Laminations and Holomorphic Dynamics

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LECTURE 1

Notion of lamination

In this lectures we will describe intimate connections between holomorphic dynamics and affine/hyperbolic laminations. They reveal many hidden geometric structures associated with rational maps. These structures, interesting per se, are also promising from the dynamical point of view. The lectures are primarily based upon two papers, [**LM**] (2nd lecture) and [**KL**] (3d and 4th lectures). For the general introduction to the theory of laminations, the reader can consult [**CC**].

1. Basic definitions

1.1. Lamination. Roughly speaking, a lamination $\mathcal{L}$ is a topological space which is decomposed into smooth immersed submanifolds ("leaves") nicely organized in a local product structure. Another viewpoint is that $\mathcal{L}$ is glued out of several "local charts" each of which is a product of a domain in $\mathbb{R}^d$ and some topological space.

A formal definition goes as follows. An d-dimensional *product lamination* is a topological space of the form $U^d \times T$, where U^d is a domain in $\mathbb{R}^d$ called a *local leaf* or a *plaque*, and T a topological space called a *transversal.* (Note that d refers to the dimension of the leaves rather than the dimension of the underlying space.)

A morphism between two product laminations is a continuous map that maps local leaves to local leaves. It is also called a *laminar map.* (This terminology can also be applied to partially defined maps.)

A *d-dimensional lamination* $\mathcal{L}$ (or, briefly, *d-lamination*) is a topological space $\mathcal{X}$ (the *underlying space*) endowed with the following structure: For any point $x \in \mathcal{X}$ there is a neighborhood $\mathcal{U} \ni x$ and a homeomorphism $\phi : \mathcal{U} \to U^d \times T$ onto a d-dimensional product lamination $U^d \times T$ such that the transit maps $\phi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \phi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ between two product laminations are laminar. These homeomorphisms ϕ are called *local charts*, the corresponding neighborhoods $\mathcal{U}$ are called *flow boxes*, and the sets $\phi^{-1}(U^n \times \{t\})$ are called *local leaves* or *plaques* of $\mathcal{L}$.

If the underlying space $\mathcal{X}$ itself is a manifold then $\mathcal{L}$ is called a *foliation.*

In what follows we will not make a notational difference between the lamination and its underlying space.

Morphisms ($\equiv$ *laminar maps*) between two laminations are continuous maps which are locally laminar.

1.2. Global leaves. Any lamination $\mathcal{L}$ is decomposed into disjoint union of *global leaves* in the following way: Two points x and y belong to the same global leaf if there is a sequence of local leaves $L_0, L_1, \dots, L_k$ such that $L_0 \ni x$, $L_k \ni y$, and $L_i \cap L_{i+1} \neq \emptyset$, $i = 0, 1, \dots k-1$. The global leaf passing through x will be denoted $L(x)$.

Global leaves can be endowed with *intrinsic topology* by declaring that plaques form its basis. This topology turns global leaves into connected topological manifolds. It is important to notice that this intrinsic topology is usually different from the one induced from $\mathcal{L}$. In fact, in most interesting cases global leaves intersect flow boxed in infinitely many plaques, so that induced neighborhoods are unbounded on the leaves.

Laminar morphisms map global leaves to global leaves and are continuous with respect to their intrinsic topology (and of course, with respect to the induced topology as well).

1.3. Holonomy. Given a local chart $\phi : \mathcal{U} \to U^n \times T$, let $\pi : \mathcal{U} \to T$ stand for the natural projection, which makes $\mathcal{U}$ a trivial bundle over T. Its sections $i : T \to \mathcal{U}$ (and their images) are called *local transversals* of $\mathcal{L}$.

Given two local transversals T_1 and T_2 in $\mathcal{U}$, there is an obvious *holonomy map* $\gamma : T_1 \to T_2$ (sliding along local leaves from one transversal to the other). Taking compositions of these holonomies along chains of flow boxes, we can (partially) define holonomy maps between local transversals sitting in different flow boxes (sliding from one transversal to the other along paths in global leaves).

In particular, on any given transversal T we obtain a *holonomy pseudo-group* of T which encodes how the leaves return back to T. This pseudo-group generalizes the Poincaré return map to a transversal of a flow. It relates topology of laminations to discrete dynamics.

1.4. Geometric structures. If the transit maps between different local charts are leafwise smooth, then the leaves of the lamination naturally become smooth manifolds. Similarly, one can consider other leafwise geometric structures (G-structures") on laminations. For instance, if $\mathcal{L}$ is a 2-lamination such that the transit maps are leafwise conformal then the leaves of $\mathcal{L}$ are endowed with a complex structure, thus becoming Riemann surfaces. Such laminations are called *Riemann surface laminations.* If the transit maps are leafwise complex affine $z \mapsto az + b$, then $\mathcal{L}$ is called a *(complex) affine lamination.* In arbitrary dimension d, if the transit maps are leafwise affine and conformal (i.e., $x \mapsto \lambda U x + b$, where U is an orthogonal operator, $\lambda > 0$, and $b \in \mathbb{R}^d$), then we will refer to $\mathcal{L}$ as a *confine* ($\equiv$ *similarity* lamination.

If local charts of a d-lamination $\mathcal{L}$ are endowed with leafwise hyperbolic structure (that is, with metric with constant negative curvature, which we assume to be equal to -1) preserved by the transit maps, then $\mathcal{L}$ is called a *hyperbolic lamination.*

Let us recall two Poincaré models of the $(d+1)$-dimensional hyperbolic space $\mathbb{H}^{d+1}$. In the first model, it is realized as the unit ball $\mathbb{D}^{d+1} \subset \mathbb{R}^{d+1}$ endowed with the metric $|dx|/4(1-|dx|^2)$. In this model, the *sphere at infinity* S^d (or the *absolute*) is realized as the unit sphere of $\mathbb{R}^{d+1}$. The second realization is the upper half-plane $\{x_{d+1} > 0\}$ endowed with the metric $|dx|/x_{d+1}$. In this model, the absolute is realized as one-point compactification of $\mathbb{R}^d = \{x_{d+1} = 0\}$. This special point, ∞, on the absolute can be viewed as an extra structure on $\mathbb{H}^{d+1}$ which makes it a *pointed at infinity* hyperbolic space. An equivalent way of describing this structure is by endowing $\mathbb{H}^{d+1}$ with a *vertical vector field* consisting of unit tangent vectors pointing at ∞. If we have an (elementary) discrete group Γ of motions of $\mathbb{H}^{d+1}$ preserving this vector field, then in quotient we obtain a more general "pointed at infinity" hyperbolic manifold, $\mathbb{H}^{d+1}/\Gamma$.

Thus, local charts of a hyperbolic lamination can be realized as maps $\mathcal{U} \to U^{d+1} \times T$, where $U^{d+1} \subset \mathbb{H}^{d+1}$, in such a way that the transit maps preserve the above metrics. A hyperbolic lamination is *pointed at infinity* if it is endowed with a (transversally continuous) leafwise vertical vector field.

If all the leaves of a G-lamination $\mathcal{L}$ are isomorphic to a G-manifold M then $\mathcal{L}$ is called an M-lamination (e.g., one can refer to an affine $\mathbb{C}$-lamination, or a hyperbolic $\mathbb{H}^{d+1}$-lamination).

