A COMPARISON THEOREM FOR THE MASS OF ALE AND ALF TORIC 4-MANIFOLDS

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Abstract. We establish sharp lower bounds for the mass of asymptotically locally Euclidean (ALE) and asymptotically locally flat (ALF) toric 4-manifolds, in terms of equilibrium geometries consisting of gravitational instantons. More precisely, the mass of a complete ALE or ALF toric 4-manifold with nonnegative scalar curvature is bounded below by a sum comprised of the following quantities: the mass of the corresponding toric gravitational instanton having the same orbit space (rod) structure as the original ALE/ALF manifold, and an expression determined by the conical angle defects of totally geodesic 2-spheres within the instanton that serve as generators for its second homology. The inequality may be generalized to the situation in which the ALE/ALF manifold also possesses conical singularities as well as orbifold singularities, and it suggests a refined notion of 'total mass' in which the result simply states that the total mass of the ALE/ALF manifold is not less than that of the corresponding gravitational instanton. Furthermore, we prove rigidity for these statements, namely the inequality is saturated only when the ALE/ALF manifold is Ricci flat and in fact agrees with the corresponding instanton. These results may be viewed in the context of positive mass theorems, providing an explanation of how positivity can fail in the ALE/ALF setting. Moreover, the main theorem may be interpreted as yielding a variational characterization of the relevant toric gravitational instantons.

1. Introduction

The positive mass theorem is a central achievement in the study of scalar curvature and mathematical relativity, and was originally established for asymptotically Euclidean (AE) manifolds with nonnegative scalar curvature by Schoen-Yau [47] and Witten [51]. Various incarnations of this theorem have been found in a variety of other settings. In the asymptotically hyperbolic case, important contributions were made by Andersson-Cai-Galloway [6], Chruściel-Herzlich [18], Wang [49], and Zhang [52]. Extensions to the asymptotically locally hyperbolic setting were obtained by Alaee-Hung-Khuri [1], Brendle-Hung [13], and Lee-Neves [38], while the complex hyperbolic case was treated by Herzlich [12,27]. In this article we will be concerned with ALE and ALF manifolds which, in particular, arise naturally in the study of *gravitational instantons* — complete, noncompact, Ricci flat 4-dimensional Riemannian manifolds with square-integrable curvature.

Definition 1.1. (ALE Manifold) A connected and complete Riemannian 4-manifold (M, g) is said to be *asymptotically locally Euclidean* (ALE) if there exists a compact set $K \subset M$, a finite subgroup $\mathcal{G} \subset O(4)$ acting freely on coordinate spheres, and a diffeomorphism $\varphi : (\mathbb{R}^4 \setminus \overline{B}_1)/\mathcal{G} \to M \setminus K$ such that

$$|\mathring{\nabla}^l(\varphi^* g - b)|_b = O(r^{-1-\kappa-l}), \qquad l = 0, 1, 2, \tag{1.1}$$

for some $\kappa > 0$ where b is the flat cone metric on $(\mathbb{R}^4 \setminus \overline{B}_1)/\mathcal{G}$ with radial distance function r, and $\mathring{\nabla}$ denotes covariant differentiation with respect to b. Moreover, the scalar curvature of g is required

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to be integrable, $R_g \in L^1(M)$. If $\mathcal{G}$ is trivial, then the manifold is called *asymptotically Euclidean* (AE).

Remark 1.2. The definition of an ALE manifold, and that of an ALF manifold given below, often allow for multiple ends. Although the results of this paper may be generalized to include more than one end, for simplicity of presentation this will not be pursued here. Note also that the model metric may be expressed in polar form $b = dr^2 + r^2 b_s$, where b_s is the metric of constant $+1$ curvature on the radial cross-section $\mathcal{S} = S^3/\mathcal{G}$. Moreover, regularity of the metric is left unspecified here and in the definition of an ALF manifold below, since mild singularities will eventually be included.

Motivated by questions in quantum gravity, the validity of the positive mass theorem was also conjectured for the ALE setting. However, the Eguchi-Hanson manifold [24] may be observed to violate the rigidity statement, and LeBrun [37] generalized their construction to find an infinite family of explicit counterexamples to the inequality. Nevertheless, positivity of mass in the ALE case has been established under additional hypotheses, notably for certain Kähler manifolds by Hein-LeBrun [26] and under suitable spin-structure matching conditions by Dahl [19] and Deruelle-Ozuch [23].

Definition 1.3 (ALF Manifold [11, Definition 1])**.** A connected and complete Riemannian 4-manifold (M, g) will be called *asymptotically locally flat* (ALF) if the following conditions are satisfied.

(i) There is a compact subset $K \subset M$ and a diffeomorphism $\varphi : \mathbb{R}_+ \times \mathcal{S} \to M \setminus K$, where $\mathcal{S}$ is a closed 3-manifold finitely covered by $S^1 \times S^2$ or S^3. If $\mathcal{S} = S^1 \times S^2$ then the manifold is called *asymptotically flat* (AF)[1].

