Problem 1

1) $a_0 + \sum_{n=0}^{k} a_n \sin(2\pi n x) + b_n \cos(2\pi n x)$, $a_i, b_j \in \mathbb{R}$.

dense set in $C(\mathbb{T})$. $f(x) = f(x+1)$.

- Stone-Weirstrass theorem.

· $\mathcal{B} \subset C(\mathbb{T})$

· $\mathcal{B}$ contains constant function.

· $\mathcal{B}$ is an algebra, that is, $f, g \in \mathcal{B}$ then $cf,\ f+g,\ f\cdot g \in \mathcal{B}$

· $\mathcal{B}$ separates points. For each ~~$x \in \mathcal{B}$, $\exists f, g$ such that~~ $x, y \in \mathcal{B}$, $x \neq y$.

$\exists f \in \mathcal{B}$, $f(x) \neq f(y)$.

Let $\mathcal{B}$ be the set of the trigonometric poly.

· $\mathcal{B}$ contains constant functions (clear).

· Separation of points. $x \neq y \Rightarrow \sin(2n\pi x) \neq \sin(2n\pi y)$.

· $f, g \in \mathcal{B} \Rightarrow af + bg \in \mathcal{B}$ (clear!).

· It's enough to show that for $f, g \in \mathcal{B}$.

$f \cdot g \in \mathcal{B}$.

$\sin(2\pi n x)\sin(2\pi m x)$

$$= \frac{1}{2}\left[-\cos(2\pi(n+m)x) + \cos(2\pi(m-n)x)\right]$$

$\cos(2\pi n x)\cos(2\pi m x)$

$$= \frac{1}{2}\left[\cos(2\pi(m+n)x) + \cos(2\pi(m-n)x)\right]$$

$\sin(2\pi n x)\cos(2\pi m x)$

$$= \frac{1}{2}\left[\sin(2\pi(m+n)x) + \sin(2\pi(n-m)x)\right]$$

implies that for $f, g \in B$, $f \cdot g \in B$.

Then $B$ is dense in $C(\mathbb{T})$.

2). $a_0 + \sum_{n=0}^{\infty} r^n (a_n \sin(2\pi n\theta) + b_n \cos(2\pi n\theta))$, $a_i, b_j \in \mathbb{R}$.

$(r, \theta) \in [0,1] \times [0,1]$ dense in $C(\mathbb{D})$.

($\because$ $r_1, r_2 \in [0,1]$ then $r_1 \cdot r_2 \in [0,1]$)

We can handle the coefficients of $\sin(2\pi n\theta)$ and $\cos(2\pi n\theta)$ to make $0 \le r \le 1$ for any trig ~~function on~~ polynomial on the unit disk.

Thus the proof of the density is same as the proof for problem 1. 1).

3). $I_2 = [0,1] \times [0,1]$. $C(I_2)$ can be unif. approximated by functions having the form.

$$\sum_{i=1}^{n} f_i(x) g_i(y), \qquad f_i, g_i \in C[0,1].$$

The proof is the proof ~~of~~ 1). Use the Stone-Weirstrass theorem.

Let $B$ ~~the~~ be the set of functions on $I_2$ on which element is $\sum_{i=1}^{n} f_i(x) g_i(y)$

- $\exists$ function separates each point on $I_2$ (clear)
- $f, g \in B \Rightarrow \alpha f + \beta g \in B$ (clear).
- $\left(\sum_{i=1}^{n} f_i g_i\right)\left(\sum_{i=1}^{m} p_i q_i\right)$

$$= \sum_i \sum_j \underbrace{f_i(x) p_j(x)} \underbrace{g_i(y) q_j(y)} \in B.$$

Then $B$ is dense in $C(I_2)$.

Problem 2.

1) $\mathbb{R}^2 \neq \bigcup_{\text{countable}} \{\text{lines}\}$.

$\mathbb{R}^2$ is a comple metric space $\subseteq$ Bairs space.

Bair's theorem. $\Rightarrow$ $\mathbb{R}^2$ is not the union of the nowhere dense set.

$\mathbb{R}^2 \setminus \{\text{line}\}$ is a dense open subset of $\mathbb{R}^2$, that is, any line is nowhere dense in $\mathbb{R}$.

$\Rightarrow$ $\mathbb{R}^2$ cannot be a countable union of lines.

! · $\mathbb{R}^2 \setminus \{\text{line}\}$ is open. (Take a line on $x$-axis.)