2. Examples

2.1. Foliations. Differential equations on manifolds provide the most classical examples of (one-dimensional smooth) foliations. Indeed, by the classical Straightening Theorem, the phase portrait of a smooth differential equation has a local product structure, with smooth transit maps. Thus we have a foliation outside the set of singular points.

Similarly, differential equations on complex manifolds with holomorphic coefficients provide us (outside singularities) with Riemann surface foliations.

More generally, a family of k linearly independent differential forms $\omega_1, \dots, \omega_n$ on an $(d+k)$-dimensional manifold M satisfying the Frobenius integrability condition generates a (smooth) d-dimensional foliation on M.

2.2. Hyperbolic dynamical systems. Hyperbolic dynamical systems with discrete time provide us with two laminations, *stable* (whose leaves are stable manifolds $W^s(x)$) and *unstable*. Partially hyperbolic dynamical systems with discrete time have one more, *neutral* lamination. A particular case of the latter are time-one maps of hyperbolic flows, where the neutral lamination coincides with the phase flow. In fact, there are two more laminations associated with a hyperbolic flow, namely *neutral-stable* and *neutral-unstable*. The leaves of these laminations are filled with forward (resp., backward) asymptotic geodesics.

A nice geometric example of a hyperbolic flow is provided by the geodesic flow on a compact manifold with negative curvature. Let us take a closer look at this example in the case of constant curvature when everything can be described explicitly, even in a non-compact situation.

2.3. Geodesic flows. Let us consider the unit ball model of the hyperbolic space $\mathbb{H}^{d+1}$. Recall that in this model the hyperbolic geodesics are represented by circles orthogonal to the absolute S^d. Note that any oriented geodesic has the beginning and the endpoint on the absolute.

The *horosphere* $O_p(x) \subset \mathbb{H}^{d+1}$ centered at a point $p \in S^d$ is the d-dimensional sphere tangent to S^d at p and passing through $x \in \mathbb{H}^{d+1}$.

The phase space of the geodesic flow g^t on $\mathbb{H}^{d+1}$ is the unit tangent bundle $U\mathbb{H}^{d+1}$, where $g^t(v)$, $v \in U_x\mathbb{H}^{d+1}$, is the tangent vector obtained by sliding v time t along the geodesic originating at x in the direction of v. Let p be the endpoint of this geodesic, and let $O = O_p(x)$. Then the stable manifold of v coincides with the set of unit tangent vectors along O orthogonal to O and pointing at the same point p at infinity (this set will also be called a "horosphere").

It is instructive to look at the neutral-stable foliation of the geodesic flow. According to the previous discussion, a neutral-stable leaf consists of geodesics with the same endpoint p on the absolute. Such a leaf can be identified with the hyperbolic space $\mathbb{H}^{d+1}$, while the space of the leaves is identified with the absolute S^d. Thus, the neutral-stable foliation of the geodesic flow on $\mathbb{H}^{d+1}$ is isomorphic to the product foliation $\mathbb{H}^{d+1} \times S^d$. In particular, it is leafwise hyperbolic.

The unstable and neutral-unstable foliations are obtained by reversing the time.

If M^{d+1} is an arbitrary hyperbolic manifold of curvature -1, then M^{d+1} is isometric to $\mathbb{H}^{d+1}/\Gamma$, where Γ is a Kleinian group (i.e., a discrete group of hyperbolic motions) acting freely on $\mathbb{H}^{d+1}$. Then the whole structure described above is Γ-invariant, so that it descends to M^{d+1}. For instance, the stable leaves of the geodesic flow on M^{d+1} are the push-forwards of the horospheres, while the neutral-stable ones are the push-forwards of the hyperbolic space. Again, the neutral-stable and neutral-unstable foliations are hyperbolic.

2.4. Solenoids. Given a sequence of topological spaces X_n, $n = 0, 1, \dots$, and surjective maps $f_n : X_n \to X_{n-1}$, the *inverse* (or *projective*) *limit* $\hat{X} = \varprojlim(f_n : X_n \to X_{n-1})$ is the space of sequences

$$\hat{x} = (x_n)_{n=0}^{\infty} : \ x_n \in X_n, \ f_n(x_n) = x_{n-1},$$

endowed with the weak topology (of coordinatewise convergence). Let $\pi_n : \hat{X} \to X_n$ stand for the natural projection $\hat{x} \mapsto x_n$; $\pi_0 \equiv \pi$. Note that π is a Cantor set fibration over X.

If the spaces X_n are d-folds and the maps f_n are coverings, then $\hat{X}$ has a natural structure of d-lamination. If additionally X is smooth or endowed with some geometric structure (conformal, affine, hyperbolic) locally preserved by the coverings f_n, then this structure naturally lifts to the corresponding leafwise structure on $\hat{X}$.

If all the X_n are the same, $X_n \equiv X$, and all the maps f_n are the same, $f_n \equiv f$, then $\hat{X}$ is the space of *backward orbits* of f. Moreover, the map f naturally lifts to a homeomorphism $\hat{f} : \hat{X} \to \hat{X}$, $\hat{f}(\hat{x}) = (fx_n)_{n=0}^{\infty}$. (Note that the inverse map $\hat{f}^{-1}$ acts on a backward orbit $\hat{x}$ by forgetting its origin.) This map is called the *natural extension* of f. The projection $\pi : \hat{X} \to X$ semi-conjugates $\hat{f}$ to f.

In particular, consider the doubling map $f_0 : z \mapsto z^2$ on the unit circle $\mathbb{T}$. Its natural extension is a one-dimensional lamination whose leaves are homeomorphic to $\mathbb{R}$. This lamination is called the (one-dimensional) *solenoid* $\mathcal{S}^1$. Since $\mathbb{T}$ has a natural (real) affine structure locally preserved by f_0, $\mathcal{S}^1$ is in fact an affine lamination, so that its leaves are affine lines and $\hat{f}_0$ is an affine isomorphism of $\mathcal{S}^1$. The fibers of the fibration $\pi : \mathcal{S}^1 \to \mathbb{T}$ can be naturally identified with the dyadic group in such a way that the holonomy of the fiber (corresponding to one revolution around the circle) becomes the translation by 1.

We can complexify this lamination by taking the natural extension $\hat{f}_0 : \mathcal{N}_0 \to \mathcal{N}_0$ of $f_0 : \mathbb{C}^* \to \mathbb{C}^*$. The leaves $L(\hat{z})$ of this lamination are (complex) affine planes isomorphic to $\mathbb{C}$, and the projections $\pi : L(\hat{z}) \to \mathbb{C}^*$ are the universal coverings of $\mathbb{C}^*$ (thus, equivalent to the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$).

If we restrict f_0 to the complement of the unit disk, $\mathbb{C}^* \smallsetminus \bar{\mathbb{D}}$, then we obtain a lamination $\mathcal{S}^2$ whose leaves have a natural hyperbolic structure (lifted from $\mathbb{C}^* \smallsetminus \bar{\mathbb{D}}$). As such, they are isomorphic to the hyperbolic plane $\mathbb{H}^2$. The map $\hat{f}_0$ acts properly discontinuously on this lamination, so that we can take the quotient, $\mathcal{S}^2/\hat{f}_0$. We obtain a compact (hyperbolic) Riemann surface lamination called *Sullivan's solenoid.*

Introducing polar coordinates on $\mathbb{C}^* \smallsetminus \bar{\mathbb{D}}$, we see that Sullivan's solenoid can be topologically realized as the *mapping torus* of the one-dimensional solenoid. In other words, $\mathcal{S}^2 \approx \mathcal{S}^1 \times [0,1]/\sim$, where $(\hat{z}, 0) \sim (\hat{f}\hat{z}, 1)$, $z \in \mathbb{T}$.