(ii) On $\mathcal{S}$ there is a 1-form τ and a vector field T such that $\iota_T \tau = 1$ and $\mathcal{L}_T \tau = 0$, where ι denotes interior product and $\mathcal{L}$ indicates Lie differentiation.

(iii) On $\mathcal{S}$ there is also a positive-semi-definite symmetric 2-tensor γ such that $\mathcal{L}_T \gamma = 0$, $\ker \gamma = \operatorname{span} T$, and which locally defines a metric of Gauss curvature $+1$ on the space of leaves of the foliation tangent to T.

(iv) $\mathbb{R}_+ \times \mathcal{S}$ is equipped with a model metric

$$b = dr^2 + r^2 \gamma + \ell^2 \tau^2, \tag{1.2}$$

where r parameterizes $\mathbb{R}_+$ and $\ell > 0$ is a constant.

(v) After pulling back via diffeomorphism φ, the metric g asymptotes to the model with decay

$$|\mathring{\nabla}^l (\varphi^* g - b)|_b = O(r^{-\frac{1}{2}-\kappa-l}), \qquad l = 0, 1, 2, \tag{1.3}$$

where $\kappa > 0$ and $\mathring{\nabla}$ denotes covariant differentiation with respect to b.

(vi) The scalar curvature of g is integrable, $R_g \in L^1(M)$.

Remark 1.4. An important special case of an ALF manifold occurs when $\mathcal{S}$ is an S^1-bundle over S^2, as exemplified by the Taub-NUT gravitational instanton in which the relevant bundle is the Hopf fibration of S^3. Such an ALF manifold, in which the bundle has Euler number e, is referred to as ALF-A_k where $k = -e - 1$. A notable case of AF manifolds occurs when the model geometry is flat and arises as the quotient $(\mathbb{R}^3 \times \mathbb{R})/\mathbb{Z}$, where the generator of the action is given by a rotation of angle $2\pi\beta\ell$ in $\mathbb{R}^3$ and a translation by distance $2\pi\ell$ in $\mathbb{R}$, for some real number β; such manifolds are referred to as AF$_{\beta\ell}$. When $\beta\ell$ is irrational the vector field T does not have closed orbits, which happens for generic members of the Kerr and Chen-Teo families of AF instantons.

[1] It should be noted that the terminology of an asymptotically flat manifold has taken on two inequivalent meanings within the context of the positive mass theorem, one arising from mathematical relativity and another from the study of gravitational instantons. In this article we will only use the latter notion.

Associated with each ALE or ALF manifold (M, g) is a well-defined notion of mass. In analogy with the classical AE setting, its expression is derived from the flux of the linearized scalar curvature operator. In particular, the mass is defined by

$$\mathrm{mass}_b(M, g) := \lim_{r\to\infty} \frac{1}{4\pi} \int_{\mathcal{S}_r} (\mathrm{div}_b\, \mathbf{e} - d\mathrm{Tr}_b\, \mathbf{e})(\partial_r) d\mathcal{V}, \qquad \mathbf{e} = \varphi^* g - b, \tag{1.4}$$

where $d\mathcal{V}$ is the volume form of the r-level set $\mathcal{S}_r$ in the model geometry. The fact that the mass is a geometric invariant in the ALE setting follows from the proof of Bartnik [8] and Chruściel [17] in the AE regime. Geometric invariance with respect to the choice of ALF structure is established in [32, Proposition 2.8] in the ALF-A_k case. An earlier statement of this type for AF_0 asymptotics was made by Minerbe [43, pg. 952]; in [43, Proposition 6] an analogous result was proven for the so called 'Gauss-Bonnet mass' (tailored to Ricci curvature) in the ALF-A_k context. Note that in the AF_0 setting, the mass (1.4) agrees with, up to normalization, those of [9, 14, 21, 41] as well as [43, Theorem 2].

