NOTE ON THE GEOMETRY OF GENERALIZED PARABOLIC TOWERS

MIKHAIL LYUBICH

Abstract. In this technical note we show that the geometry of generalized parabolic towers cannot be essentially bounded. It fills a gap in [L1].

1. Introduction

In 1985 Milnor posed a problem of existence of "wild attractors" for real quadratic polynomials [M]. In this situation a quadratic polynomial f would have a topologically transitive invariant interval I such that almost all orbits on I would converge to a Cantor set $A \subset I$. It was proven in [L1] that this is impossible (Theorem II). This result was deduced from a fundamental property of *exponential decay of geometry* (Theorem I) which has found many further applications since then.

The above results were stated in [L1] for a more general class of maps, S-unimodal maps with non-degenerate critical point. The S-unimodal case was reduced to the quadratic one by taking (generalized) quadratic-like limits of (generalized) renormalizations and applying to them a "quasiconformal trick". In this reduction one special case when the limits of generalized renormalizations become parabolic was missed from the consideration (this gap was pointed out by Oleg Kozlovski). In this note we will show how to fill this gap.

To this end we replace the generalized renormalization with the "generalized parabolic renormalization" relating different parabolic maps obtained in the limit. In this way we obtain a "parabolic tower" of generalized quadratic-like maps. Applying the quasi-conformal trick to such a tower, we conclude that it has exponentially decaying geometry. (The idea of the quasiconformal trick is to show that any two towers with the same combinatorics are quasiconformally equivalent and then to provide one example of a tower with a given combinatorics and decaying geometry.)

Note that the part of the argument which contains the gap is concerned with the smooth case only. In the case of quadratic polynomials

Date: January 5, 2002.

(or, more generally, analytic maps of Epstein class), the proof of [L1] does not need any adjustments.

In §2 we will outline, for reader's convenience, the original argument of [L1]. In the following two sections, §3 and §4, we will review the theory of generalized parabolic towers, real and complex. In §5 we will prove, by means of the quasiconformal trick, that such towers have decaying geometry. In the Appendix we will give another proof of this result, by adjusting complex estimates from [L2, §6].

Remark. Yet another way to prove decay of geometry of S-unimodal maps is by means of the asymptotically conformal extension of smooth maps as proposed in [L2, §12.2].

Acknowledgement. I thank the participants of the seminar in IMPA (August 2000), particularly A. Avila, O. Kozlovski, W. de Melo, and M. Shishikura, for their attention and valuable comments.

2. Outline of the original argument

Let $f : [-1, 1] \to [-1, 1]$ be a C^3 unimodal map with negative Schwarzian derivative and non-degenerate critical point at 0 such that $f(1) = 1$ and $f'(1) > 1$. Assume for simplicity that f is even. Such a map has a unique fixed point $\alpha \in (-1, 1)$. Assume this point is also repelling and f is non-renormalizable with recurrent critical point.

Let $I^0 = [\alpha, -\alpha]$. Define inductively the *principal nest* of 0-symmetric intervals

$$I^0 \supset I^1 \supset I^2 \supset \dots$$

as follows. Let r_n be the first landing time of the critical point at I^{n-1}. Then I^n is the component of $f^{-r_n}(I^{n-1})$ containing 0. Let $g_n = f^{r_n}|I^n$.

There are two different types of returns of the critical point to I^{n-1}, central and non-central. The return to I^{n-1} (and the corresponding level $n-1$) is called *central* if $g_n(0) \in I^n$. Let $\{n_k - 1\}$ be the sequence of non-central levels in the principal nest.

The ratios $\lambda_n = |I^n|/|I^{n-1}|$ are called the *scaling factors* of f.

Theorem 2.1 (Theorem II of [L1]). *There exist constants $\rho \in (0, 1)$ and $C > 0$ such that $\lambda_{n_k+1} \le C\rho^k$.*

We say that maps satisfying the conclusion of Theorem II have *(exponentially) decaying geometry.*

The proof of this theorem given in [L1] contained a gap which will be explained and filled below.

Theorem II was first proven under the assumption that one of the scaling factors is sufficiently small:

Lemma 2.2. *There exists an absolute constant $\delta > 0$ such that if $\lambda_N < \delta$ for some N then f has exponentially decaying geometry.*

The further analysis depended on the combinatorics of f defined in terms of generalized renormalizations $g_n : \cup I_i^n \to I^{n-1}$ of f. Consider the sequence of first return maps $f_n : \cup I_i^n \to I^{n-1}$ to the intervals of the principal nest. For each n, I_i^n are disjoint closed intervals, $I_0^n \equiv I^n$, the map $f_n | I_0^n$ coincides with the previously introduced g_n, and f_n diffeomorphically maps each non-central interval I_i^n, $i \neq 0$, onto I^{n-1}. The *generalized renormalization* g_n is the restriction of f_n to the union of intervals I_i^n intersecting the critical orbit.