3. Associated notions

3.1. Foliated bundles. Let $\mathcal{L}$ be a lamination, M be a manifold, and let $\pi : \mathcal{L} \to M$ be a fibration with fiber F. If any point $x \in M$ has a neighborhood U such that the preimage $\pi^{-1}U$ is a flow box in $\mathcal{L}$ and the corresponding local chart $\phi : \pi^{-1}U \to U \times F$ fibers over $\mathrm{id} : U \to U$, then $\mathcal{L}$ is called a *foliated bundle.* In this case, the projection π is a leafwise covering. Any geometric structure on M can be lifted to the corresponding leafwise structure on $\mathcal{L}$.

Geodesic flows and solenoids give examples of foliated bundles.

3.2. Orbifold laminations. An *orbifold* is an object modeled on manifolds modulo finite group actions (see [**Th, Sc**] for the precise definition). Similarly, an *orbifold lamination* is an object modeled on regular laminations modulo finite groups of laminar isomorphisms.

As in the regular case, orbifold laminations can be endowed with different geometric structures (smooth, conformal, hyperbolic, etc.). Such a structure is locally represented by a regular structure on an orbifold flow box (which is a regular product lamination) preserved by the corresponding finite group.

An instructive example of a Riemann surface lamination is the product lamination $\mathbb{D} \times (-1, 1)$ modulo the involution $\sigma : (z, t) \to (-z, -t)$. Similarly, $\mathbb{C} \times (-1, 1)$ modulo σ is naturally an affine orbifold lamination. Note that these laminations have only one singular leaf, $L(0, 0)$.

LECTURE 2

Laminations associated with rational functions

In this lecture we will construct several natural laminations associated with a rational function $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$.

1. Regular leaf space

1.1. Leaves. The construction begins with considering the natural extension $\hat{f} : \mathcal{N}_f \to \mathcal{N}_f$ of f. It is a homeomorphism of a compact space $\mathcal{N}_f$ semi-conjugate to f by means of the natural projection $\pi : \mathcal{N}_f \to \bar{\mathbb{C}}$. As the solenoidal examples from the first lecture suggest, the space $\mathcal{N}_f$ tends to have a laminar structure. However, this structure is well hidden, so it requires some effort to reveal it.

First, we should remove some bad points. (Even for $f_0 : z \mapsto z^2$, two points, $\hat{0} = (\dots, 0, 0)$ and $\hat{\infty} = (\dots, \infty, \infty)$, should be removed).

Given a backward orbit $\hat{z} = (z_{-n})_{n=0}^{\infty} \in \mathcal{N}_f$ and a neighborhood $U \ni z_0$, let U_{-n} stand for the component of $f^{-n}U$ containing z_{-n}. This sequence of neighborhoods, $\hat{U} = (U_{-n})_{n=0}^{\infty}$, is called the *pullback* of U_0 along $\hat{z}$. In general, the maps $f : U_{-n-1} \to U_{-n}$ are branched coverings.

A point $\hat{z}$ is called *regular* if there is a neighborhood $U \ni z_0$ whose pullback along $\hat{z}$ is eventually univalent, that is, the maps $f : U_{-n-1} \to U_{-n}$ are univalent for $n \geq N$. Note that in this case the map

$$\pi_{-N} : \hat{U} \to U_{-N} \tag{2.1}$$

is one-to-one.

Let $\mathcal{R}_f \subset \mathcal{N}_f$ stand for the set of regular points. This set can be decomposed into the leaves $L(\hat{z})$ endowed with a natural conformal structure. By definition, $L(\hat{z})$ is the *path connected component* of $\mathcal{R}_f$ containing $\hat{z}$. The maps (2.1) serve as local charts on the leaves.

1.2. Type Problem. There are some special leaves obtained by lifting Siegel disks or Herman rings to the natural extension. They will also be called "Siegel disks" or "Herman rings".

LEMMA 2.1. *All the leaves except Herman rings are simply connected.*

It is easy to see that the leaves are not compact. Hence by the Uniformization Theorem, there are only two possibilities left: any leaf $L(\hat{z})$ is conformally equivalent to either the complex plane $\mathbb{C}$ (*parabolic* or *affine case*) or to the unit disk $\mathbb{D}$ (*hyperbolic case*). A Siegel disk provides an example of a hyperbolic leaf. By now, there is quite a few other known examples: for instance, if a critical point is dense in the Julia set $J(f)$, then there is always a hyperbolic leaf in $\mathcal{R}_f$ (Kahn, unpublished).

However, it is much simpler to construct parabolic leaves, which exist for any rational function. For instance, if $\hat{z} = (z_{-n})_{n=0}^{\infty}$, where $z_0 = z_{-pn}$ ($n \in \mathbb{N}$) is a repelling periodic point of period p, then the leaf $L(\hat{z})$ is parabolic. Similarly, all leaves of a hyperbolic (in the dynamical sense!) rational function are parabolic (in the conformal sense!)

Since the only conformal automorphisms of the complex plane $\mathbb{C}$ are affine $z \mapsto az + b$, the plane is endowed with a unique affine structure compatible with the conformal structure. If $L(\hat{z})$ is an affine leaf, the map $\hat{f} : L(\hat{z}) \to L(\hat{f}\hat{z})$ is affine with respect to this structure.

Let $\mathcal{A}_f^n \subset \mathcal{R}_f$ stand for the union of affine leaves in the regular leaf space. (The upper script "n" indicates that this space is considered in the topology of the natural extension which should be distinguished from a finer topology introduced below.) An immediate question is whether $\mathcal{A}_f^n$ is an affine lamination?

2. Affine lamination

2.1. Chebyshev example. If the map f is hyperbolic then it is easy to see that $\mathcal{A}_f^n$ is indeed an affine lamination. However, already simple non-hyperbolic examples cause a problem. Consider the simplest such an example, the Chebyshev polynomial $f : z \mapsto z^2 - 2$. For this map, the second iterate of the critical point 0 is the repelling fixed point $\beta = 2$. Notice that f can be realized as the quotient of the map $f_0 : z \mapsto z^2$ modulo the involution $\sigma : z \mapsto 1/z$. It is easy to see that all the leaves of $\mathcal{N}_f \setminus \hat{\infty}$ are parabolic.

Let $\hat{\beta} = (\dots, 2, 2)$ be the lift of β to the natural extension, and let $\hat{z}_n = (\cdots - 2, 2 \dots 2)$, where the string of 2's has length n. Then $\hat{z}_n \to \hat{\beta}$. However, the leaf $L(\hat{\beta})$ is unbranched over $\mathbb{C}$ at $\hat{\beta}$, once the leaves $L(\hat{z}_n)$ are branched over $\mathbb{C}$ at $\hat{z}_n$. This is not compatible with the local product structure.

