Homework 3

(1) Compute the singular homology groups of:
  (a) A torus $S^1 \times S^1$,
  (b) A genus 2 surface.
  (c) $S^2/\sim$ where $\sim$ is an equivalence relation identifying the north pole with the south pole.

(2) Calculate the singular homology groups of

$$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \ : \ -\frac{1}{2\pi} < x < \frac{1}{2\pi},\ x \neq 0\right\} \cup \{(0,y) \ : \ -1 \leq y \leq 1\} \subset \mathbb{R}^2$$

(3) Let $X = [0,1]$ and $A = \{0\} \cup \{1/n \ : \ n \in \mathbb{N}_{>0}\}$. Show that $H_1(X,A)$ is not isomorphic to $\widetilde{H}_1(X/A)$ (Hatcher Section 2.1, Exercise 26).

(4) **Optional.** Consider the set $X = \{0,1,2,3\}$ with the topology

$$\mathcal{T} := \{\emptyset, \{0\}, \{2\}, \{0,2\}, \{0,1,2\}, \{0,3,2\}, \{0,1,2,3\}\}.$$

What are its singular homology groups?

(5) Prove the *five lemma*. I.e. given a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \alpha\big\downarrow & & \beta\big\downarrow & & \gamma\big\downarrow & & \delta\big\downarrow & & \epsilon\big\downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where the horizontal arrows are exact, β,δ are isomorphisms, α is surjective and ϵ is injective, show that γ is an isomorphism.