Let us say that f has *essentially bounded geometry* if all the non-central intervals I_i^n, $i \neq 0$, and all the gaps in between (i.e., the components of $I^{n-1} \setminus \cup I_i^n$) are commensurable (uniformly on all levels). Note that in the case of essentially bounded geometry, there are only finitely many intervals I_i^n on each level (and their number is bounded).

Lemma 2.3 ([L1], §3)**.** *Maps with essentially unbounded geometry have exponentially decaying geometry.*

To treat the bounded case we passed to limits of the renormalizations. These limits have better qualities than the original maps.

A map $g : \cup_{i=0}^l V_i \to \Delta$ is called a (real) *generalized quadratic-like map* (see [L2]) if V_i and Δ are ($\mathbb{R}$-symmetric) topological disks in $\mathbb{C}$, $V_i \Subset \Delta$, $\bar{V}_i$ are pairwise disjoint, $g : V_0 \to \Delta$ is a double branched covering, while the maps $g : V_i \to \Delta$ are conformal isomorphisms (preserving the real line). We put the critical point of $g|V_0$ at the origin. In what follows, all generalized quadratic-like maps are assumed to be real.

Lemma 2.4 ([L1], §§3 - 4)**.** *If f has essentially bounded geometry then the renormalizations g_n have an analytic limit which admits an extension to a generalized quadratic-like map $g : \cup V_i^n \to \Delta$.*

If the limit g of renormalizations has decaying geometry then the original map f has arbitrary small scaling factors and hence has exponentially decaying geometry by Lemma 2.2. This reduces Theorem 2.1 to the quadratic-like case. We treated this case by means of the "quasiconformal trick" based on the following lemma:

Lemma 2.5 (Kahn, see Lemma 5.3 of [L1])**.** *Any two non-renormalizable generalized quadratic-like maps g and $\tilde{g}$ with the same combinatorics are qc conjugate.* [1]

[1] "qc" stands for "quasiconformal".

Since the property of decaying geometry is qc invariant, it is enough to give one example of a quadratic-like map with giving combinatorics and decaying geometry to conclude that this property is valid for all maps with that combinatorics. But it is easy to show (using a version of the kneading theory, see [L1], §1) that for any given combinatorics, there exists a generalized quadratic-like map $g : \cup V_i^n \to \Delta$ with this combinatorics whose central branch $g|V_0^n$ is purely quadratic while all non-central branches $g|V_i^n$ are linear. Moreover, the central domain V_0^n can be selected arbitrary small in Δ, so that g has exponentially decaying geometry by Lemma 2.2.

What is overlooked in the above argument is that the limit quadratic-like map g in Lemma 2.4 does not have to be non-renormalizable. However, as the following simple lemma shows (compare [[L2], Theorem V]), the only alternative is parabolic:

Lemma 2.6. *Assume f has essentially bounded geometry. Let $n-1$ be a central level of the principal nest and $m-1$ be the following non-central level, $m > n+N$. If N is sufficiently big then $0 \notin g_n(I^n)$ ("low return"), and $g_n|I^n$ is close to a unimodal map with parabolic fixed point.*

Passing to limits of generalized renormalizations on consecutive non-central levels, we obtain a "parabolic tower" consisting of maps related by either the above generalized renormalization or by its parabolic version. Applying the quasiconformal trick to such towers, we complete the proof of Theorem 2.1.

Below we will elaborate on this argument.

In the case when f is a quadratic polynomial (or more generally, an analytic map of Epstein class), Lemma 2.4 is not needed since f itself has a generalized quadratic-like renormalization g to which we can apply the quasiconformal trick. So, in this case, the proof of Theorem II given in [L1] was complete.

3. Real parabolic towers

3.1. **Definition.** Let $\mathcal{R}$ be the class of C^3-smooth one-dimensional maps (considered up to rescaling)

$$g : \bigcup_{i=0}^{l} I_i \to T,$$

with negative Schwarzian derivative such that

• the I_i are disjoint closed intervals contained in $\operatorname{int} T^n$; $I_0 \ni 0$;

• the restrictions $g : I_i \to T$ are diffeomorphisms for $i \neq 0$;
• the restriction $g : (I_0, \partial I_0) \to (T, \partial T)$ is an even unimodal map with a *non-degenerate* critical point at 0;

We let $\mathcal{R}(\epsilon)$ be the subclass of maps which admit an extension $\hat{I}_i \to \hat{T}$ of class $\mathcal{R}$ with $\hat{T} = (1+\epsilon)T$ (ϵ is called the "extension parameter").