The mass may be viewed as a geometric invariant that connects scalar curvature with the global geometry and topology of the manifold. The Euclidean Reissner-Nordström metrics on $\mathbb{R}^2 \times S^2$ are AF_0, complete, and scalar flat, but they can have negative mass for certain choices of parameters. Similarly, the charged Taub-Bolt Einstein-Maxwell instanton is an example of a complete ALF-A_0 manifold with zero scalar curvature, that admits negative mass for certain ranges of parameters. These examples, as well as others, are discussed in detail in Section 8. Thus, as in the ALE setting, the positive mass theorem dramatically fails for ALF manifolds. On the other hand, like Dahl's result [19] in the ALE spin case, Minerbe [43, Theorem 2] considered AF_0 manifolds with nonnegative scalar curvature and a matching condition for the spin structure at infinity, to establish a positive mass theorem. In a different direction, Liu-Shi-Zhu [41, Theorem 1.2] (see also Chen-Liu-Shi-Zhu [14, Theorem 1.8]) show that for AF_0 manifolds of dimensions less than 8 the positive mass theorem holds if the circle at infinity is homotopically nontrivial. Moreover, Khuri-Wang [32, Theorem 1.2] obtain the same conclusion in the AF_0 setting under the hypothesis that a codimension-two coordinate sphere in the asymptotic end is trivial within the homology of M. Related results were additionally found by Dai [21], Dai-Sun [22], and Barzegar-Chruściel-Hörzinger [9]. In the case of ALF-A_k manifolds, Minerbe [43, Theorem 1] proved positivity of mass under the assumption of nonnegative Ricci curvature; note that Minerbe's definition of mass in this result does not coincide with the standard one. While in the same setting, Khuri-Wang [32, Theorem 1.7] establish a positive mass lower bound in terms of bundle degree with the hypotheses of nonnegative scalar curvature and an almost free $U(1)$ action. Furthermore, an ALF-D_2 positive mass theorem was established by Kim-Ozuch [34, Theorem 0.5].

Despite this progress, positivity properties and more generally geometric inequalities involving the mass remain poorly understood in the ALE and ALF regimes. The purpose of the present paper is to investigate to what extent a positive mass style theorem can be achieved in the presence of such robust counterexamples, as described above. More precisely, we seek a result that 'applies to' such counterexamples, rather than avoiding them with exclusionary hypotheses, in the hope of understanding the reason for the presence of negative mass. In this regard, we shall restrict attention to toric ALE and ALF manifolds, and will show that a full and satisfactory answer to this question may be given in this setting.

Definition 1.5. A *toric ALE or toric ALF manifold* is an ALE or ALF manifold that admits an effective isometric T^2 action, which is compatible with the ALE or ALF structure in the following sense. Let η_1, η_2 denote Killing field generators of this action.

(i) In the ALE case, it asymptotes to an effective isometric T^2 action on the model flat cone geometry of the end, which preserves radial cross-sections. Moreover, there exist generators ξ_1, ξ_2 of the limiting action such that

$$|\mathring{\nabla}^l \left((\varphi^{-1})_*\eta_i - \xi_i\right)|_b = O(r^{-\kappa-l}), \tag{1.5}$$

for $l = 0, 1,\ i = 1, 2$.

(ii) In the ALF case, it asymptotes to an effective isometric T^2 action on the model geometry of the end, which preserves the radial cross-sections and leaves τ as well as γ invariant. Moreover, there exist generators ξ_1, ξ_2 of the limiting action such that $T = a^i\xi_i$ where $a^i \in \mathbb{R}$ and

$$|\mathring{\nabla}^l \left((\varphi^{-1})_*\eta_i - \xi_i\right)|_b = O(r^{\frac{1}{2}-\kappa-l}), \qquad |\mathring{\nabla}^l \left((\varphi^{-1})_*(a^i\eta_i) - T\right)|_b = O(r^{-\frac{1}{2}-\kappa-l}), \tag{1.6}$$

for $l = 0, 1,\ i = 1, 2$.

Remark 1.6. The compatibility condition (i) for ALE manifolds is actually a consequence of the first statement in Definition 1.5 concerning the existence of a torus action [29]. Furthermore as explained in Section 2 below, in both settings the toric condition implies that the radial cross-section $\mathcal{S}$ must be a lens space, or alternatively in the ALF case, $S^1 \times S^2$.

Toric symmetries play an important role in the study of gravitational instantons [10, 39], and have been used for AE mass lower bounds [2–5]. In a dramatic recent development Li-Sun [40] have discovered toric AF instantons on infinitely many new diffeomorphism types of 4-manifolds, which are not locally Hermitian. It is then natural to consider positive mass theorems in the ALE and ALF settings which assume this symmetry, and to expect that toric instantons are featured in an essential way. To this point, we recall that among AE manifolds with zero scalar curvature the critical points of the mass are Ricci flat [16, Proposition 7.1], and note that the same conclusion holds in the ALE and ALF contexts. Therefore, the search for a global minimizer of the mass among such manifolds (or more generally those with nonnegative scalar curvature) leads inevitably to gravitational instantons. In fact, loosely speaking, an indication as to why the positive mass theorem fails in the ALE and ALF settings and on the other hand is valid in the AE setting, is that there are many Ricci flat manifolds with the same ALE/ALF structure (possibly with conical singularities) but there is only one Ricci flat AE manifold, namely Euclidean space.