~~x-axis~~. any pt $\notin$ line has the definite distance between pt and line. Take an open ball $B_\varepsilon(pt)$ such that $\varepsilon <$ distance.

· $\mathbb{R}^2 \setminus \{\text{line}\}$ is dense. in $\mathbb{R}^2$.

Let $p \in$ line. and take $B_\varepsilon(p)$. then for each fixed $\varepsilon > 0$, we can choose $q \in \mathbb{R}^2 \setminus \{\text{line}\}$. Then we can make a seq $\{q_n\} \subseteq \mathbb{R}^2 \setminus \{\text{line}\}$ such that $q_n \to p$ as $n \to \infty$. Then $p$ is accumulation pt.

2). $\mathbb{R} \setminus \mathbb{Q}$ is not a union of closed sets, none of which contain an open subset.

Suppose not, then $\mathbb{R} = \cancel{\mathbb{Q}} (\mathbb{R} \setminus \mathbb{Q}) \cup (\mathbb{Q})$

$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ countable union of the closed set

and so is $\mathbb{R} \setminus \mathbb{Q}$.

Then $\mathbb{R}$ is countable. ~~It's contradiction.~~
union of the nowhere dense subsets.

But $\mathbb{R}$ contains an open set (open interval) but countable union of the nowhere dense subset cannot. It's the contradiction.

3) Closet is dense.

Any top. space $X$. $X \subseteq X$
open and close set

and $\overline{X} = X$

then $X$ is dense in $X$.

But any proper closed set cannot be dense.

Problem 3: For every $x \in [0,1]$

$\lim_{n \to \infty} f_n(x) = f(x)$ ; $\boxed{\text{! } f \text{ is conti. at some pt on } [0,1]}$

Let $A_n = \{x \mid |f_\cdot(x) - f_n(x)| \le \varepsilon\}$.

Problem 2. $F \subseteq X$

closed , on $X$

$\mathbb{R}$ is homeomorphic to $(-1,1)$

(eg. $g(x) = \frac{2}{\pi} \arctan(x)$ on $\mathbb{R}$.), let $f_0 = g \circ f$.

By Tietze extension theorem, $f_0$ can be extended for the whole $X$ into the $[-1,1]$. Let this extended function be $\bar{f}_0$. Then $\underbrace{\bar{f}_0^{-1}(\{-1\} \cup \{1\})}_{= D}$ is the ~~disj~~ closed subset of $X$, which is disjoint from $F$.

By the Urysohn's lemma, there exists a function $\phi : X \to [0,1]$ such that

~~$\phi(0) =$~~ $\phi(F) = \{1\}$ and $\phi(D) = \{0\}$.

Take $h(x) = \phi(x) \cdot \bar{f}_0(x)$

Then $h$ is an extension of $\bar{f}_0$

Problem 5 Find an interval on which there is a solution

- Fundamental local existence theorem of ODE.

Let $S = [t_0 - h, t_0 + h] \times B_R(Y_0) \subseteq \mathrm{Dom}(F)$

for $\frac{dy}{dt} = F(t, y)$ with $y(t_0) = Y_0$

- $\mathrm{Range}(F) \subset \mathbb{R}^n$ . $F$ is conti on $S$
- $|F(t_1, Y_1) - F(t_1, Y_2)| \leq K|Y_1 - Y_2|$ for some $K$,

$\Rightarrow$ $\exists!$ a solution of the above ODE, $y(t)$ on $(t_0 - r, t_0 + r)$ for some $r > 0$.

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1) $y' = x + y^3$. $y(0) = 0$,

Let $F(x, y) = x + y^3$ Take a domain of $F$
$S = [-N, N]^2$ for some $N > 0$

- $F$ is conti. on $S$
- $\mathrm{Rang}(F) \subseteq \mathbb{R}^2$
- $|F(x_1, y_1) - F(x_1, y_2)| = |y_1^3 - y_2^3|$

$$= |y_1 - y_2|\,|y_1^2 + y_1 y_2 + y_2^2|$$

$$\leq 3N^2 |y_1 - y_2|.$$

Then by the local existence theorem.

$\exists\ y(x)$ of the ODE.

2) $y' = x + \exp(y)$, $y(1) = 0$.

Let $F(x,y) = x + e^y$

and take the domain of $F$, $S = [-N+1, N+1] \times [-N, N]$.

