MAT 534 FALL 2000
FINAL

NAME : SSN :

THERE ARE 8 PROBLEMS
SHOW YOUR WORK!!!

1	
2	
3	
4	
5	
6	
7	
8	
Total	

1. Let G be a group and $C_G = \{c \in G \mid \forall g \in G \; cg = gc\}$.

a) Pove that C_G is a normal subgroup of G.

Sol's. Check that if $c \in C_G$ then $c^{-1} \in C_G$. Check that if $c, d \in C_G$, then $cd \in C_G$. Check that $gC_Gg^{-1} \subseteq C_G$.

b) Prove that if G contains exactly one element a of order two, then $a \in C_G$.

Sol's. $ag = ga$ iff $g^{-1}ag = a$. $(g^{-1}ag)^2 = g^{-1}a^2g = e_G$. It follows that the order of $g^{-1}ag$ is two for every $g \in G$ and by the uniqueness of a we are done.

2. Let $f : G \to H$ be a group homomorphism with kernel N. Prove that for every subgroup K of G we have that $f^{-1}(f(K)) = K$ if and only if $N \subseteq K$.

Sols. First prove that $f^{-1}(f(K)) = KN$. Then prove that $K = f^{-1}f(K)$ iff $N \subseteq K$.

3. Let p be the smallest prime number dividing the order of a finite group G.

Show that any subgroup H of G of index p is normal in G.

(Hint. Consider the action of G on G/H.)

Sols. Consider G acting on the set of left cosets G/H by left translation: $g * (aH) := gaH$. $|G/H| = p$. As discussed in class this defines a group homomorphism $G \to S_p$, where S_p is identifyed with the permutations of the set G/H. The kernel K must be contained in H.

The image is isomorphic to the group G/K. The image is a subgroup of S_p so that $|G/K| = [G : K]$ divides $p!$.

$[G : K]$ divides $|G|$.

The minimality of p implies that either $[G : K] = 1$ or $[G : K] = p$.

But $K \subseteq H \neq G$. So $[G : K] = p$. It follows that $[H : K] = 1$, i.e. $H = K$.

But K is normal.

4. Let $A = ||a_{ij}|| \in M_{n\times n}(K)$ be a $n \times n$ matrix over a field K. Assume that $a_{ii} = a$ for some $a \in K$ and every $1 \leq i \leq n$. Assume also that $a_{ij} = 0$ for every $i < j$.

a) Prove that $(aI_n - A)^n = 0$.

Sol's. A is lower triangular. The characteristic polynomial is, by expansion, $(x-a)^n$. Apply Cayley-Hamilton.

b) Can one always find an integer n' such that $0 < n' < n$ and $(A - aI_n)^{n'} = 0$?

Sol's. No:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

5. Let

$$B = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

a) Find the characteristic polynomial and the eigenvalues of B.

Sol's. The char. polynomial is $|xI - B| = x^3 - 12x - 16 = (t+2)^2(t-4)$. The eigenvalues are $l_1 = -2$ and $l_2 = 4$.

b) Find a maximal set S of linearly independent eigenvectors of B.

Sol's. Solve the linear system $(-2I - B)X^t = 0$. The solutions are multiples of $v_1 :=< 1, 1, 0 >$. There is a unique eigenvector for $l_1 = -2$. Similarely, there is a unique eigenvector, $v_2 :=< 0, 1, 1 >$ for $l_2 = 4$. $S = \{v_1, v_2\}$ is a set of the required form.

c) Is B diagonalizable? If yes, find and invertible matrix P such that PBP^{-1} is diagonal.

Sol's. No: B has two eigenvalues. One, $l_1 = -2$ has algebraic multiplicity two, but the corresponding eigenspace, $< v_1 >$, has dimension one.

It follows that there cannot be a basis of eigenvectors and this implies that B is not diagonalizable.

6. Let

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

a) Find an orthogonal matrix P such that P^tAP is diagonal.

Sol's. By the spectral theorem (real case) there is a orhtonormal basis v_1, v_2, v_3 of eigenvectors of A such that if P is the matrix with *columns* v_1, v_2, v_3 then $P^{-1}AP = P^tAP$ is diagonal with the corresponding eigenvectors l_1, l_2, l_3 on the diagonal.