The variational approach suggests that a meaningful replacement for the positive mass theorem in the current setting should take the form of a rigid mass comparison result between ALE and ALF manifolds of nonnegative scalar curvature, and certain gravitational instantons. In order to realize such a concept, it is necessary to have a mechanism to produce a large variety of gravitational instantons with prescribed structure. Toric symmetry provides a robust solution to this problem by reducing the Ricci flat equations to an axisymmetric harmonic map into the hyperbolic plane with prescribed singularities. More precisely, it follows from [30, 35, 36, 40] that given a rod data set — an embellished orbit space boundary that characterizes the toric action — and the ALE/ALF asymptotic structure, there exists a unique corresponding harmonic map giving rise to a toric gravitational instanton admitting these properties; the method is explained in Section 2.3. We refer to this instanton as an *equilibrium geometry*, and note that it typically will have conical singularities on the axes, see Section 2. For this reason, our main result below is naturally stated with conical angle defects. As described in more detail in the next section, a rod data set consists of a collection of intervals called axis rods whose union is the boundary of the orbit space M/T^2, and an associated collection of 'weights' that detail the degeneration of the torus action; the intervals are parameterized by a coordinate z.

Theorem 1.7. *Let (M, g) be a simply connected toric ALE or toric ALF manifold with nonnegative scalar curvature, possibly having conical singularities and corners*[2] *along finite axes. Consider the corresponding toric gravitational instanton (M, g_o) sharing the same asymptotic ALE or ALF structure, and the same rod data set consisting of intervals $\{\Gamma_n\}_{n=1}^{N+1}$. Then*

$$\mathrm{mass}_b(M, g) - \mathrm{mass}_b(M, g_o) \geq 2\pi \sum_{n=1}^{N+1} \int_{\Gamma_n} (\boldsymbol{\vartheta}^n - \boldsymbol{\vartheta}_o^n) dz, \tag{1.7}$$

where $\boldsymbol{\vartheta}^n$ and $\boldsymbol{\vartheta}_o^n$ are the logarithmic angle defects on axis rod Γ_i with respect to g and g_o, respectively. Furthermore, equality holds if and only if (M, g) is isometric to the Ricci flat equilibrium geometry (M, g_o).

The precise meaning of conical singularities in the context of toric ALE/ALF manifolds, as well as their logarithmic angle defects, will be given in the next section. When conical singularities are not present, the inequality (1.7) simply states that the mass of the given ALE/ALF manifold is bounded below by the mass of its corresponding instanton equilibrium geometry. Even in the general case when conical singularities are present, we are motivated to define a new mass

$$\mathbf{mass}_b(M, g) := \mathrm{mass}_b(M, g) - 2\pi \sum_{n=1}^{N+1} \int_{\Gamma_n} \boldsymbol{\vartheta}^n dz, \tag{1.8}$$

and again the result yields the simple statement that $\mathbf{mass}_b(M, g) \geq \mathbf{mass}_b(M, g_o)$. In this way, up to multiplication by 2π, the total logarithmic angle defect along Γ_n may be viewed as the mass of the rod, or rather the mass of the corresponding totally geodesic 2-sphere lying within M. As is shown in Section 8, these contributions are responsible for and explain the negative mass present in the Reissner-Nordström manifolds, since the equilibrium geometry associated with these examples must possess conical defects. It should also be noted that Theorem 1.7 will continue to hold if conical singularities are present on the semi-infinite axes, as long as the difference $\boldsymbol{\vartheta}^n - \boldsymbol{\vartheta}_o^n$ is integrable on such rods. Moreover, the difference of masses on the left-hand side of (1.7) is equivalent to $\mathrm{mass}_{g_o}(M, g)$, the mass of g with respect to the Ricci flat background g_o, and thus the main inequality may then be viewed as giving a sharp lower bound for this interpretation of mass. In fact, the concept of using Ricci flat backgrounds to define the mass in ALF contexts has previously been put forward by Kim-Ozuch [34, Introduction]. Finally we mention that in the ALE case, it follows from Bando-Kasue-Nakajima [7] that the instanton metric will fall-off at order 4, which implies that $\mathrm{mass}_b(M, g_o) = 0$ and hence simplifies the inequality (1.7).

In order to illustrate a delicate aspect of this theorem, and the necessity of including some hypothesis beyond fixing the asymptotic structure, we may compare with the stability result of Dahl-Kröncke [20, Theorem 1.8]. Recall that an open Einstein manifold is called *linearly unstable* if the linearized Ricci operator, restricted to compactly supported transverse-traceless tensors, has a negative bottom of the spectrum. When this occurs for a gravitational instanton, Dahl-Kröncke show that there exist compactly supported perturbations of the instanton metric which have nonnegative scalar curvature that is not identically zero. This perturbation may then be conformally changed back to zero scalar curvature, as in Schoen-Yau's [47] approach to the AE positive mass theorem, while preserving the asymptotics and resulting in a smaller mass than the original instanton. Thus, if applied to a Schwarzschild AF instanton $(\mathbb{R}^2 \times S^2, g_{sc})$ which is linearly unstable [20, Example 1.13], while taking care to preserve toric symmetry in the deformation, we obtain a new toric AF manifold $(\mathbb{R}^2 \times S^2, \tilde{g}_{sc})$ of nonnegative scalar curvature with $\mathrm{mass}_b(\mathbb{R}^2 \times S^2, \tilde{g}_{sc}) < \mathrm{mass}_b(\mathbb{R}^2 \times S^2, g_{sc})$. This appears to