A way around this problem is to endow the leaf $L(\hat{\beta})$ with the *orbifold* affine structure modeled on $\mathbb{C}$ modulo $z \mapsto -z$, which will turn $\mathcal{A}_f^n$ into an orbifold affine lamination. In fact, this lamination can be realized as the global quotient of $\mathcal{N}_0$ (see §2.4 of the 1st lecture) modulo the involution $\hat{\sigma}$ (the lift of σ to $\mathcal{N}_0$).

More generally, consider a postcritically finite rational map, e.g., $f : z \mapsto z^2 + i$ for which the second iterate of the critical point lands at the repelling cycle $\beta = (-1+i,\, -i)$. Let $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)$ be the lift of this cycle to $\mathcal{N}_f$. Then the leaves $L(\hat{\beta}_i)$ are unbranched at the $\hat{\beta}_i$ but they can be approximated by both branched and unbranched leaves. To turn this leaf space into a lamination, we need to double each leaf $L(\hat{\beta}_i)$, to put the orbifold affine structure on one copy and the regular affine structure on the other, and to endow this space with an appropriate topology making the leafwise affine structure continuous with respect to the transverse parameter.

Even in such a simple example this procedure is quite involved if to do it by hands. However, there is a self-organizing construction which makes it work automatically for an arbitrary rational map. This construction is based on the "universal" space of meromorphic functions.

2.2. Construction of the affine lamination. Let us consider some leaf $L(\hat{z})$ in $\mathcal{A}^n_f$, and let $\pi(\hat{z}) = z$. Since the leaf is parabolic, it can be uniformized by the complex plane, $\gamma_{\hat{z}} : (\mathbb{C}, 0) \to (L(\hat{z}), \hat{z})$. Then the composition $\phi_{\hat{z}} = \pi \circ \gamma_{\hat{z}}$ is a non-constant meromorphic function $(\mathbb{C}, 0) \to (\bar{\mathbb{C}}, z)$, well defined up to a pre-composition with a complex scaling, $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$. It is easy to see that vise versa, any point $\hat{z} \in \mathcal{A}^n_f$ can be uniquely recovered from the corresponding meromorphic function. Thus we can identify the points with the meromorphic function. The idea now is to transfer the topology from the space of meromorphic functions to $\mathcal{A}^n_f$. Completion of this new space gives us the desired orbifold lamination.

Let us now outline this construction more precisely. Consider the universal space $\mathcal{U}$ of all non-constant meromorphic functions on $\mathbb{C}$ with metrizable topology of uniform convergence on compact subsets. It is foliated into four-dimensional orbits of the right action $\phi \mapsto \phi \circ A$ of the group Aff of complex affine maps $\mathbb{C} \to \mathbb{C}$. On the other hand, any rational map f acts on $\mathcal{U}$ on the left: $\phi \mapsto f \circ \phi$. Let

$$\mathcal{K}_f = \bigcap_{n \in \mathbb{N}} f^n \mathcal{U}$$

be the "global attractor" of f on $\mathcal{U}$, and let $\hat{f} : \hat{\mathcal{K}}_f \to \hat{\mathcal{K}}_f$ be its natural extension. The group Aff still naturally acts on this space turning it into a 4-lamination. Factoring $\hat{\mathcal{K}}_f$ with respect to the action of $\mathbb{C}^* \subset$ Aff turns it into a *universal affine lamination* $\boldsymbol{\mathcal{A}}_f$ endowed with affine $\hat{f}$-action.

There is a natural $\hat{f}$-equivariant laminar embedding $\iota : \mathcal{A}^n_f \to \boldsymbol{\mathcal{A}}_f$. (Notice that in general this embedding is discontinuous!) The closure of $\iota(\mathcal{A}^n)$ in $\boldsymbol{\mathcal{A}}_f$ is $\hat{f}$-invariant and leaf saturated. This is the desired affine orbifold lamination $\mathcal{A}_f$ associated with f.

It is worthy to look closer at where the orbifold points of $\mathcal{A}_f$ come from. The group $\mathbb{C}^*$ is the direct product of the group $\mathbb{T}$ of rotations and the group $\mathbb{R}$ of real rescalings. The orbifold points are developed when we take the quotient by the $\mathbb{T}$-action. To see it, consider a local transversal T to an orbit $\mathbb{T}\phi$. The first return map to T generates a finite group action on T. It may happen that the return time at ϕ is essentially smaller than at nearby points on T (in this case the meromorphic function ϕ has an "accidental" rotational symmetry). Then ϕ corresponds to an orbifold point of the quotient.

Such a factorization procedure by a circle action is familiar to topologists under the name of "Seifert fiber bundle" (compare [**Sc**]).

Note finally that the lamination $\mathcal{A}_f$ is not necessarily locally compact but it is such in many interesting cases, for instance, in the postcritically non-recurrent case, or in the Feigenbaum case, or in the Fibonacci case. A rational function is called *tame* if $\mathcal{A}_f$ is locally compact.

3. Hyperbolic 3-lamination

3.1. Hyperbolization functor. Any (orbifold) confine d-lamination can be functorially extended to an (orbifold, pointed at infinity) hyperbolic $(d+1)$-lamination. To see it, let us describe functorially an extension of the confine plane $\mathbb{R}^d$ to the (pointed at infinity) hyperbolic space $\mathbb{H}^{d+1}$. Let us realize the latter us an upper half-space $\{(z,t) : z \in \mathbb{R}^d, t > 0\}$. Consider a horosphere $O(c) = \{t = c\}$ centered at ∞. Restriction of the hyperbolic metric to $O(c)$ can be identified (via the vertical projection) with a Euclidean metric $\varepsilon_c = |dz|/c$ on $\mathbb{R}^d$ compatible with the confine structure of $\mathbb{R}^d$. Vice versa, selection of such a metric specifies a horosphere centered at ∞. Hence a point x of the hyperbolic space $\mathbb{H}^{d+1}$ can be defined as a pair (z, ε) where $z \in \mathbb{R}^d$ and ε is a Euclidean metric on $\mathbb{R}^d$ compatible with its confine structure. So, the (pointed at infinity) hyperbolic space $\mathbb{H}^{d+1}$ can be defined as the $\mathbb{R}$-bundle over $\mathbb{R}^d$ of such Euclidean metrics. Moreover, any affine map between two planes uniquely extends to a hyperbolic isomorphism between the spaces, and this extension is functorial.

If now $\mathcal{A}$ is a confine (orbifold) d-lamination then applying the hyperbolization functor leafwise we obtain a hyperbolic (orbifold, pointed at infinity) $(d+1)$-lamination, and this extension is functorial. Applying it to the 2-lamination $\mathcal{A}_f$, we obtain an (orbifold) hyperbolic 3-lamination $\mathcal{H}_f$ associated with a rational map f.

There is another path leading to the lamination $\mathcal{H}_f$. If in the above construction we quotient $\hat{\mathcal{K}}_f$ by the action of the unit circle $\mathbb{T} \subset \mathbb{C}^*$ rather than $\mathbb{C}^*$, we will obtain the universal orbifold *hyperbolic* lamination $\boldsymbol{\mathcal{H}}_f$ associated with f. On the other hand, we can apply the hyperbolization functor to the leaf space $\mathcal{A}_f^n$ to obtain a hyperbolic leaf space $\mathcal{H}_f^n$. As in the affine case, there is a natural equivariant (generally, discontinuous) embedding $\mathcal{H}_f^n \to \boldsymbol{\mathcal{H}}_f$. The closure of its image is the desired hyperbolic orbifold lamination.