- $F$ is conti on $S$

- Range $(F) \subseteq \mathbb{R}^2$

- $|F(x_1, y_1) - F(x_1, y_2)| = |e^{y_1} - e^{y_2}|$

mean value thm. $= e^{\xi}|y_1 - y_2|$ for some $\xi \in [-N, N]$

$\leq e^N |y_1 - y_2|$.

Problem 6. Picard method. (2nd approximation).

$y' = x - y^2, \quad 0 \le x \le \frac{1}{2}, \; y(0) = 0.$

$F(x,y) = x - y^2. \; y(0) = 0.$

$y_1(x) = y(0) + \int_0^x (x - y_0^2)\,dx$ where $y_0(x) = 0$

$= 0 + \int_0^x x - 0\,dx = \frac{1}{2}x^2.$ ( constant function)

$y_2(x) = y(0) + \int_0^x x - y_1^2\,dx$

$= \int_0^x x - \frac{1}{4}x^4\,dx = \boxed{\frac{1}{2}x^2 - \frac{1}{20}x^5}$

Error. bounds.

Caculate $|y_2' - x + y^2|$

$= \left|\cancel{x} - \frac{1}{4}x^4 - \cancel{x} + \left(\frac{1}{2}x^2 - \frac{1}{20}x^5\right)^2\right|$

$= \left|\cancel{\frac{1}{2}x^2 \; \frac{1}{4}x^4 \; \frac{1}{20}x^5}\right|$

$= \left|-\frac{1}{20}x^7 + \frac{1}{400}x^{10}\right| \le \frac{1}{20}\cdot\frac{1}{2^7} + \frac{1}{400\cdot 2^{10}}$

$\le \frac{1}{1280}$

Problem 3

$x' = |t|^{\alpha} + |x|^{\beta}$ Let $F(t,x) = |t|^{\alpha} + |x|^{\beta}$

1. local existence 2. local uniqueness.

① $\alpha \geq 0$, $\beta = 1$

$$|F(t_1, x_1) - F(t_1, x_2)| = \big||x_1| - |x_2|\big| \leq |x_1 - x_2|$$

F is $\boxed{\text{global}}$ Lipschitz function.

Then the solution of ODE exists on $\mathbb{R} \times \mathbb{R}^n$ (globally). Furthermore, $x(t_0) = x_0$ for any $t_0 \in \mathbb{R}$.

The uniqueness of the solution of the ODE near to $,($ $, t_0+\varepsilon)$ can be extended on the whole $\mathbb{R}$.

~~because $\alpha \geq 0$, implies $x(t)$ Do the domain of~~

$\Rightarrow$ The solution is unique globally.

② $\alpha \geq 0$, $\beta > 1$.

$F(t,x)$ is locally Lipschitz. Then there exists the unique local solution depending on initial condition.

For example, $\alpha = 0$ and $\beta = 2$.

$$\frac{dx}{dt} = 1 + x^2, \text{ take the initial condition}$$

$$x(0) = 0.$$

Then $$\frac{dx}{1+x^2} = dt.$$

$$\int \frac{dx}{1+x^2} = \int dt$$

$$\arctan x = t + C, \quad C = 0.$$

$$x = \tan(t).$$

It has a local solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

③ $\alpha < 0$, then there is no global solution because $F(t, x)$ is not defined at $t = 0$.

solution of the ODE cannot be defined at 0

even if $|F(t, x_1) - F(t, x_2)| \leq K |x_1 - x_2|$ for some $K > 0$ globally.

For example, $\alpha = -2$, $\beta = 0$. $x(t) = -\frac{1}{t} + t + C$,

$F(t,x) = |t|^{\alpha} + |x|^{\beta}$

is locally Lipschitz, that is, $|F(t_1, x_1) - F(t_1, x_2)| \le K|x_1 - x_2|$ where $(t_1, x_1), (t_1, x_2) \in [(-\varepsilon + t_0, t_0 + \varepsilon] \times B_\varepsilon(x_0)] \cap D$

D is a set containing $(t_0, x_0)$ of the domain F.

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Locally Lipschitz function F, there exists locally unique solution of the ODE.