Consider a nest of closed intervals T^n containing 0, and a sequence of maps

$$g_n : \bigcup_{i=0}^{l_n} I_i^n \to T^n,$$

of class $\mathcal{R}(\epsilon)$ for some $\epsilon > 0$ (independent of the level). We assume that the next map, g_{n+1}, is related to the previous one, g_n, by a generalized renormalization of three possible types, standard, cascade, or parabolic. We will now describe these renormalizations.

The *standard* renormalization is applied in the non-central case, i.e., when $g_n(0) \notin I_0^n$. Then $T^{n+1} = I_0^n$, and g_{n+1} is the first return map to T^{n+1} under iterates of g_n restricted to some intervals I_i^{n+1}.

The *cascade* renormalization is applied in the central escaping case i.e., when $g_n(0) \in I_0^n$ but there is some $t > 0$ and $j \neq 0$ such that $(g_n \mid I_0^n)^{\circ t}(0) \in I_j^n$. In this case, T^{n+1} is the pull-back of I_j^n under $g_n^{\circ t}$ containing 0 (i.e., the component of $g_n^{-t} I_j^n$ containing 0), and $g_{n+1}|I_i^{n+1}$ is composed of some branches of the first return map to T^{n+1}.

In this case we also consider a bigger family $\mathcal{L}^n$ of intervals L_s^n obtained from the non-central intervals I_j^n, $j \neq 0$, as the pull-backs under iterates of $g_n|I_0^n$. We define the *transit map* as follows: $\Psi_n|L_s^n = g_n^{\circ t}|L_s^n$ where $g_n^{\circ t}$ maps L_s^n into I_j^n. One of these intervals coincides with T^{n+1}; it will be denoted L_0^n. The times $t = t(n,s)$ will be called the *transit times* of the intervals L_s^n.

Let $G_n = g_n \circ \Psi_n$. This is a "Bernoulli map" which diffeomorphically maps each non-central interval L_j^n onto T^n and unimodally maps $(L_0^n, \partial L_0^n)$ into $(T^n, \partial T^n)$.

The *parabolic* renormalization is applied when $g_n : I_0^n \to T^n$ is a parabolic map supplied with a Lavours map τ_n through the parabolic point (see [D1, Sh]). In this case we consider pull-backs L_s^n of the non-central intervals I_j^n under joint iterates of $g_n|I_0^n$ and τ_n. Thus, for any $L = L_s^n$, there exist $k = k$ and l such that the transit map

$$\Psi_n|L \equiv g_n^{\circ l} \circ \tau_n \circ g_n^{\circ k}|\ L$$

is either a diffeomorphism onto some I_j^n (for $s \neq 0$) or is a unimodal map (for $s = 0$). Let $T^{n+1} = L_0^n$ and let $G_n = g_n \circ \Psi_n$ be the associated

Bernoulli map. Then g_{n+1} is composed of some branches of the first return map to T^{n+1} under iterates of G_n.

Let us call such a sequence of maps (g_n, Ψ_n) a (one-sided *parabolic) tower* $\mathcal{G}$ of class $\mathcal{T}(\epsilon)$. (We will usually skip the transit maps Ψ_n from the notation.) Let $\mathcal{T} = \cup \mathcal{T}(\epsilon)$. In the case when there are no parabolic maps among the g_n, we we also refer to the tower as a *principal nest* (of generalized renormalizations of the top map).

We will consider towers up to rescaling of the base interval T^0.

Let $\mathcal{I}^n$ stand for the family of intervals I_i^n and let $\mathcal{I}_*^n = \mathcal{I}^n \smallsetminus \{I_0^n\}$. In the standard case, we let $G_n \equiv g_n$ and $L_j^n \equiv I_j^n$.

3.2. **Combinatorics of towers.** The combinatorics $\kappa(g_n)$ is determined by the order of the intervals L_s^n on the real line and by the itineraries of the intervals I_i^{n+1} through the L_s^n under the iterates of G_n until the first return back to T^{n+1}. The combinatorics $\kappa = \kappa(\mathcal{G})$ of a tower $\mathcal{G} = \{g_n\}$ is the sequence $\{\kappa(g_n)\}$.

Given an interval $L \in \mathcal{L}^n$, let k_+ be defined by the property that $g_n^{\circ k_+} L \in \mathcal{I}_*$ (if such k_+ does not exist, we let $k_+ = \infty$), and let k_- be the biggest k such that $L = g_n^{\circ k} K$ for some interval $K \in \mathcal{L}^n$. We define the *depth* of L (and of any point $x \in L$) as $\min(k_-, k_+)$.

Two combinatorics on a given level are called *essentially equivalent* if they become the same after removing all intervals of sufficiently big depth. Two tower combinatorics are essentially equivalent if they are such on every level.