Note that the unit tangent bundle $U\mathbb{H}^3$ (viewed as the hyperbolic lamination, see §2.3 of the 1st lecture) can also be obtained by applying the hyperbolization functor to an affine lamination. Namely, every hyperbolic leaf $L(x)$ of $U\mathbb{H}^3$ is associated to some point $p \in S^2$, and can be recognized as the hyperbolization of $S^2 \smallsetminus \{p\} \approx \mathbb{C}$. In fact, this affine lamination is just $S^2 \times S^2 \smallsetminus \text{diag} \approx \mathbb{C} \times \mathrm{S}^2$.

3.2. Quotient lamination. It turns out that the action of $\hat{f}$ on the hyperbolic lamination $\mathcal{H}_f$ is properly discontinuous, so that we can take the quotient $\mathcal{M}_f = \mathcal{H}_f/\hat{f}$. This is the hyperbolic (orbifold) 3-lamination associated with f. In many respects it is similar to the unit tangent bundle of a hyperbolic 3-orbifold (associated with a Kleinian group). In particular, it supports a natural flow analogous to the geodesic flow. Indeed, since the vertical vector field V_f on $\mathcal{H}_f$ is invariant under $\hat{f}$, it descends to the quotient lamination $\mathcal{M}_f$ (turning $\mathcal{M}_f$ into a pointed at infinity lamination). This vector field and the corresponding flow are still called *vertical*.

3.3. Examples. The 3-lamination $\mathcal{M}_0$ associated with $f_0 : z \mapsto z^2$ has a relatively simple topological structure: $\mathcal{H}_0 \approx \mathcal{S}^2 \times (0,1)$, where $\mathcal{S}^2$ is Sullivan's solenoid (see §2.4 from the 1st lecture). In fact, it is still true for the laminations $\mathcal{M}_\varepsilon$ corresponding to $f_\varepsilon : z \mapsto z^2 + \varepsilon$ when ε is inside the main cardioid of the Mandelbrot set. ($\mathcal{M}_\varepsilon$ for $\varepsilon \neq 0$ is not conformally equivalent to $\mathcal{M}_0$, though.)

However, even for $f_{-1} : z \mapsto z^2 - 1$, the structure of the corresponding 3-lamination $\mathcal{M}_{-1}$ is not fully understood. In some way this lamination remembers the doubling bifurcation that created f_{-1}. Even more interesting is the structure of the 3-lamination corresponding to the Feigenbaum point: the whole cascade of doubling bifurcations is reflected in that single lamination.

4. Seifert fibration

If we quotient $\hat{\mathcal{K}}_f$ by the $\mathbb{R}$-action instead of $\mathbb{T}$-action, we will obtain a 3-lamination $\mathcal{S}_f$ fibered over $\mathcal{A}_f$, with circle fibers. This fibration is not locally trivial, though, but rather a "Seifert fiber bundle". Unlike $\mathcal{A}_f$, the lamination $\mathcal{S}_f$ is *regular* (recall that the orbifold points in $\mathcal{A}_f$ appear exactly because of factorization by the circle action). The map f lifts to a laminar map $\hat{f} : \mathcal{S}_f \to \mathcal{S}_f$ fibered over $\mathcal{A}_f$. The circle fibers play the role of "neutral lamination" for this lift. Roughly speaking, we obtain a "regular partially hyperbolic" dynamics on $\mathcal{S}_f$ fibered over "orbifold hyperbolic" dynamics on $\mathcal{A}_f$.

LECTURE 3

Poincaré series and basic cocycles on laminations

1. Transverse invariant measures

Given a laminations $\mathcal{L}$, assume that we have associated to each transversal T a Borel measure $\mu|T$. Then μ is referred to as a *transverse measure* on $\mathcal{L}$. A transverse measure is called *holonomy invariant* if for any holonomy $\gamma : T \to S$ between two transversals, $\mu|S = \gamma_*(\mu|T)$ (see [**CC, Pl**] and further references therein).

THEOREM 3.1 (see [**Su, KL**]). *There is a unique holonomy invariant transverse measure m on $\mathcal{A}_f$ concentrated on $\iota(\mathcal{A}^n_f)$.*

To construct m, take the measure κ of maximal entropy of f ($\equiv$ *balanced* measure) (see [**Br, L**]), lift it to the natural extension $\mathcal{N}_f$, disintegrate it over the fibers of the projection $\pi : \mathcal{N}_f \to \bar{\mathbb{C}}$, and then transfer it to $\mathcal{A}_f$ by ι.

This measure serves as a "counting measure" on the transversals.

2. Poincaré series and critical exponent

2.1. Kleinian groups. Let Γ be a Kleinian group acting isometrically on the hyperbolic space $\mathbb{H}^{d+1}$. Take a base point $x \in \mathbb{H}^{d+1}$ and some other point $y \in \mathbb{H}^{d+1}$, and consider the orbit Γy of the latter. Then the Poincaré series is defines as follows:

$$\Xi_{x,y}(\delta) = \sum_{\gamma \in \Gamma} \exp(-\delta \operatorname{dist}(x, \gamma y)), \tag{3.1}$$

were dist stands for the hyperbolic distance in $\mathbb{H}^{d+1}$.

The critical exponent $\delta_{\mathrm{cr}} = \delta_{\mathrm{cr}}(\Gamma)$ separates convergent and divergent values of the exponent: For $\delta > \delta_{\mathrm{cr}}$, the Poincaré series converges, while for $\delta < \delta_{\mathrm{cr}}$ it is divergent (independently of x and y). For δ_{cr} itself, either event can occur. Depending on which one occurs, the group is called of *convergent* or *divergent* type.

One can show that $\delta_{\mathrm{cr}} \in (0, d]$.

2.2. Rational maps. Consider now a rational map f whose Julia set $J(f)$ does not coincide with the whole sphere $\bar{\mathbb{C}}$. Take some base point $x \in \bar{\mathbb{C}} \smallsetminus J(f)$ which is not post-critical. Then the Poincaré series for f can be defined as follows:

$$\Xi_x(\delta) = \sum_{n=0}^{\infty} \sum_{z \in f^{-n}x} \frac{1}{\|Df^n(z)\|^{-\delta}} . \tag{3.2}$$

Again, the critical exponent is the one separating convergent values of δ from divergent once. Unfortunately, it may depend on x (and it is not defined when $J(f) = \bar{\mathbb{C}}$). To fix these problems, we will pass to laminations.

2.3. Laminations.

2.3.1. *Leafwise conformal metrics.* There are three viewpoints on the notion of conformal metric ρ on a confine lamination $\mathcal{A}$.

First definition is suggested by the mere name of the object: it is a continuous leafwise Riemannian metric compatible with the leafwise confine structure. In local charts $D \times T$ it is written as $\rho(\mathbf{z}, t)|d\mathbf{z}|^2$, where $\rho(\mathbf{z}, t)$ is continuous in two variables (note that we do not require ρ to be smooth in the leafwise direction).

