Homework 2

(1) Calculate the singular homology groups of

$$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \;:\; -\frac{1}{2\pi} < x < \frac{1}{2\pi},\ x \neq 0\right\} \cup \{(0, y) \;:\; -1 \leq y \leq 1\} \subset \mathbb{R}^2$$

(2) Let $X = [0,1]$ and $A = \{0\} \cup \{1/n \;:\; n \in \mathbb{N}_{>0}\}$. Show that $H_1(X, A)$ is not isomorphic to $\widetilde{H}_1(X/A)$ (Hatcher Section 2.1, Exercise 26).

Definition: Define the *kth Betti number of a topological space* X to be the rank of the homology group $H_k(X)$. Define the *Euler characteristic of* X to be the integer

$$\chi(X) := \sum_{k=0}^{\infty} (-1)^k \text{ (the } k\text{th Betti number of } X).$$

(3) (Hatcher Section 2.2, Question 22). Let $\pi : \widetilde{X} \to X$ be a covering space with m sheets. Show $\chi(\widetilde{X}) = m\chi(X)$.

(4) (Hatcher Section 2.2 Question 28).

(a) Use the Mayer–Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times x_0$ in the torus.

(b) Do the same for the space obtained by attaching a Möbius band to $\mathbb{R}P^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{R}P^1 \subset \mathbb{R}P^2$.

(5) **Optional.** Consider the set $X = \{0, 1, 2, 3\}$ with the topology

$$\mathcal{T} := \{\emptyset, \{0\}, \{2\}, \{0, 2\}, \{0, 1, 2\}, \{0, 3, 2\}, \{0, 1, 2, 3\}\}.$$

What are its singular homology groups?

(6) Let X be the quotient space of S^2 obtained by identifying the north and south poles to a point. Compute the CW homology groups of X.

(7) Compute the CW homology groups of $\mathbb{R}P^n$.