MAT 534: Solutions for problem Set 4

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These are solutions for **some** of the HW problems. If you didn't solve the problem yourself, be sure to look through the solutions.

3. (a) T is nilpotent iff $\chi_T(\lambda) = \lambda^n$.

   (b) T is nilpotent $\implies T^{\dim V} = 0$.

   Proof: If T is nilpotent, then its eigenvalues are zero. Indeed, write T in upper tringular form; then T^k will have λ_i^k on diagonal, so $T^k = 0$ implies that all $\lambda_i = 0$.

   Conversely: assume that all $\lambda_i = 0$. Write T in an upper triangular form; it will be strictly upper triangular, i.e. will have zeros on the diagonal. Explicit calculation shows that T^2 will have zeros on the diagonal and immediately above it; T^3 will have zeros on the diagonal and the two adjacent subdiagonals, etc. This implies $T^{\dim V} = 0$, proving both (a) and (b).

4. Prove $\operatorname{tr} A^i = 0$ for all $i \implies A$ is nilpotent.

   Idea of proof: Writing A in upper-triangular form and using the previous problem, we see that $\sum \lambda_i^k = 0$ for all k. Now we need the following lemma:

   Coefficients of the polynomial $\prod(\lambda - \lambda_i)$ can be written as polynomials without constant term in $\sigma_1 = \sum \lambda_i, \sigma_2 = \sum \lambda_i^2, \ldots$ (For example: for $n = 2$, the coefficients are

$$-(\lambda_1 + \lambda_2) = -\sigma_1$$
$$\lambda_1\lambda_2 = \frac{1}{2}[(\lambda_1 + \lambda_2)^2 - \lambda_1^2 - \lambda_2^2] = \frac{1}{2}(\sigma_1^2 - \sigma_2)$$

   This lemma is not easy to prove, but it can be done by induction. Using this lemma, we see that the chracteristic polynomial of A is λ^n; by previous probelm, it means that A is nilpotent.

5. Prove: $\det(e^A) = e^{\operatorname{tr} A}$

   Idea of proof: suffices to check for upper triangular matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$. In this case, A^k is also upper-triangular with eigenvalues $\lambda_1^k, \dots, \lambda_n^k$. Thus, $e^A = \sum A^k/k!$ is also upper triangular with $e^{\lambda_1}, \dots, e^{\lambda_n}$ on the diagonal.

7. Let A be a diagonalizable operator such that $\lambda_1 = 1$ and $|\lambda_i| < 1$ for $i > 1$. Prove that $P = \lim_{n\to\infty} A^n$ exists and satisfies $P^2 = P$. Describe $\operatorname{Im} P$.

   Idea of proof: In a suitable basis, $A = diag(1, \lambda_2, \dots)$. Thus, $A^n = diag(1, \lambda_2^n, \dots) \to diag(1, 0, \dots) = P$. It is easy to see that $\operatorname{Im} P = v_1$ – the first eigenvector.

8. Let A, B be commuting linear operators: $AB = BA$. Prove that

   (a) they have a common eigenvector.

   (b) they have a common invariant flag, i.e., there exists a basis in which both A and B are upper-triangular.

   (c) the eigenvalues of AB are products of eigenvalues of A and B.

   (d) Which of these statements still hold if $AB \neq BA$?

   Idea of proof: (a) Let λ be an eigenvalue of A, and $V_\lambda = \operatorname{Ker}(A - \lambda)$ the space of eigenvectors. We claim that V_λ is invariant under B. Indeed: if $v \in V_\lambda$, then $A(Bv) = BAv = B\lambda v = \lambda Bv$ and thus, Bv is an eigenvector for A with eigenvalue λ.

   Consider the restriction of B to V_λ. This restricted operator has at least one eigenvector (say, w) in V_λ. On the other hand, every vector in V_λ is an eigenvector for A, so w is an eigenvector for both A and B.

   (b) This is done in exactly the same way as for one opertor, by induction in dimension of V. That is: let v_1 be a common eigenvector for A, B. Consider the space $V' = V/\mathbb{C}v_1$. The operators A, B act on V' and commute. By induction assumption, there exists a basis $v'_1, \dots, v'_{n-1}$ in V' in which these operators have upper triangular form. Lift v'_1 to a vector in V (that is: choose a representative in the equivalence class v'_1); denote it by v_2. Do the same with all other basis elements v'_i; this will give us vectors $v_2, \dots, v_n \in V$. As discussed before, the vectors $v_1, v_2, \dots, v_n$ form a basis in V, and A, B are upper-triangular in this basis.

(c) is obvious from (b)

(d) None of these statements hold: take $A = diag(1,2), B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.