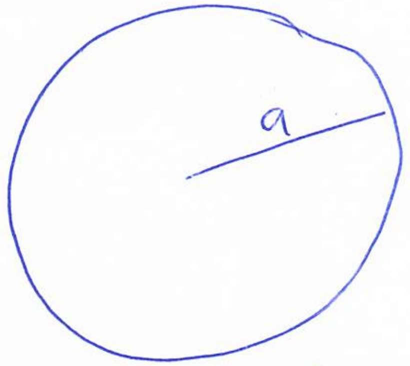
XII a 5.7]

Symmetric Membrane



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$u(a,t) = 0$$

$$u(r,0) = f(r)$$

$$\frac{\partial u}{\partial t}(r,0) = g(r)$$

no θ-dependence

$u(r,t) = \phi(r)T(t)$ $\qquad$ $\phi(a) = 0$ $\qquad$ ϕ bounded.

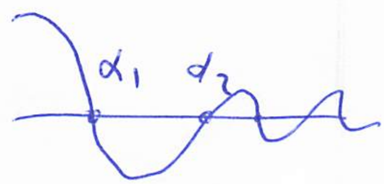
$$\frac{1}{r}(r\phi')'T = \frac{1}{c^2}\phi T''$$

$$\frac{(r\phi')'}{r\phi} = \frac{1}{c^2}\frac{T''}{T} = -\lambda^2$$

$$\begin{cases} T'' + \lambda^2 c^2 T = 0 \\ (r\phi')' + \lambda^2 r\phi = 0 \end{cases}$$ $\leftarrow$ Bessel Eq $\nu = 0$

$$\phi(r) = J_0(\lambda r)$$

$J_0(\lambda a) = 0$ $\qquad$ $\lambda_n a = \alpha_n$



$$\lambda_n^2 = \left(\frac{\alpha_n}{a}\right)^2$$

$$\phi_n(r) = J_0(\lambda_n r)$$

$$T_n(t) = a_n \cos\lambda_n t + b_n \sin\lambda_n t$$

$$u_n(r,t) = J_0(\lambda_n r)\left[a_n \cos\lambda_n t + b_n \sin\lambda_n t\right]$$

General Sol.

$$u(r,t) = \sum_{n\geq 1} J_0(\lambda_n r)\left[a_n \cos\lambda_n t + b_n \sin\lambda_n t\right].$$

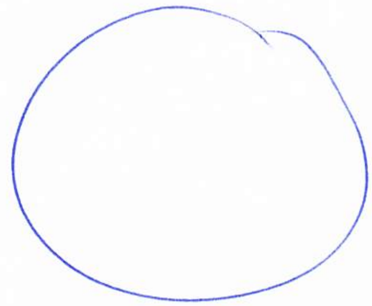
$$f(r) = u(r,0) = \sum_{n\geq 1} a_n J_0(\lambda_n r)$$

$$g(r) = \frac{\partial u}{\partial t}(r,0) = \sum_{n\geq 1} b_n \lambda_n c J_0(\lambda_n r).$$

$$a_n = \int_0^a f(r) J_0(\lambda_n r)\, dr \Big/ \int_0^a J_0(\lambda_n r)^2\, dr.$$

$$b_n = \frac{1}{\lambda_n c} \int_0^a g(r) J_0(\lambda_n r)\, dr \Big/ \int_0^a J_0(\lambda_n r)^2\, dr.$$

General Membrane (No Symmetry).



$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} \\[2ex] u(a,\theta,t) = 0 \\[1ex] u \text{ bounded.} \\[1ex] u(r,\theta+2\pi,t) = u(r,\theta,t) \\[1ex] u(r,\theta,0) = f(r,\theta) \\[1ex] \dfrac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{cases}$$

$$u(r,\theta,t) = \phi(r,\theta)\,T(t)$$

$$\begin{cases} T'' + \lambda^2 c^2 T = 0 \\[1ex] \cancel{\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}} \quad \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = -\lambda^2 \phi \end{cases}$$