[2] See Definition 2.1.

violate inequality (1.7), if we naively use the original Schwarzschild instanton as the equilibrium geometry. However, in order to apply Theorem 1.7, the equilibrium geometry must be chosen to have the same rod data set as the perturbation. Ultimately, the deformation disturbs the rod lengths, so that a new Schwarzschild instanton with different mass must be used for the comparison.

This paper is organized as follows. In Section 2 we derive consequences of the toric action, and analyze the asymptotic model geometries. In Section 3, scalar curvature identities are exploited to obtain a relation between a reduced harmonic energy and certain flux integrals. Convexity properties of the reduced energy are studied in Section 4, and then used to produce a gap lower bound. Section 5 is dedicated to showing that the flux integrals yield the desired difference of masses, while the main theorem is proved in Section 6. Asymptotics at infinity, the axes, and corners are derived in Section 7. Finally, several examples are detailed in Section 8 and an appendix is included to record miscellaneous calculations and formulae.

2. Background and Setup

Let (M,g) be a simply connected toric ALE or toric ALF manifold. It follows from [46] and the proof of [28, Proposition 3] that the orbit space M/T^2 is diffeomorphic to a half-plane $\{(\rho,z)\mid \rho\geq 0, z\in\mathbb{R}\}$, with certain 'weights' embellishing the boundary. More precisely, the z-axis Γ is decomposed into an exhaustive sequence of closed intervals referred to as *axis rods* and denoted by

$$\Gamma_1=(-\infty,z_1],\quad \Gamma_2=[z_1,z_2],\ \dots\ \Gamma_n=[z_{N-1},z_N],\quad \Gamma_{N+1}=[z_N,\infty), \tag{2.1}$$

where $z_n<z_{n+1}$, such that the interior of each Γ_n corresponds to points in M with 1-dimensional isotropy subgroup. The intersection point of two adjacent axis rods is called a *corner* and represents a point in M with 2-dimensional isotropy subgroup, whereas all interior points of the half-plane correspond to principal orbits. Note that in the gravitational instanton literature, corners are referred to as *nuts* and the collection of orbits over an axis rod is referred to as a *bolt*; these latter objects are totally geodesic 2-spheres in M that represent the generators of its second homology.

Let η_1, η_2 denote Killing field generators of the T^2-action and consider the Gram matrix G with components $G_{ij}=g(\eta_i,\eta_j)$. Associated with each Γ_n is an element $\mathbf{v}_n=(v_n^1,v_n^2)\in\mathbb{Z}^2$ called a *rod structure*, whose components are relatively prime, and which generates the kernel of G on this rod or equivalently the Killing field $v_n^1\eta_1+v_n^2\eta_2$ vanishes on Γ_n. The collection $\mathcal{R}=\{(\mathbf{v}_n,\Gamma_n)\}_{n=1}^{N+1}$ of axis rods and their rod structures is referred to as the *rod data set*, and completely encodes the topology of M, see [31] for further discussion. For instance, by examining the torus fibration over a semi-circle in the orbit space that connects the two semi-infinite rods Γ_1 and Γ_{N+1}, we find that the only allowable cross-sectional topologies [28, Proposition 2] for the asymptotic end of M are $S^1\times S^2$ and the lens spaces $L(p,q)$; these two cases occur for the pair of rod structures $\{((1,0),\Gamma_1),((1,0),\Gamma_{N+1})\}$ and $\{((1,0),\Gamma_1),((q,p),\Gamma_{N+1})\}$, respectively. Moreover, it will be assumed that any rod data set satisfies the *admissibility condition* at corner points:

$$\det\begin{pmatrix} v_n^1 & v_n^2\\ v_{n+1}^1 & v_{n+1}^2\end{pmatrix}=\pm 1. \tag{2.2}$$

This condition preserves the manifold structure in a neighborhood of corner points. Without it, such neighborhoods admit an orbifold structure, and although our results and proofs should continue to hold in that situation we will not pursue this direction here.

