i (x, \tilde{e}_i)\tilde{e}_i \qquad \{\tilde{e}_i\} \quad \text{"}$$

$$Ux = U\left(\sum_i (x, e_i) e_i\right) = \sum_i (x, e_i)\, \tilde{e}_i$$

and

$$\|x\|^2 = \sum_i |(x, e_i)|^2 = \|Ux\|$$

Then U is an isometry iff U is unitary.

Consider $A \circ U : H \circlearrowleft$. By the 1.

$\sum |a_{ij}|$ is independent of the basis.

#6. $\nexists$ measure μ on $[0,1]$ such that

$\mu([0,1]) < \infty$ and $\mu(\{x\}) > 0$ for any $x \in [0,1]$.

Suppose not, $\mu([0,1]) = C < \infty$

Let $A_n = \{x \mid \frac{C}{n+1} < \mu(x) \leq \frac{C}{n}\}$

$A_n \cap A_m = \emptyset$ if $n \neq m$. and

$\bigcup_{n=1}^{\infty} A_n = [0,1]$.

However, each A_n contains only finitely many points. If not, $\mu([0,1]) = \infty$.

But the countable union of finite set cannot be $[0,1]$ because $[0,1]$ is uncountable.

$\not\downarrow$.