$$\phi(r,\theta) = R(r)\,Q(\theta)$$

$$\frac{1}{r}(rR')'Q + Q''R\frac{1}{r^2} = -\lambda^2 RQ.$$

$$\frac{r(rR')'}{R} + \lambda^2 r^2 = -\frac{Q''}{Q} = \nu^2$$

$$\begin{cases} Q'' + \nu^2 Q = 0 \\ Q(\theta + 2\pi) = Q(\theta) \end{cases}$$

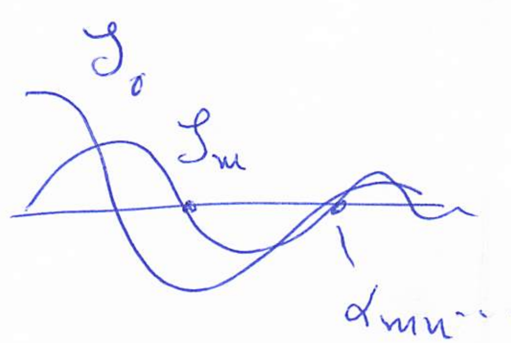
$$\begin{cases} (rR')' - \frac{\nu^2}{r} R + \lambda^2 r R = 0 \\ R(a) = 0 \\ R \text{ bounded} \end{cases}$$

$\nu_0^2 = 0$: $\quad Q_0 = 1.$

$\nu_m^2 = m^2$: $\quad Q_m = \cos m\theta, \ \sin m\theta \quad m \geq 1$

$$R(r) = C J_m(\lambda r) + \cancel{D\, Y_m(r)}$$

$$R(r) = J_m(\lambda r)$$



$R(a) = 0 \qquad J_m(\lambda a) = 0$

$$J_m(\alpha_{mn}) = 0$$

$$\lambda_{mn} = \frac{\alpha_{mn}}{a} \qquad \begin{matrix} m \geq 0 \\ n \geq 1 \end{matrix}$$

$$\left\{ \begin{matrix} J_m(\lambda_{mn} r) \cos m\theta , & m \geq 1 , n \geq 1 \\ J_m(\lambda_{mn} r) \sin m\theta & \end{matrix} \right.$$

$$\left\{ J_0(\lambda_{0n} r) \right.$$

$$T_{mn} = \cos \lambda_{mn} t \quad , \sin \lambda_{mn} t . \qquad \begin{matrix} m \geq 0 \\ n \geq 1 \end{matrix}$$

— // —

Complete Selection of "Standing Waves"

$$\left\{ \begin{matrix} \left. \begin{matrix} J_m(\lambda_{mn} r) \cos m\theta \cos \lambda_{mn} t \\ J_m(\lambda_{mn} r) \cos m\theta \sin \lambda_{mn} t \\ J_m(\lambda_{mn} r) \sin m\theta \cos \lambda_{mn} t \\ J_m(\lambda_{mn} r) \sin m\theta \sin \lambda_{mn} t \end{matrix} \right\} & m, n \geq 1 \\ \\ \begin{matrix} J_0(\lambda_{0n} r) \cos \lambda_{0n} t \\ J_0(\lambda_{0n} r) \sin \lambda_{0n} t \end{matrix} & n \geq 1 \quad (m = 0) \end{matrix} \right.$$

The general sol. consists of all linear combinations of these standing waves.