Consider the hyperbolization $\mathcal{H}$ of $\mathcal{A}$ (see §3.1 of the 2nd lecture). Recall that points of $\mathcal{H}$ are pairs $(\mathbf{z}, \varepsilon)$, where $\mathbf{z} \in \mathcal{A}$ and ε is the Euclidean metric on the leaf $L(\mathbf{z})$. A choice of ε is equivalent to a choice of a conformal metric in the tangent space $T_{\mathbf{z}}L(\mathbf{z})$. Thus, a leafwise conformal metric is the same as a continuous section $\sigma : \mathcal{A} \to \mathcal{H}$.

The third viewpoint comes from the uniformizations $\gamma : \mathbb{R}^d \to L(\mathbf{z})$ of the leaves. Assume that all the leaves are isomorphic to $\mathbb{R}^d$. Then γ is defined up to an orthogonal operator composed with rescaling. The choice of the scaling factor determines a Euclidean structure on $L(\mathbf{z})$. Thus, a selection of a Riemannian metric ρ amounts to a continuous choice of normalizations of the uniformizations γ (up to an orthogonal operator). This viewpoint is fruitful in the dynamical setting.

2.3.2. *Special metric.* Let us go back to the dynamical affine lamination $\mathcal{A}_f$. To any point $\mathbf{z} \in \mathcal{A}_f$ corresponds a meromorphic function $\phi_{\mathbf{z}} = \pi \circ \gamma_{\mathbf{z}}$ (see §2.2 of the 2nd lecture). Normalizing the uniformization $\gamma_{\mathbf{z}}$ amounts to normalizing $\phi_{\mathbf{z}}$. Here is a natural choice: a meromorphic function $\phi : \mathbb{C} \to \bar{\mathbb{C}}$ is *normalized* if

$$\int_{\mathbb{D}} \|D\phi\|^2 = 1,$$

where $\|\cdot\|$ is calculated with respect to the Euclidean norm on the source $\mathbb{C}$ and the spherical norm on the target $\bar{\mathbb{C}}$ (so that the integral represents the spherical area of the image $\phi(\mathbb{D})$ counted with multiplicities).

This metric is called *special*; it makes the dynamics expanding:

LEMMA 3.2. *The map $\hat{f}^{-1} : \mathcal{A}_f \to \mathcal{A}_f$ is locally uniformly contracting with respect to the special metric ρ in the sense that $\|D\hat{f}^{-n}(\mathbf{z})\|_\rho \to 0$ locally uniformly in $\mathbf{z}$ as $n \to \infty$.*

2.3.3. *Backward Poincaré exponent.* The idea is that in the definition of the Poincaré series points should be replaced by transversals:

$$\Xi_T(\delta) = \sum_{n=0}^{\infty} \int_{\hat{f}^{-n}T} \|D\hat{f}^n\|_\rho^{-\delta}\, dm,$$

where m is the transverse invariant measure from Theorem 3.1.

Then the critical exponent separates convergent cases from divergent ones. It is the same for all precompact transversals, so that for tame rational functions (see §2.2 of the 2nd lecture), $\delta_{\rm cr}(f)$ is well defined. *In what follows we will assume that f is tame.*

THEOREM 3.3. *For any tame rational function, $\delta_{\rm cr} \in (0, 2]$.*

2.3.4. *Forward Poincaré exponent.* It is obtained by the dual construction using leaves in place of transversals. This leads to the following definition of the forward Poincaré exponent:

$$\Theta_D(\gamma) = \sum_{n=0}^{\infty} \int_{\hat{f}^n D} \|D\hat{f}^{-n}\|_\rho^{\gamma}\, d\tilde{\kappa},$$

where D is a local leaf and $\tilde{\kappa}$ is the leafwise lift of the balanced measure κ. The forward critical exponent, $\gamma_{\rm cr}$, separates convergent values of γ from the divergent ones. It is independent of D.

It is important to know whether $\delta_{\rm cr} = \gamma_{\rm cr}$ (see §3.2 of the 4th lecture).

3. Cocycles

3.1. Busemann cocycle on the hyperbolic space. Given the hyperbolic space $\mathbb{H}^{d+1}$ and a point $p \in S^d$ at infinity, $\beta_p(x, y)$ stand for the signed hyperbolic distance between the horospheres $O_p(x)$ and $O_p(y)$, $x, y \in \mathbb{H}^{d+1}$, where $\beta_p(x, y) > 0$ if y is "closer" to p then x. This function satisfies the cocycle rule

$$\beta_p(x, y) + \beta_p(y, z) = \beta_p(x, z), \tag{3.3}$$

and is called the *Busemann cocycle*.

3.2. Cohomology of laminations. Let $\mathcal{L}$ be a lamination, and let $\mathrm{Graph}(\mathcal{L}) = \{(x,y) \in \mathcal{L} \times \mathcal{L} : \ y \in L(x)\}$. A function β : $\mathrm{Graph}(\mathcal{L}) \to \mathbb{R}$ is called a *cocycle* if it is locally continuous (i.e., continuous on the graphs of the flow boxes treated as laminations) and satisfies the cocyclic rule (3.3).

A cocycle is called *trivial* if there is a continuous function $\alpha : \mathcal{L} \to \mathbb{R}$ such that $\beta(x,y) = \alpha(y) - \alpha(x)$.

The space of cocycles modulo the trivial ones is called the *first cohomology group* of the lamination, $H^1(\mathcal{L})$.

3.3. Busemann cocycle on a pointed at infinity hyperbolic lamination. Let $\mathcal{H}$ be a pointed at infinity $\mathbb{H}^{d+1}$-lamination. Then the Busemann cocycle of $\mathcal{H}$ is defined as the leafwise Busemann cocycle with respect to the marked at infinity point.

If $\mathcal{M}$ is a quotient of $\mathcal{H}$ modulo some group action preserving the vertical flow, then the Busemann cocycle *locally* descends to flow boxes of $\mathcal{M}$.

3.4. Basic class of a confine lamination. Consider now a confine $\mathbb{R}^d$-lamination $\mathcal{A}$ with hyperbolization $\mathcal{H}$. Take some leafwise conformal metric ρ on $\mathcal{A}$ represented by a section $\sigma : \mathcal{A} \to \mathcal{H}$. Then the *basic cocycle* corresponding to this metric is obtained by restricting the Busemann cocycle to the graph of this section:

$$\beta_\rho(x,y) = \beta(\sigma(x), \sigma(y)).$$

If we replace ρ with some other metric ρ', then the basic cocycle is replaced by the cohomologous one (with $\alpha = \log(\rho'/\rho)$. Thus, we have a well defined cohomology class $\mathsf{b} = [\beta_\rho] \in H^1(\mathcal{A})$. It is called the *basic class* of $\mathcal{A}$. Its geometric significance is explained by the following statement:

PROPOSITION 3.4. *A confine lamination $\mathcal{A}$ is Euclidean if and only if $\mathsf{b} = 0$.*

(To be *Euclidean* or *flat* means that the confine structure can be refined to the Euclidean one). Thus, b gives the cohomological obstruction to flatness.