Let us consider a combinatorics κ of a parabolic tower and a sequence of essentially equivalent combinatorics κ_n of principal nests. We say that $\kappa_n \to \kappa$ if the transit times of the κ_n on the parabolic levels of κ grow to ∞. (See §3 of [Hi] for a more detailed discussion of the space of combinatorics.)

The combinatorics of a tower is called *essentially bounded* if:
- The numbers l_n of the intervals I_i^n are bounded;
- If an interval $J \subset I_j^n$ belongs to the itinerary of some $I \in \mathcal{I}^{n+1}$ then $g_n J$ lands at an interval $L \in \mathcal{L}^n$ of bounded depth;
- The return times of the intervals $I \in \mathcal{I}^{n+1}$ back to T^{n+1} under the iterates of G_n are bounded.

If all the above numbers are bounded by p we say that the essential combinatorics of the tower is bounded by p.

3.3. **Essentially bounded geometry.** We say that a parabolic tower has an *essentially bounded geometry* if:

- the intervals I_j^n and the gaps in between them (i.e., connected components of $T^n \smallsetminus \cup I_j^n$) are all commensurable with T^n;

• all the maps $(g_n|I^n_j)^{-1}$ $(j \neq 0)$, τ_n^{-1} admit a diffeomorphic extension with positive Schwarzian derivative to a definitely bigger neighborhoods of their domains of definition;
• $g_n|I^n_0$ is a composition of the quadratic map and a diffeomorphism h_n whose inverse admits an extension as above. b

By an obvious quantification of this notion (which also incorporates the extension parameter ϵ of the tower), we can make sense of the notion of "essentially C-bounded geometry" Moreover, this notion makes sense on any particular level of the tower. Let $\mathcal{T}(p, C)$ stand for the space of parabolic towers with essentially C-bounded geometry on the top level and essentially p-bounded combinatorics. The following statement is a slightly modified Lemma 8.8 of [L2].

Lemma 3.1. *Given C and p, towers $\mathcal{G} \in \mathcal{T}(p, C)$ have essentially bounded geometry on every level.*

We are ready to formulate the main result of this note:

Theorem 3.2. *There are no parabolic towers with essentially bounded geometry.*

This theorem together with Lemma 2.3 imply Theorem 2.1.

3.4. **Scaling factors.** The ratios $\lambda_n = |T^n|/|T^{n-1}|$ are called the *scaling factors* of the tower.

Lemma 3.3. *There exists an absolute constant $\delta > 0$ such that if $\lambda_1 < \delta$ then $\lambda_n \leq C\rho^n$ for some $\rho \in (0, 1)$, $C > 0$.*

This is a straightforward generalization of Lemma 2.2. Note that in the cascade case, the proof of Lemma 2.2 uses only the transit map through the cascade, so that it applies to the parabolic case as well.

4. Complex parabolic towers

4.1. **Analytic limits of Epstein class.** A sequence of towers $\mathcal{G}_m = \{g_{m,n}, \Psi_{m,n}\}$ converges to a tower $\mathcal{G} = \{g_n, \Psi_n\}$ if on every level n the domains of the maps $g_{m,n}$ converge to the domain D_n of g_n in the Hausdorff topology, $g_{m,n} \to g_n$ in the C^1-topology on the components of D_n, and $\Psi_{m,n} \to \Psi_n$ in the C^1-topology on the components of the domain of Ψ_n.

A parabolic tower is called a tower of *Epstein class* if
• all the maps g_n and Ψ_n are analytic;
• the inverse maps $(g_n|I^n_j)^{-1}$, $j \neq 0$, admit a holomorphic extension to the slit plane $\mathbb{C} \setminus (1 + \epsilon)T^n$, and the similar statement holds for the transit maps Ψ_n ;

• the map $g_n|I_0^n$ is a composition of the quadratic map and a diffeomorphism which admits a holomorphic extension as in the previous item.

By the m-shifted tower $\mathcal{G}^m$ we mean the tower $\{g_n\}_{n\geq m}$. Now, the Shuffling Lemma (see [MS, Ch. VI, Theorem 2.3]) yields:

Lemma 4.1. *Let $\mathcal{G}$ be a parabolic tower with bounded geometry. Then any sequence of shifted towers $\mathcal{G}^m$ contains a subsequence converging to a tower of Epstein class.*

4.2. **Generalized quadratic-like maps.** We will now complexify the discussion of §3.

Let $\mathcal{C}$ be the space of (real symmetric) generalized quadratic-like maps (up to rescaling of Δ). Let $\mathcal{C}(\epsilon)$ be the subspace of maps which admit an extension $\hat{V}_i \to \hat{\Delta}$ of class $\mathcal{C}$ with $\mathrm{mod}(\hat{\Delta} \smallsetminus \Delta) > \epsilon$.