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$V_\varepsilon^c$ $I$ and open in $I$ and $N \in \mathbb{N}$ s.t. $|f(x) - f_n(x)| \le \varepsilon$

whenever $x \in V_\varepsilon$ and $n \ge N$

pf: $\forall\, N \in \mathbb{N}$, let $\varepsilon' = \varepsilon/2$

Let $A_N = \{ x \in I \mid |f_m(x) - f_n(x)| \le \varepsilon' \text{ if } m \ge N \text{ and } n \ge N \}$

$A_N$ is closed since if $x \notin A_N$, i.e. $\exists\, m, n \ge N$ s.t.

$$|f_m(x) - f_n(x)| = \alpha \varepsilon' \quad \text{where } \alpha > 1$$

since $f_m - f_n$ is continuous,

$\exists\, U$ open s.t. $y \in U \Rightarrow |(f_m - f_n)(y) - (f_m - f_n)(x)| < \frac{\alpha - 1}{3} \varepsilon'$

$$\begin{aligned}
\Rightarrow |(f_m - f_n)(y)| &= |(f_m - f_n)(y) - (f_m - f_n)(x) + (f_m - f_n)(x)| \\
&\ge |f_m - f_n(x)| - |(f_m - f_n)(x) - (f_m - f_n)(y)| \\
&= \alpha \varepsilon' - |(f_m - f_n)(x) - (f_m - f_n)(y)| \\
&\ge \alpha \varepsilon' - \frac{\alpha - 1}{3} \varepsilon' = \frac{3\alpha - \alpha + 1}{3} \varepsilon' \\
&= \frac{2\alpha + 1}{3} \varepsilon' > \varepsilon' \\
&\qquad \because \alpha > 1
\end{aligned}$$

$\Rightarrow y \notin A_N \Rightarrow A_N^c$ is open $\Rightarrow$ $A_N$ is closed.

Moreover, $\lim_{n \to \infty} f_n(x) = f(x)$ so $\{f_n(x)\}_{n \in \mathbb{N}}$ is a Cauchy Sequence for every $x \in [0,1]$

$\Rightarrow \forall x \in I$, $x \in A_N$ for some $N \in \mathbb{N}$

$\Rightarrow I = \bigcup_{N \in \mathbb{N}} A_N$

Thus, $\exists\, N \in \mathbb{N}$ s.t. $A_N$ contains an open set $V_\varepsilon$

if $x \in V_\varepsilon$ and $n \geq N$

$\exists N' \in \mathbb{N}$ s.t $|f_m(x) - f(x)| < \frac{\varepsilon}{2}$ $\forall m \geq N'$

pick $m \geq \max\{N, N'\}$

$$\Rightarrow |f(x) - f_n(x)| \leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)|$$
$$\leq \frac{\varepsilon}{2} + \varepsilon' = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(2) In the $V_\varepsilon$ defined above, we can find an open ball $U_\varepsilon \subset V_\varepsilon$ s.t $|f(x) - f(y)| < 3\varepsilon$ $\forall x, y \in U_\varepsilon$

pick $x \in V_\varepsilon$, since $f_N$ is continuous on $[0,1]$

$\Rightarrow$ $f_N$ is uniformly continuous

$\Rightarrow \exists U_\varepsilon \subset V_\varepsilon$ s.t $|f_N(x) - f_N(y)| < \varepsilon$ $\forall x, y \in U_\varepsilon$

$$\Rightarrow |f(x) - f(y)| \leq \underbrace{|f(x) - f_N(x)|}_{\text{by (1)} < \varepsilon} + |f_N(x) - f_N(y)| + \underbrace{|f_N(y) - f(y)|}_{\text{by (1)} < \varepsilon}$$
$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

(3) Now we use (1) & (2) to find a continuity of $f$.
We define some intervals recursively:

(i) let $I_1 = [0,1]$, $U_1$ be an open set in $I$, $\forall$

s.t $|f(x) - f(y)| < 1$ , $x, y \in U_1$

(ii) pick a closed interval $I_2 \subset U_1$,

Let $N_2 \in \mathbb{N}$, $U_2$ open in $I_2$ s.t

$|f(x) - f(y)| < \frac{1}{2}$ , $x, y \in U_2$

(iii) If $I_{i-1}$, $U_{i-1}$, are now chosen so that

$\forall j = 1, 2, \cdots, i-1$ $U_j$ is open in $I_j$, where $I_j$ is a c interval an closed interval

Then we pick $I_i \subset U_{i-1}$

and $U_i \subset I_i$ s.t. $|f(x)-f(y)| < \frac{1}{i}$ $\forall x,y \in U_i$

$\Rightarrow$ Thus we may pick $x \in \bigcap_{i=1}^{\infty} I_i \neq \emptyset$

Notice that $x \in \bigcap_{i=1}^{\infty} U_i$ since $x \in I_{i+1} \subset U_i$ $\forall i \in \mathbb{N}$

$\Rightarrow$ $x$ is a continuity of $f$.