3.5. Dynamical formula. In the case of the dynamical lamination $\mathcal{A}_f$, there is a nice dynamical formula for the basic cocycle. Let $\mathbf{z}, \boldsymbol{\zeta} \in \mathcal{A}_f$ and $\mathbf{z}_{-n} = \hat{f}^{-n}\mathbf{z}$, $\boldsymbol{\zeta}_{-n} = \hat{f}^{-n}\boldsymbol{\zeta}$. Then

$$\beta_\rho(\mathbf{z}, \boldsymbol{\zeta}) = \sum_{n=1}^{\infty} \log \frac{\|D\hat{f}(\mathbf{z}_{-n})\|_\rho}{\|D\hat{f}(\boldsymbol{\zeta}_{-n})\|_\rho}.$$

Expressions of this kind appeared in dynamics as densities of SRB measures on the unstable foliation (see [**AS, Le**]).

3.6. Flatness criterion. Naturally, some affine laminations are not flat (see [**Gh1**]). Perhaps, one can expect that a "generic" affine lamination is not flat. At least, this is the case for the dynamical laminations:

THEOREM 3.5. *The lamination $\mathcal{A}_f$ is flat if and only if f is a post-critically finite rational function with parabolic Thurston orbifold, that is, f is either $z \mapsto z^n$, or f is a Chebyshev polynomial, or f is a Lattès example.*

LECTURE 4

Streams and measures on laminations

1. Conformal streams

1.1. Conformal "measures" on manifolds. What is usually called "conformal measures" actually are not measures but other tensor objects. To avoid confusion we call them "conformal streams".

Let M be a conformal manifold, that is a manifold endowed with a class of conformally equivalent Riemannian metrics. Assume that to any metric ρ in this class we have associated a Radon measure η_ρ on M in such a way that all measures are equivalent and

$$\frac{d\eta_{\rho'}}{d\eta_\rho} = \left(\frac{\rho'}{\rho}\right)^\delta \tag{4.1}$$

for some $\delta \geq 0$. Then η is called a *δ-conformal stream* on M.

For instance, volume of the metric ρ is an example of an d-conformal stream, where $d = \dim M$. More generally, let $J \subset M$ be a closed subset in M of Hausdorff dimension δ such that the Hausdorff measure in dimension δ is finite. In fact, this measure depends on the choice of the conformal metric ρ generating a δ-conformal stream.

If $f : M \to M'$ is a conformal isomorphism between two conformal manifolds, then any stream η on M can be naturally transferred to M' (so that we obtain the *push-forward* stream $f_*(\eta)$). Note that if we select conformal metrics ρ and ρ' on M and M' respectively, then the transformation rule is the following:

$$\eta_{\rho'} = \|Df\|^\delta f_*(\eta_\rho), \tag{4.2}$$

where $f_*(\eta_\rho)$ is the usual push-forward of the measure η_ρ.

In particular, if we have a conformal map $f : M \to M$ then a δ-conformal stream η on M is f-invariant if a representative measure η_ρ is transformed according to (4.2) (with $\rho' = \rho$). This is what is usually called a "conformal measure" in dynamics.

1.2. Patterson construction.

THEOREM 4.1 ([**Pa, S1**]). *Any Kleinian group Γ acting on the sphere S^d has an invariant $\delta_{\rm cr}$-conformal stream supported on the limit set $\Lambda(\Gamma)$, where $\delta_{\rm cr}$ is the critical exponent of Γ.*

Let us outline a remarkable construction of this stream due to Patterson [**Pa**]. Consider the extension of Γ to the hyperbolic space $\mathbb{H}^{d+1}$. To any point $x \in \mathbb{H}^{d+1}$ corresponds the *visual metric* ρ_x on S^d obtained by projecting the unit tangent sphere in $T_x\mathbb{H}^{d+1}$ to the sphere S^d at infinity along the geodesics. All these metrics are conformally equivalent to ρ_0, the Euclidean metric on S^d (in the standard realization of S^d as the unit sphere in $\mathbb{R}^{d+1}$).

To construct the measure η_x corresponding to this metric, take some $\delta > \delta_{\mathrm{cr}}$, some $y \in \mathbb{H}^{n+1}$, and consider the following finite measure supported on the orbit Γy:

$$\mu_{x,y,\delta} = \frac{1}{\Xi_{y,y}(\delta)} \sum_{\gamma \in \Gamma} \exp(-\delta \operatorname{dist}(\gamma y, x))\, D_{\gamma y},$$

where $\Xi_{x,y}(\delta)$ is the Poincaré series (3.1) and D_z is the Dirac mass supported at z.

If Γ is of divergent type then the measure $\eta_x \equiv \eta_{\rho_x}$ is defined as the weak limit of $\mu_{x,y,\delta}$ as $\delta \searrow \delta_{\mathrm{cr}}$, which depends conformally (i.e., as required for an invariant conformal stream) on the metric ρ_x.

In the convergence case, some regularization procedure should be incorporated to turn the Poincaré series into a divergent one.

1.3. Sullivan's conformal measure. It is a measure μ of the Julia set $J(f)$ of a rational function f which satisfies the transformation rule

$$\frac{d(f^{-1}\mu)}{d\mu} = \|Df\|^{\delta}.$$

The reader can recognize this rule as f-invariance of the associated stream.

THEOREM 4.2 ([**S2**]). *Any rational function has a conformal measure supported on the Julia set.*

This measure can be constructed by a version of Patterson's method applied to the Poincaré series (3.2) from the 3d lecture.

1.4. Conformal streams on laminations. We have two versions of conformal streams on laminations, transverse and leafwise. A *transverse conformal stream* associates to any (leafwise) conformal metric ρ a transverse measure η_ρ in such a way that

$$\frac{d\mu_{\rho'}}{d\mu_\rho} = \left(\frac{\rho'}{\rho}\right)^{-\delta}.$$

A *leafwise conformal stream* associates to any (leafwise) conformal metric ρ a leafwise measure η_ρ satisfying the transformation rule (4.1). Notice that the product of a leafwise and transverse streams is a global *measure* on $\mathcal{A}$ *independent* of ρ.

Any confine lamination $\mathcal{A}$ is endowed with a natural *leafwise connection*: leafwise tensors are parallel with respect to this connection if they are leafwise constant in the local charts. A transverse stream η_ρ on $\mathcal{A}$ is called *parallel* if for any metric ρ which is parallel on a flow box $\mathcal{B}$, the measure η_ρ is holonomy invariant on $\mathcal{B}$.

THEOREM 4.3. *If f is tame then there is an $\hat{f}$-invariant parallel transverse δ_{cr}-conformal stream μ on $\mathcal{A}_f$.*

Notice that taken a representative μ_ρ of the stream, the property of being $\hat{f}$-invariant is expressed by the transformations rule:

$$\frac{d(\hat{f}^{-1}\mu_\rho)}{d\mu_\rho} = \|D\hat{f}\|_\rho^{-\delta},$$

while the property of being parallel is expressed by the rule

$$\frac{d(\gamma^{-1}\mu_\rho)}{d\mu_\rho} = \exp \delta_{\text{cr}}\beta_\rho(x, \gamma x),$$

where $\gamma : T \to T'$ is the holonomy between two transversals and β_ρ is the basic cocycle.