This notion immediately allows us to complexify the notion of parabolic tower. Such a tower consists of generalized quadratic-like maps $g_n : \cup V_i^n \to \Delta^n$ related by either standard, or cascade, or parabolic renormalization, and associated transit maps $\Psi_n : U_s^n \to L_j^n$, where U_s^n are topological disks whose real slices coincide with the intervals L_s^n. We will also consider the complexification of the associated Bernoulli maps $G_n : \cup U_s^n \to \Delta^n$.

4.3. **A priori bounds and rigidity.** By repeating the argument of [L1, §4] (using on parabolic levels the parabolic transit maps instead of almost parabolic transit maps), we obtain:

Lemma 4.2 (Complex bounds)**.** *If $\{g_n\}$ is a parabolic tower of Epstein class then one of the maps g_n admits a generalized quadratic-like extension.*

Remark. One can also use complex bounds of [LY] to prove the above statement.

Proposition 4.3 ([Hi], Prop. 6.1)**.** *A complex parabolic tower is uniquely determined by its combinatorics and the top map g_1.*

4.4. **Space of towers.** Similarly to the real case, the space of generalized quadratic-like maps $g_m : \cup_{i=0}^{l} V_i \to \Delta$ and the space of complex towers are endowed with natural topologies.

Given a generalized quadratic-like map $g : \cup V_i \to \Delta$, let $A_i \Subset \Delta \smallsetminus \overline{\cup V_i}$ be an annulus of maximum modulus surrounding V_i but not surrounding the other domains V_j, $j \neq i$. Let $\mathrm{mod}(g)$ $\min \mathrm{mod}(A_i)$.

Lemma 4.4. *The space of complex parabolic towers $\mathcal{G} \in \mathcal{T}(p, C)$ with a definite modulus on the top level is compact.*

Proof. A simple estimate shows that $\operatorname{mod}(g_{n+1}) \geq 1/2 \operatorname{mod}(g_n)$ so that the moduli of the tower maps on every level are definite. Moreover, the extensions for the inverse maps $(g_n|V_i^n)^{-1}$ provide us with extensions of the transit maps with a definite modulus. Now the assertion follows from the Carathéodory compactness of the family of pointed domains with bounded from below inner radius (see [McM, Thm 5.2]) and normality argument. ☐

Let us consider a holomorphic family of generalized quadratic-like maps $g_\lambda : \cup V_{i,\lambda} \to \Delta_\lambda$ over a topological disk $\Lambda \subset \mathbb{C}$ such that the boundary $\partial\Delta_\lambda$ moves holomorphically with λ. Such a family is called *proper* if $g_\lambda(0) \to \partial\Delta_\lambda$ as $\lambda \to \partial\Lambda$. A proper family is called *unfolded* if for a Jordan curve $\gamma \subset \Lambda$ close to $\partial\Lambda$ in the Hausdorff topology, the curve $\lambda \mapsto g_\lambda(0)$, $\lambda \in \gamma$, has winding number 1 around 0 (see [DH, L2]).

Lemma 4.5. *Let $\mathcal{G}$ be a complex parabolic tower with combinatorics κ. Let $\{\kappa_l\}$ be a sequence of essentially equivalent nest combinatorics converging to κ. If these combinatorics have a sufficiently big transit time on the top parabolic level, then there exists a sequence $\mathcal{G}_l$ of complex principal nests with combinatorics κ_l converging to $\mathcal{G}$.*

Proof. Let g_s be the first parabolic map in the tower. Include the map $g_1 \equiv g_{1,0}$ into a one-parameter family of generalized quadratic-like maps $g_{1,\lambda}$ over some domain Λ with efficiently changing multiplier of the parabolic point of g_s (i.e., the parameter derivative of this multiplier does not vanish). Then there is a sequence of domains $\Omega_m \subset \Lambda$ converging to 0 such that the combinatorics of the tower does not change on levels $1, 2, \ldots, s$, and the associated Bernoulli maps $F_{m,\lambda}$ on level s form proper unfolded families over the Ω_m (compare [D2]). These domains are labelled by the transit time m of the critical point through the almost parabolic region until it lands at the appropriate non-central domain of $g_{s,\lambda}$. If the transit time of the combinatorics κ_l on level s is sufficiently big, then we can select m to be equal to this transit time.

Restricting the family $F_{m,\lambda}$ to the real slice of Ω_m, we obtain a full family of maps of class $\mathcal{R}$. Since any admissible nest combinatorics is realizable in such a family (see [L1, §1]), the desired combinatorics κ_l are all realizable by some principle nests $\mathcal{G}_l$. .

By Lemma 4.4, the sequence $\{\mathcal{G}_l\}$ is pre-compact. Moreover, any limit parabolic tower has combinatorics κ and by Proposition 4.3 coincides with $\mathcal{G}$. Hence $\mathcal{G}_l \to \mathcal{G}$. ☐

5. Quasi-conformal trick

We will now adjust the argument of [L1, §5] to the case of parabolic towers.