Conical singularities and corners arise naturally in this setting. In particular, let $\pi: M\to M/T^2$ be the quotient map, then conical singularities can occur along the axes or rather the 2-sphere bolts

$\pi^{-1}(\Gamma_n)$. Consider a model cone metric on $D^2 \times S^1 \times (0,1)$ given by

$$g_{\mathrm{cone}} = ds^2 + c^2 s^2 (d\psi^1 + a d\psi^2)^2 + g_{\mathrm{cyl}}(y), \tag{2.3}$$

where $c = c(t) > 0$ and $a = a(t)$ are smooth functions, (s, ψ^1) are polar coordinates on the open unit disk D^2, and $g_{\mathrm{cyl}}(y)$ is a metric on the cylinder $(0,1) \times S^1$ parameterized by coordinates $y = (t, \psi^2)$. Here ψ^1, ψ^2 are 2π-periodic and their coordinate vector fields generate a T^2 action by isometries. Note that $s = 0$ corresponds to the axis, and that $c = e^{-\boldsymbol{\vartheta}}$ where $\boldsymbol{\vartheta}$ is the *logarithmic angle defect* of each cone along the axis. Consider also a flat model corner metric on a 4-dimensional ball B_1 in polar-Hopf coordinates $(r, \theta, \psi^1, \psi^2)$ given by

$$g_{\mathrm{corner}} = dr^2 + r^2 \left(d\theta^2 + c_1^2 \sin^2\theta (d\psi^1)^2 + c_2^2 \cos^2\theta (d\psi^2)^2 \right), \tag{2.4}$$

where $c_1, c_2 \in (0, \infty)$ are constants, and $r \in [0,1)$, $\theta \in [0, \pi/2]$, while ψ^1, ψ^2 are again 2π-periodic. The values c_1 and c_2 yield angle defects of the neighboring axes in the usual way.

Definition 2.1. We say that a toric ALE or toric ALF manifold (M, g) possesses *conical singularities and corners* if the metric is globally L^∞ and smooth away from the axes $\pi^{-1}(\Gamma)$, with the following two types of model asymptotics.

(i) At each point of any bolt $\pi^{-1}(\operatorname{int}\Gamma_n)$ there exists a neighborhood of the form $D^2 \times S^1 \times (0,1)$ with coordinates (s, ψ^1, y) and an associated cone metric g_{cone} such that after pullback

$$|\mathring{\nabla}^l (g - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} = O(s^{1+\zeta-l}), \qquad l = 0, 1, \tag{2.5}$$

for some $\zeta > 0$ where $\mathring{\nabla}$ denotes covariant differentiation with respect to g_{cone}.

(ii) At each corner (nut) point $\pi^{-1}(\Gamma_n \cap \Gamma_{n+1})$ there exists a 4-ball neighborhood with coordinates $(r, \theta, \psi^1, \psi^2)$ and an associated corner metric g_{corner} such that after pullback

$$|\mathring{\nabla}^l (g - g_{\mathrm{corner}})|_{g_{\mathrm{corner}}} = O(r^{2-l}), \qquad l = 0, 1, \tag{2.6}$$

where $\mathring{\nabla}$ denotes covariant differentiation with respect to g_{corner}.

In both cases, after a pushforward, the coordinate vector fields ∂_{ψ^i}, $i = 1, 2$ correspond to generators of the T^2 action on M.

2.1. **Asymptotic model geometries.** The toric hypothesis places strong restrictions on the asymptotic model geometries of ALE and ALF manifolds. In fact we will show that the model metric b can be assumed to take an explicit form, which falls into one of three types that we now describe.

2.1.1. *Asymptotically locally Euclidean (ALE).* Consider the asymptotic end $(\mathbb{R}^4 \setminus \overline{B}_1)/\mathbb{Z}_p = (1, \infty) \times L(p, q)$ where $p, q \in \mathbb{Z}$ are relatively prime with $p \geq 1$, equipped with the flat metric

$$b_{\mathrm{ALE}} = dr^2 + r^2 \left[d\theta^2 + p^{-2} \sin^2\theta (d\psi^1)^2 + \cos^2\theta \left(d\psi^2 + p^{-1} q d\psi^1 \right)^2 \right], \tag{2.7}$$

in which $r > 1$ and (θ, ψ^1, ψ^2) are Hopf coordinates on the lens space with $\theta \in [0, \pi/2]$ and ψ^1, ψ^2 are 2π-periodic. Here the toric symmetry is generated by the Killing fields ∂_{ψ^1}, ∂_{ψ^2}, and the semi-infinite rod structures are given by $\mathbf{v}_1 = (0, 1)$ and $\mathbf{v}_{N+1} = (p, -q)$.