For a leafwise stream, the condition of being parallel should be replaced by the condition of being invariant under the "vertical holonomy" (sliding from one leaf to another along the fibers of the projection π):

THEOREM 4.4. *There exists a leafwise γ_{cr}-conformal stream λ invariant under the vertical holonomy.*

1.5. Construction. The idea is to replace points in the Patterson construction with transversals. Take a reference precompact transversal S and some other precompact transversal T. Consider a preimage $\hat{f}^{-n}T$, put on it the measure $\|D\hat{f}^n\|^{-\delta}m$, and push it forward to T:

$$\mu^{n,\delta}|T = (\hat{f}^n)_*(\|D\hat{f}^n\|^{-\delta}m).$$

Then for $\delta > \delta_{\text{cr}}$, the measure

$$\mu^\delta|T = \frac{1}{\Xi_S(\delta)}\sum_{n=0}^{\infty}\mu^{n,\delta}|T$$

is well defined. If f is of divergent type then define $\mu|\bar{T}$ as a weak limit of measures $\mu^\delta|T$ as $\delta \searrow \delta_{\text{cr}}$. In the convergent case, a regularization procedure should be incorporated before passing to a limit.

The construction of the leafwise stream λ is dual to the above one.

2. Harmonic measures

2.1. Riemannian laminations. Given a Riemannian manifold M, let Δ be the Laplace-Beltrami operator on M. A function ϕ on M is called λ-*harmonic* if $\Delta\phi = \lambda\phi$. A measure ω on M is called λ-harmonic if $\Delta^*\omega = \lambda\omega$. It is equivalent to saying that $\omega = \phi\mathrm{vol}$, where vol is the Riemannian volume and ϕ is a λ-harmonic function.

If now $\mathcal{L}$ is a (leafwise) Riemannian lamination then let Δ be the leafwise Laplace-Beltrami operator, that is, Δ acts on leafwise smooth transversally continuous functions. Notions of λ-harmonic functions and λ-*harmonic measures* are defined in terms of this operator in the same way as in the case of manifold. A measure ω on $\mathcal{L}$ is λ-harmonic if and only if its conditional measures on almost all leaves of $\mathcal{L}$ are λ-harmonic.

THEOREM 4.5 (Garnett [**Ga**]). *On any compact Riemannian lamination there is a harmonic (that is,* $\lambda = 0$*) measure.*

However, on non-compact laminations the question of existence of a λ-harmonic measure is non-trivial.

2.2. Harmonic measures and conformal streams. Harmonic measures are intimately related to transverse conformal streams. Let us consider a confine lamination $\mathcal{A}$ and its hyperbolization $\mathcal{H}$. Then any parallel transverse δ-conformal stream η on $\mathcal{A}$ can be lifted to a transverse measure $\tilde{\eta}$ on $\mathcal{H}$ satisfying the transformation rule

$$\frac{d(\gamma^{-1}\tilde{\eta})}{d\tilde{\eta}} = \exp \delta\beta(x, \gamma x) \tag{4.3}$$

where γ is a holonomy map and β is the Busemann cocycle.

Paring this transverse measure with the leafwise hyperbolic volume, we obtain a λ-harmonic measure ω on $\mathcal{H}$, where $\lambda = \delta(\delta - d)$ (so ω is harmonic iff $\delta = d$).

If the conformal stream η is invariant under some affine action on $\mathcal{A}$ then the corresponding harmonic measure ω is invariant under the hyperbolic lift of this action. Hence ω descends to a conformal measure on the quotient lamination.

Under the vertical flow, ω is contracted with exponential rate $d - \delta$ (so that ω is invariant iff $\delta = d$).

2.3. Harmonic measures for geodesic flows. Let us now apply the above constructions to the geodesic flow on a hyperbolic d-fold M. If we start with the volume stream on S^d then we come up with the classical Liouville measure on UM. Since $\delta = d$, this measure is harmonic and invariant under the geodesic flow.

If we start with the Patterson conformal measure, we come up with another λ-harmonic measure on UM, where $\lambda = \delta_{\text{cr}}(\delta_{\text{cr}} - d)$ is the bottom eigenvalue of the Laplacian on M.

2.4. Harmonic measures on $\mathcal{H}_f$. Similarly, the transverse conformal measure from Theorem 4.3 can be promoted to a λ-harmonic measure on the hyperbolic lamination $\mathcal{M}_f$, where $\lambda = \delta_{\text{cr}}(\delta_{\text{cr}} - 2)$. This gives us a promising link between fractal geometry of Julia sets (reflected in δ_{cr}) and spectral theory of hyperbolic laminations (reflected in λ). This kind of connection proved to be very useful in the theory of Kleinian groups (see [**S3, BJ**] and references therein).

3. Invariant measures

There exist other natural measures associated to conformal streams. They are always invariant under the dynamics in question but rarely absolutely continuous on the leaves.

3.1. Margulis measure for geodesic flows. A Γ-invariant Patterson conformal stream on S^d can be naturally transferred to the family of *measures* on the stable/unstable horospheres. The stream property implies that these measures are uniformly contracted/expanded under the geodesic flow with rate δ. Hence the product of these measures with the hyperbolic measure on the flow lines is invariant under the flow. It is also invariant under Γ, so that it descends to an invariant measure for the geodesic flow on $M = \mathbb{H}^{d+1}/\Gamma$. This invariant measure is called *Margulis*. In the case of compact manifold M, this is the unique measure of maximal entropy of the geodesic flow (see [**Ma, K**]).

This construction works in higher dimensions and for variable curvature as well.

3.2. Gibbs invariant measure for rational maps. If the backward and forward critical exponents (see §2.3.3) are equal, then the product of the transverse and leafwise conformal streams, $\lambda \times \eta$, is a *measure* $\tilde{\nu}$ invariant under $\hat{f}$. By construction, the conditional measures of $\tilde{\nu}$ on the leaves of $\mathcal{A}_f$ are absolutely continuous with respect to the leafwise conformal stream. Pushing $\tilde{\nu}$ down to the sphere $\bar{\mathbb{C}}$, we obtain an f-invariant measure μ absolutely continuous with respect to a conformal measure on the Julia set $J(f)$. It is called *Gibbs conformal measure.*

This construction can be applied to postcritically non-recurrent rational maps (i.e., such that all critical points on the Julia set are non-recurrent and there are no parabolic points):

LEMMA 4.6. *If f is a postcritically non-recurrent rational map then* $\delta_{\text{cr}} = \gamma_{\text{cr}}$.

Thus, for postcritically non-recurrent rational maps, we come up with a new construction of the Gibbs conformal measure ν, first constructed in [**DU, U**] by means of thermodynamical formalism. In fact, in this case the lift $\hat{f} : \mathcal{A}_f \to \mathcal{A}_f$ has a compact Julia set and due to Lemma 3.2 of the 3d lecture it can be viewed as an "orbifold hyperbolic" map. This opens an opportunity to derive statistical properties of the Gibbs conformal measure ν from the (orbifold) hyperbolic theory (a project under way, joint with M. Urbanski). In this approach, the Seifert fibration from §4 can help to deal with the orbifold points.

3.3. An invariant measure for the vertical flow. The Gibbs conformal measure when exists can be lifted to an invariant measure for the vertical flow. It is also invariant under $\hat{f}$, so that it descends to a V^t-invariant measure on $\mathcal{M}_f$. In the case when $\mathcal{M}_f$ is compact, we obtain the measure of maximal entropy for the vertical flow V^t. For hyperbolic maps, this measure was constructed in [**BR, BFU**].

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