We first find the eigenvalues: $l_1 = 0$, $l_2 = -1$ and $l_3 = 3$. We then find eigenvectors by solving $(A - l_i I)X^t = 0$. We find $v_1' =< 1, -1, 1 >$ for $l_1 = 0$; $v_2' =< 0, 1, 1 >$ for $l_2 = -1$; $v_3' =< -2, -1, 1 >$ for $l_3 =$. They are orthogonal. We normalize them and find $v_1 =< 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3} >$, $v_2 =< 0, 1/\sqrt{2}, 1/\sqrt{2} >$, $v_3 =< -2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6} >$.

It follows that

$$P^t = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

is a matrix of the required form.

b) Let $f(\mathbf{x}, \mathbf{y}) = \sum_{ij} A_{ij} x_i y_j$ be the corresponding bilinear form on $\mathbb{R}^3$. Find the null space and the signature (p, m) of f.

Sol's. By part a), we have one zero, one positive and one negative eigenvalue. The null space is the one generated by the eigenvector corresponding to $l_1 = 0$, that is: v_1. The signature is $(1, 1)$.

7. Let $M = M_{n\times n}(\mathbb{R})$, $S = \{C \in M |\ C = C^t\}$ and $A = \{C \in M |\ C = -C^t\}$.

a) Prove that every $D \in M$ can be written uniquely as $D = D_S + D_A$, where $D_S \in S$ and $D_A \in A$.

Sol's. Define $D_S = \frac{1}{2}(D + D^t)$ and $D_A = \frac{1}{2}(D - D^t)$. Clearly $D_S + D_A = D$. We have $S \cap A = \{\text{the zero matrix}\}$. Let $D_S' + D_A' = D$ be another decomposition. Then $D_S - D_S' = D_A' - D_A$ and both terms are symmetric and anti-symmetric. It follows they are both zero.

b) Prove that $\dim_{\mathbb{R}} S = \frac{1}{2} n(n+1)$.

Sol's. It is the number of pairs (i, j) with $i \leq j$ which can be counted as $1 + 2 + 3 + \ldots + n$.

c) Find the dimension of the trace zero linear transformations on a n-dimensional real vector space which are symmetric with respect to the dot product.

Sol's. Pick an orthonormal basis for the dot product. Using this basis, the matrices of the linear transformations f symmetric with respect to the dot product are symmetric and, "viceversa." Under this identification the required matrices are the trace-zero symmetic matrices. Their dimension is one less than the dimension of the space of symmetric matrices computed above.

8. Let V and W be finite dimensional vector spaces over a field.
a) Prove, using the properties of the tensor product, that $(V \otimes W)^* \simeq V^* \otimes W^*$.

(*If you introduce a map, you must check that it is well-defined and that it has the properties that you state and use. The map must be independent of any choice of bases*).

Sol's. Define a bilinear map $g : V^* \times W^* \to (V \otimes W)^*$ by setting $g(f,h)(v \otimes w) := f(v)h(w)$.

By the basic property of the tensor product there exists a unique linear map $g' : V^* \otimes W^* \to (V \otimes W)^*$ sending $f \otimes h$ to $g(f,h)$.

Pick bases v_i for V, w_α for W and consider the dual bases: v_i^* and w_α^*. Let $t := \sum_{i\alpha} a_{i\alpha} v_i^* \otimes w_\alpha^*$ be an element in $V^* \otimes W^*$. $g'(t)(v_j \otimes w_\beta) = a_{j\beta}$. It follows that $t \in Ker g'$ iff $t = 0$ and g' is injective.

Since g' is injective between vector spaces of the same finite dimension $\dim V \times \dim W$, g' is also surjective, i.e. it is an isomorphism.

b) Prove that there is a canonical isomorphism

$$Hom(V, W^*) \simeq (Hom(V^*, W))^*.$$

Sol's. Recall the isomorphisms $Hom(A,B) \simeq A^* \otimes B$ and $C^{**} \simeq C$. The RHS is isomorphic to $((V^*)^* \otimes W)^* \simeq (V \otimes W)^*$, which, by part a), is isomorphic to $V^* \otimes W^*$ which is isomorphic to the LHS.