A homeomorphism $h : \Delta \to \tilde{\Delta}$ is called a *pseudo-conjugacy* between two generalized quadratic-like maps $g : \cup V_i \to \Delta$ and $\tilde{g} : \cup \tilde{V}_i \to \tilde{\Delta}$ if h is equivariant on the boundary, i.e., $h(gz) = \tilde{g}(hz)$ for $z \in \cup \partial V_i$.

Remark. Under the circumstances of Lemma 2.5, any qc pseudo-conjugacy between g_0 and $\tilde{g}_0$ promotes to a qc conjugacy with the same dilatation.

Lemma 5.1. *Any two parabolic towers with the same combinatorics are qc conjugate.*

Proof. Consider two parabolic towers $\mathcal{G}$ and $\tilde{\mathcal{G}}$ with the same combinatorics. By Lemma 4.5, they can be approximated by combinatorially equivalent principal nests $\mathcal{G}_n$ and $\tilde{\mathcal{G}}_n$. By the above Remark, these nests are qc equivalent with bounded dilatation. Passing to limits, we conclude that the towers $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are qc equivalent as well. □

Proof of Theorem 3.2. If there exists a real parabolic tower with essentially bounded geometry, then by Lemmas 4.1 and 4.2, there exists a complex parabolic tower with this property as well. We will now show by means of the quasi-conformal trick that such complex towers do not exist.

It is easy to create a complex parabolic tower G with a given combinatorics and an arbitrary small initial scaling factor λ_1. Namely, consider a generalized quadratic-like map $g : \cup V_i \to \Delta$ with the given combinatorics whose central branch $g|\Delta$ is purely quadratic, while the non-central branches $g|\mathbb{V}_i$, $i \neq 0$, are linear. The central domain V_0 of this map can be selected as an arbitrary small round disk. Changing the transit map, we can produce a full family of towers on the consecutive levels (compare the proof of Lemma 4.5), so that there is a tower with the required combinatorics among them.

By Lemma 3.3, the scaling factors of $\mathcal{G}$ decay exponentially.

Let $\tilde{\mathcal{G}}$ be an arbitrary parabolic tower with the same combinatorics. By Lemma 5.1, $\tilde{\mathcal{G}}$ is qc equivalent to $\mathcal{G}$. Since the exponential decay of the scaling factors is a qc invariant property, it holds for $\tilde{\mathcal{G}}$ as well. □

6. Appendix

6.1. **Isles and asymmetric moduli.** Here we reproduce for convenience some definitions from §5 of [L2].

Let $\{V_i\}_{i\in\mathcal{I}} \subset \mathcal{V}$ be a finite family of disjoint domains consisting of at least two elements (that is $|\mathcal{I}| \geq 2$) and containing a critical domain V_0. Let us call such a family *admissible*. We will freely identify the label set $\mathcal{I}$ with the family itself.

Given a domain $D \subset \Delta$ whose boundary does not intersect V_i, let $\mathcal{I}|D$ denote the family of domains of $\mathcal{I}$ contained in D. Let D contain at least two domains of family $\mathcal{I}$. For $V_i \subset D$, let

$$R_i \equiv R_i(\mathcal{I}|D) \subset D \setminus \bigcup_{j\in\mathcal{I}|D} V_j$$

be an annulus of maximal modulus enclosing V_i but not enclosing other domains of the family $\mathcal{I}$. We will briefly call it the *maximal annulus* enclosing V_i in D (rel the family $\mathcal{I}$).

Let us define *the asymmetric modulus of the family* $\mathcal{I}$ *in* D as

$$\sigma(\mathcal{I}|D) = \sum_{i\in\mathcal{I}} \frac{1}{2^{1-\delta_{i0}}} \operatorname{mod} R_i(\mathcal{I}|D),$$

where δ_{ji} is the Kronecker symbol. So the critical modulus is supplied with weight 1, while the off-critical moduli are supplied with weights 1/2 (if D is off-critical then all the weights are actually 1/2).

For $D = \Delta$, let $\sigma(\mathcal{I}) \equiv \sigma(\mathcal{I}|\Delta)$. The asymmetric modulus of $\mathcal{V}$ is defined as follows:

$$\bar{\sigma}(\mathcal{V}) = \min_{\mathcal{I}} \sigma(\mathcal{I}),$$

where $\mathcal{I}$ runs over all admissible sub-families of $\mathcal{V}$.

Let $\{V_i'\}_{i\in\mathcal{I}'}$ be an admissible sub-family of $\mathcal{V}'$. Let us organize the domains of this family in *isles* in the following way. A domain $D' \subset \Delta'$ is called an *island* (for family I') if

- D' contains at least two domains of family $\mathcal{I}'$;
- There is a $t \geq 1$ such that $G^k D' \subset V_{i(k)},\ k = 1, \dots t-1$, with $i(k) \neq 0$, while $G^t D = \Delta$ (where G is the Bernoulli map from §4.2).