<u>Initial Data</u>

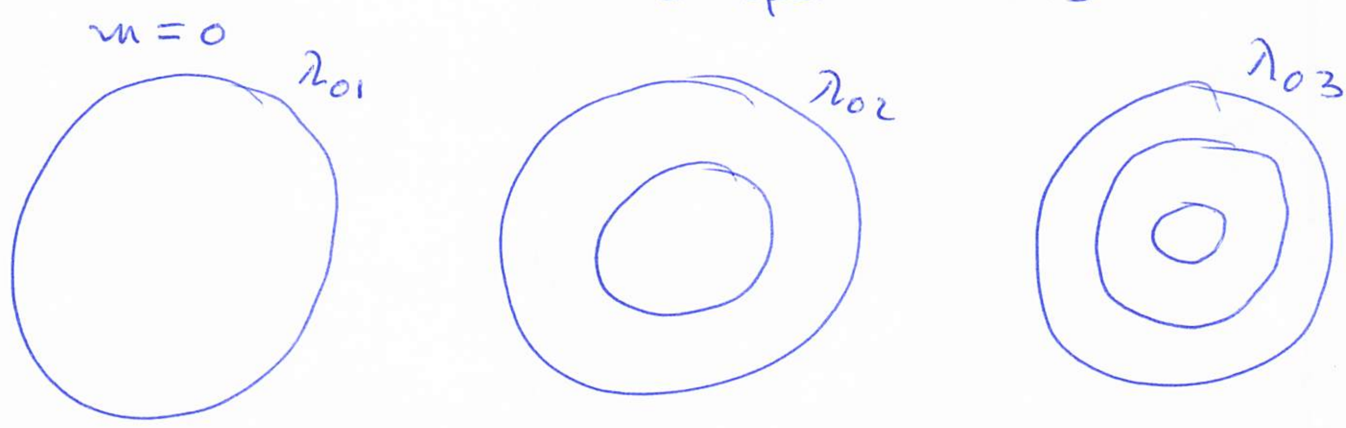
$$
\begin{aligned}
u(r,\theta,0) = f(r,\theta) = {} & \sum a_{0n} J_0(\lambda_{0n} r) + \\
& \sum a_{mn} J_m(\lambda_{mn} r) \cos m\theta + \\
& \sum b_{mn} J_m(\lambda_{mn} r) \sin m\theta
\end{aligned}
$$

J_0, $J_{mn} \cos m$, $J_{mn} \sin m\theta$ orthogonal basis

etc. . .

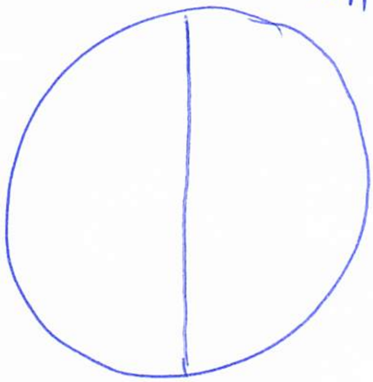
Standing Waves $J_n(\lambda_{mn} r) \cos m\theta \cos \lambda_{mn} t$