2.1.2. *Asymptotically locally flat (ALF-A_{k-1}).* Let $k \in \mathbb{Z}_+$ and consider the asymptotic end $\mathbb{R}_+ \times L(k, 1)$ equipped with the metric

$$b_{\mathrm{ALF}} = \ell^2 \left(d\psi^2 + k \cos^2(\theta/2)\, d\psi^1 \right)^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta (d\psi^1)^2 \right), \tag{2.8}$$

where $r > 0$, $\theta \in [0, \pi]$, and ψ^1, ψ^2 are 2π-periodic. The induced metric on radial level sets exhibits the lens space as an S^1-bundle over the 2-sphere with Euler number $e = -k$, and thus this model geometry is associated with type ALF-A_{k-1}. By multiplying the second and third terms of (2.8) by $h(r) = 1 + \frac{k\ell}{2r}$ and multiplying the first term by $h(r)^{-1}$ as in (2.12) one obtains a new metric $\tilde{b}_{ALF}$ which is Ricci flat, and coincides in the case of $k = 1$ with the Taub-NUT gravitational instanton on $\mathbb{R}^4$ after adding the origin point $r = 0$. The toric symmetry is again generated by the Killing fields ∂_{ψ^1}, ∂_{ψ^2}, and the semi-infinite rod structures are given by $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_{N+1} = (-1, k)$. Note that the metric (2.8) may be placed into the context of Definition 1.3 (iv) by setting

$$T = \ell^{-1}\partial_{\psi^2}, \qquad \tau = \ell\left(d\psi^2 + k\cos^2(\theta/2)\, d\psi^1\right), \qquad \gamma = d\theta^2 + \sin^2\theta (d\psi^1)^2. \tag{2.9}$$

We may also consider the case when $k = p/q$ for relatively prime positive integers p, q. In this situation the radial level sets have topology $L(p, q)$, however the metric b_{ALF} admits conical singularities on the axis rod Γ_{N+1} when $q \neq 1$.

2.1.3. *Asymptotically flat ($AF_{\beta\ell}$).* Consider the asymptotic end $\mathbb{R}_+ \times S^1 \times S^2$ equipped with the flat metric

$$b_{\mathrm{AF}} = \ell^2 (d\psi^2)^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta (d\psi^1 + \beta\ell d\psi^2)^2\right), \tag{2.10}$$

where $\ell > 0$, $\beta \geq 0$ are constants and $r > 0$, $\theta \in [0, \pi]$, and ψ^1, ψ^2 are 2π-periodic. The radial level sets are topologically $S^1 \times S^2$ as realized by the rod structures $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_{N+1} = (1, 0)$ on the semi-infinite rods, associated with the toric symmetry generated by the Killing fields ∂_{ψ^1}, ∂_{ψ^2}. This model geometry is of the type $\mathrm{AF}_{\beta\ell}$. Note that the metric (2.10) may be placed into the context of Definition 1.3 (iv) by setting

$$T = \ell^{-1}\left(\partial_{\psi^2} - \beta\ell\partial_{\psi^1}\right), \qquad \tau = \ell d\psi^2, \qquad \gamma = d\theta^2 + \sin^2\theta (d\psi^1 + \beta\ell d\psi^2)^2. \tag{2.11}$$

Observe that T has closed orbits if and only if $\beta\ell$ is rational.

Remark 2.2. We have chosen to distinguish the AF and ALF cases since many explicit examples fall into one of these two classes described above. However, they can be treated together as members of a larger toric family of Ricci flat geometries, possibly with conical singularities. Namely, using the previous notation consider the following metric

$$\tilde{b} = h^{-1}\ell^2\left(d\psi^2 + k\cos^2(\theta/2)\left(d\psi^1 + \beta\ell d\psi^2\right)\right)^2 + h\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta (d\psi^1 + \beta\ell d\psi^2)^2\right)\right), \tag{2.12}$$

where $h(r)$ is the radial function from Section 2.1.2. If $k = 0$ this reduces to the setting of b_{AF}, so assume that $k > 0$. In order to have the structure of a manifold in the vicinity of rod Γ_{N+1}, we require that $k^{-1}(1 + k\beta\ell) = \frac{q}{p}$ for some relatively prime integers $p \neq 0$ and q. In this case, the semi-infinite rod structures are $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_{N+1} = (-q, p)$, and the topology of the asymptotic end on which this metric is defined is given by $\mathbb{R}_+ \times L(p, q)$. Moreover, conical singularities occur on Γ_{N+1} unless $k = p$ and $1 + k\beta\ell = q$.

Remark 2.3. When h is replaced by 1 in (2.12), we will refer to the resulting metric as $\tilde{b}_{ALF}$. Although the results of this paper continue to hold for the more general model metric $\tilde{b}_{ALF}$, for simplicity of exposition we will mostly restrict attention to the less embellished version b_{ALF} from (2.8).

We will now show that in the toric setting, the model geometries must take one of the above three forms up to negligible error.

Proposition 2.4. *Let (M, g) be a toric ALE or ALF manifold.*

(i) *In the ALE case, the asymptotic model geometry is of the form* $(\mathbb{R}_+ \times L(p,q), b)$ *for some relatively prime integers* $p \geq 1$, q, *and there exists an explicit model metric* (2.7) *such that*

$$|\mathring{\nabla}^l (b - b_{ALE})|_{b_{ALE}} = O(r^{-2-l}), \qquad l = 0,1,2, \tag{2.13}$$

where $\mathring{\nabla}$ *denotes covariant differentiation with respect to* b_{ALE}.