Given an island D', let $\phi_{D'} = G^t : D' \to \Delta$. This map is either a double covering or a biholomorphic isomorphism depending on whether D' is critical or not. In the former case, $D' \supset V_0'$.

We call a domain $V_j' \subset D'$ $\phi_{D'}$*-precritical* if $\phi_{D'}(V_j') = V_0$. There are at most two precritical domains in any D'. If there are actually two of them, then they are off-critical and symmetric with respect to the critical point 0. In this case D' must also contain the critical domain V_0'.

Let $\mathcal{D}' = \mathcal{D}(\mathcal{I}')$ be the family of isles associated with $\mathcal{I}'$. Let us consider the asymmetric moduli $\sigma(\mathcal{I}'|D')$ as a function on this family.

This function is clearly monotone:

$$\sigma(\mathcal{I}'|D') \geq \sigma(\mathcal{I}'|D'_1) \quad \text{if} \quad D' \supset D'_1, \tag{6.1}$$

Let us call an island D' *innermost* if it does not contain any other isles of the family $\mathcal{D}(\mathcal{I}')$. As this family is finite, innermost isles exist.

6.2. **Growth of moduli.** Let $\mu = \operatorname{mod}(A) \equiv \operatorname{mod}(\Delta \smallsetminus V_0)$, $\mu' = \operatorname{mod}(A') \equiv \operatorname{mod}(\Delta' \smallsetminus V'_0)$, $\nu = \operatorname{mod}(\Delta \smallsetminus U_0)$.

Lemma 6.1. $\mu' \geq \frac{1}{2}\nu \geq \mu$.

Proof. Let B be the parabolic basin of $g_0 \equiv g|V_0$. Then by the Grötzsch inequality,

$$\nu \geq \operatorname{mod}(\Delta \smallsetminus B) \geq \sum_{n=0}^{\infty} \operatorname{mod}(g_0^{-n} A) = 2\mu,$$

which is the second required inequality.

To prove the first one, consider a double covering $G : \Delta' \smallsetminus V'_0 \to \Delta \smallsetminus \Omega_k$ (where Ω_k is an appropriate domain of the first landing map L from §4.2). Since $\operatorname{mod}(\Delta \smallsetminus \Omega_k) \geq \nu$, the first inequality follows as well. $\square$

The following statement is Lemma 5.4 from [L2].

Lemma 6.2. *Let $\mathcal{I}'$ be an admissible family of domains of $\mathcal{V}'$. Let D' be an innermost island associated to the family $\mathcal{I}'$, and let $\mathcal{J}' = \mathcal{I}'|D$, $\phi = \phi_{D'}$. For $j \in \mathcal{J}'$, let us define $i(j)$ by the property $\phi(V'_j) \subset U_{i(j)}$, and let $\mathcal{I} = \{i(j) : j \in \mathcal{J}'\} \cup \{0\}$. Then $\{U_i\}_{i\in\mathcal{I}}$ is an admissible family of domains of $\mathcal{U}$, and*

$$\sigma(\mathcal{I}'|D') \geq \frac{1}{2}\left((|\mathcal{J}'| - s)\nu + s \ \operatorname{mod} P_0 + \sum_{j\in\mathcal{J}',\, i(j)\neq 0} \operatorname{mod} P_{i(j)}\right),$$

where $s = \#\{j : i(j) = 0\}$ is the number of ϕ-precritical pieces, and P_i are the maximal annuli enclosing U_i in Δ rel $\mathcal{I}$.

Using the transit map Ψ from §4.2, we see that the annuli $P_{i(j)}$ from Lemma 6.2 are univalent pull-backs of some off-critical annuli $R_{k(j)}$ of the family $\mathcal{V}$ in Δ. Incorporating it into Lemma 6.2, we obtain:

$$\sigma(I') \geq \sigma(I'|D') \geq (\frac{1}{2}\nu - \mu) + \mu + \frac{1}{2}\sum \operatorname{mod} R_{k(j)} \geq$$

$$\geq (\frac{1}{2}\nu - \mu) + (\mu - \operatorname{mod}(R_0)) + \bar{\sigma}(\mathcal{V}) \geq \bar{\sigma}(\mathcal{V}) \tag{6.2}$$

Taking the infimum over all admissible families I', we see that

$$\bar{\sigma}(\mathcal{V}') \geq \bar{\sigma}(\mathcal{V}). \tag{6.3}$$

Consider now the principal nest $g_n : \cup V_i^n \to \Delta^n$ of generalized quadratic-like maps. The next map of the nest is related to the previous one by the usual or parabolic generalized renormalization. We assume that if g_{n+1} is the parabolic renormalization of g_n, then the previous renormalization level is non-central, i.e., $g_{n-1}(0) \notin V_0^{n-1}$ (for otherwise $g_n|V_0^n = g_{n-1}|V_0^n$ and we could skip g_n from the nest). Denote by $\mu_n = \mathrm{mod}(\Delta_n \smallsetminus V_n^0)$ the principal moduli of the g_n and by $\bar{\sigma}_n$ their asymmetric moduli.