Shape. nodes Vibration

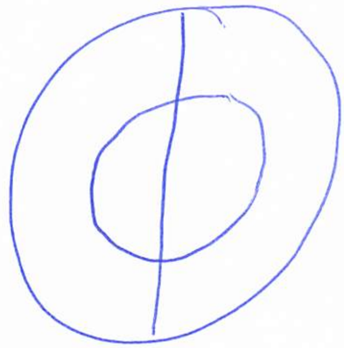


$m=1$

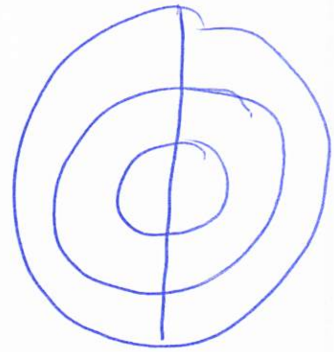
λ_{11}



λ_{12}

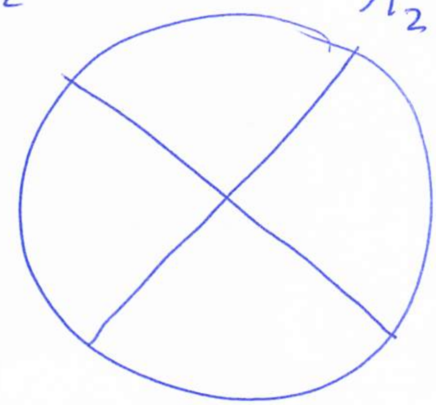


λ_{13}

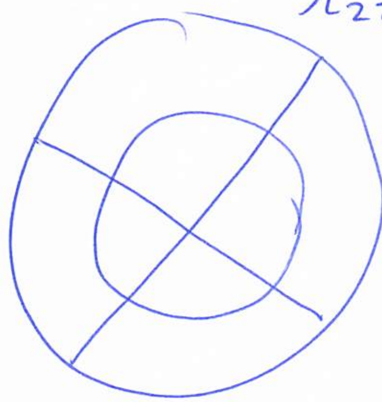


$m=2$

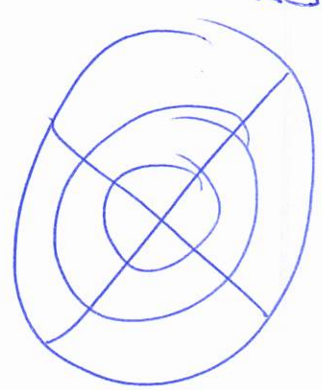
λ_{21}



λ_{22}



λ_{23}



XIIb §6.1

f piece wise C^0 $\quad \cancel{0 \le t \le T}$

$$Lf(s) = \int_0^\infty e^{-st} f(t)\, dt. = F(s)$$

$$F = Lf.$$

F Laplace transform f.

Ex

$$L1 = \frac{1}{s}$$

$$Le^{at} = \frac{1}{s-a}$$

Good f: f piecewise C^0 on every intv $[0,T]$

and

$\lim e^{-kt} f(t) = 0$ $\exists k$ $\Big\}$ exp order.

$\Longrightarrow$

$Lf(s)$ exists for $s > k$.

$\mathcal{L}$ is a linear operator

$$\mathcal{L}(af+bg) = a\mathcal{L}f + b\mathcal{L}g.$$

Ex) $\mathcal{L}\left(\frac{e^{at}+e^{-at}}{2}\right) = \cdots = \frac{s}{s^2-a^2}$

$$\mathcal{L}(\sin\omega t) = \mathcal{L}\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)$$

$$= \cdots = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}(e^{bt}f(t))(s) = \mathcal{L}f(s-b).$$

$$\mathcal{L}(e^{bt}\sin\omega t) = \frac{\omega}{(s-b)^2+\omega^2}$$

$$\mathcal{L}(f')(s) = \int_0^{a} e^{-st} f'(t)\,dt =$$

$$= e^{-st}f(t)\Big|_0^{\infty} - \int_0^{\infty} f(t)\cdot(-s)\,e^{-st}\,dt =$$

$$\boxed{L(f')(s) = -f(0) + sLf(s)}$$

$$L(f'')(s) = -f'(0) - sf(0) + s^2 Lf(s)$$

$$\boxed{L(f^{(n)})(s) = -f^{(n-1)}(0) - sf^{(n-2)}(0) + \cdots - s^{n-1}f(0) + s^n Lf(s)}$$

Ex $f(t) = t^k$

$$0 = f(0) = \cdots = f^{(k-1)}(0) = 0 \qquad f^{(k)}(0) = k!$$

$$f^{(k+1)}(t) \equiv 0$$

$n = k+1$:

$$0 = -k! + s^{k+1} L(t^k)$$

$$L(t^k) = \frac{k!}{s^{k+1}}$$

$$g(t) = \int_0^t f(x)\,dx \qquad g(0) = 0$$

$$g'(t) = f(t)$$

$$\underset{\displaystyle Lf(s)}{\underset{\parallel}{Lg'(s)}} = -\underset{0}{\underset{\parallel}{g(0)}} + \underset{\displaystyle sL\left[\int_0^t f(x)\,dx\right].}{\underset{\parallel}{sLg(s)}}$$

$$\boxed{L\left[\int_0^t f(x)\,dx\right] = \frac{1}{s} Lf(s)}$$

$$L(tf(t))(s) = \int_0^\infty t f(t) e^{-st}\,dt$$

$$= \int_0^\infty f(t) \frac{\partial}{\partial s}(e^{-st})\,dt = -\int_0^\infty \frac{\partial}{\partial s}\left(f(t) e^{-st}\right) dt$$

$$= \frac{\partial}{\partial s} \int_0^\infty f(t) e^{-st}\,dt = -\frac{\partial}{\partial s} Lf(s)$$

$$\boxed{L(tf(t))(s) = -\frac{\partial}{\partial s} Lf(s)}$$

Ex] $\mathcal{L}(t\sin\omega t)(s) = -\frac{\partial}{\partial s}\mathcal{L}(\sin\omega t)(s)$

$$= -\frac{\partial}{\partial s}\frac{\omega}{s^2+\omega^2} = \frac{2s\omega}{(s^2+\omega^2)^2}$$