Corollary 6.3. *The asymmetric moduli $\bar{\sigma}_n$ stay away from 0. The principal moduli $\mu_n = \mathrm{mod}(\Delta_n \smallsetminus V_n^0)$ stay away from 0 for all parabolic maps g_n .*

Proof. The first statement immediately follows from (6.3) and the corresponding statement for the usual renormalization (Corollary 5.5 of [L2]).

Take a parabolic map g_n. If it is obtained from g_{n-1} by the usual generalized renormalization, then the previous renormalization level is non-central. By Corollary 5.3 of [L2], $\mu_n \geq \frac{1}{2}\bar{\sigma}_{n-1}$. Hence these principal moduli μ_n stay away from 0.

If g_n is obtained from g_{n-1} by the parabolic renormalization then by Lemma 6.1, $\mu_n \geq \mu_{n-1}$. Putting these together, we obtain the assertion. □

Let us go back to the estimate (6.2). If the hyperbolic distance between the domains V_0 and $V_{k(j)}$ (in the hyperbolic metric of Δ) is bounded by some d, then $\mu \geq \mathrm{mod}(R_0) + \alpha(d)$, where $\alpha(d) > 0$, and we observe a definite increase of the asymmetric modulus:

$$\sigma(\mathcal{I}') \geq \bar{\sigma}(\mathcal{V}) + \alpha(d).$$

Since the tower under consideration has essentially bounded geometry, the intervals I_k stay a bounded hyperbolic distance from I_0 in the hyperbolic metric of J. If at the same time the hyperbolic distance from some $V_{k(j)}$ to V_0 in Δ is big ($\geq d$), then the curve $\partial\Delta$ must be pinched near the real line. In this case we have a big adding in the Grötzsch inequality when we estimate ν by 2μ (see §§13.2- 13.3 in [L2, Appendix A]): $\nu \geq 2\mu + \beta(d)$, where $\beta(d) \to \infty$ as $d \to \infty$. This yields a big increase of the asymmetric modulus:

$$\sigma(\mathcal{I}') \geq \bar{\sigma}(\mathcal{V}) + \frac{1}{2}\beta(d).$$

Thus the asymmetric moduli $\bar{\sigma}_n$ grow. Since it is incompatible with the essentially bounded geometry on the real line, we arrive at a contradiction.

References

[D1] A. Douady. Does a Julia set depend continuously on the polynomial? In: "Complex dynamical systems", Proceed. Symp. Apllied Math., v. 49, AMS, 91 - 138.

[D2] A. Douady (with the participation of X. Buff, R. Devaney and P. Sentenac). Baby Mandelbrot sets are born in cauliflowers. In the book "The Mandelbrot set, themes and variations", London Math. Soc. Lect. Note Series, v. 274 (2000), 19-36.

[DH] A. Douady & J.H. Hubbard. On the dynamics of polynomial-like maps. Ann. Sc. Éc. Norm. Sup., v. 18 (1985), 287-343.

[Hi] B. Hinkle. Parabolic limits of renormalization. Preprint IMS at Stony Brook # 1997/7. To appear in Erg. Th. & Dynam. Syst.

[L1] M. Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. Ann. Math., v. 140 (1994), 347 - 404.

[L2] M. Lyubich. Dynamics of quadratic polynomials, I-II. Acta Math., v. 178 (1997), 185-297.

[L2] M. Lyubich. Dynamics of quadratic polynomials, III. Parapuzzle and SBR measure. Asterisque volume in honor of Douady's 60th birthday "Géométrie complexe et systémes dynamiques", v. 261 (2000), 173 - 200.

[LY] M. Lyubich & M. Yampolsky. Dynamics of quadratic polynomials: Complex bounds for real maps. Ann. Institut Fourier, v. 47 (1997), 1219 - 1254.

[M] J. Milnor. On the concept of attractor. Comm. Math. Physics., v. 99 (1985), 177-195, and v. 102 (1985), 517-519.

[McM] C. McMullen. Complex dynamics and renormalization. Annals of Math. Studies, # 135 (1994), Princeton Univ. Press.

[MS] W. de Melo & S. van Strien. One dimensional dynamics. Springer Verlag, 1993.

[Sh] M. Shishikura. The parabolic bifurcation of rational maps. Colóqio 19 Brasileiro de Matemática.