(ii) *In the ALF/non-AF case, the asymptotic model geometry is of the form* $(\mathbb{R}_+ \times L(p,q), b)$ *for some relatively prime positive integers* p, q, *the bounded Killing field* T *has closed orbits, and there exists an explicit model metric from Remark 2.3 with* $k = p$ *and* $1 + p\beta\ell = q$ *such that*

$$|\mathring{\nabla}^l (b - \tilde{b}_{ALF})|_{\tilde{b}_{ALF}} = O(r^{-1-l}), \qquad l = 0,1,2, \tag{2.14}$$

where $\mathring{\nabla}$ *denotes covariant differentiation with respect to* $\tilde{b}_{ALF}$.

(iii) *In the AF case, the asymptotic model geometry is of the form* $(\mathbb{R}_+ \times S^1 \times S^2, b)$, *the bounded Killing field* T *may not have closed orbits, and there exists an explicit model metric* (2.10) *such that*

$$|\mathring{\nabla}^l (b - b_{AF})|_{b_{AF}} = O(r^{-1-l}), \qquad l = 0,1,2, \tag{2.15}$$

where $\mathring{\nabla}$ *denotes covariant differentiation with respect to* b_{AF}.

Proof. We will treat the ALF and AF cases here, and simply note that the ALE case may be proved similarly. According to Definition 1.5 there is an effective isometric T^2 action on the model geometry of the end, which induces a toric symmetry on its radial cross-sections $\mathcal{S}$. By [46, Section 2] the orbit space $\mathcal{S}/T^2$ is a closed interval which we may parameterize by $\theta \in [0,\pi]$. Let ψ^1, ψ^2 be 2π-periodic coordinates parameterizing the torus fibers, then by expressing the cross-section metrics in Riemannian submersion format we find that

$$b|_{\mathcal{S}} = r^2\gamma + \ell^2\tau^2 = A(r)d\theta^2 + B_{ij}(r,\theta)d\psi^i d\psi^j, \tag{2.16}$$

for some coefficient functions A and B_{ij} which are independent of the torus coordinates since ∂_{ψ^1}, ∂_{ψ^2} generate the toric symmetry. Note also that there are no cross-terms between θ and ψ^i since the horizontal distribution is integrable.

The bounded Killing field is a linear combination of the action generators, and thus by rescaling ℓ if necessary, we may assume without loss of generality that $T = a\partial_{\psi^1} + \partial_{\psi^2}$ for some $a \in \mathbb{R}$. Let

$$\omega^1 = d\theta, \qquad \omega^2 = d\psi^1 - a d\psi^2, \qquad \omega^3 = d\psi^2, \tag{2.17}$$

be a co-frame tailored to T in the sense that $\omega^1(T) = \omega^2(T) = 0$ and $\omega^3(T) = 1$. Then we may write

$$\tau = \sum_{\mathrm{i}=1}^{3} \tau_{\mathrm{i}}(\theta)\omega^{\mathrm{i}}, \qquad \gamma = \sum_{\mathrm{i,j}=1}^{3} \gamma_{\mathrm{ij}}(\theta)\omega^{\mathrm{i}}\omega^{\mathrm{j}}. \tag{2.18}$$

Since $\tau(T) = 1$ and $\gamma(T,\cdot) = 0$ we find that $\tau_3 = 1$ and $\gamma_{13} = \gamma_{23} = \gamma_{33} = 0$. Next, by inserting the resulting expressions into (2.16) we find that γ_{11} is constant, and using that there are no cross-terms between θ and ψ^i on the right-hand side it follows that $\tau_1 = \gamma_{12} = 0$. Moreover, since γ locally defines a metric of Gauss curvature $+1$ on the space of leaves of the foliation tangent to T, we conclude that $\gamma_{11} = 1$ and $\gamma_{22} = \sin^2\theta$. By setting $\beta = -\ell^{-1}a$ the model metric may now be expressed as

$$b = dr^2 + r^2\left(d\theta^2 + \sin^2\theta(d\psi^1 + \beta\ell d\psi^2)^2\right) + \ell^2\left(d\psi^2 + \tau_2(d\psi^1 + \beta\ell d\psi^2)\right)^2. \tag{2.19}$$

By initially choosing coordinates on the torus appropriately, it may be assumed that the rod structures for the asymptotic end are given by $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_{N+1} = (-q,p)$, for two coprime nonnegative integers p, q. Note that the first rod structure implies $\tau_2(\pi) = 0$. If $\mathbf{v}_{N+1} = (-1,0)$

then $\mathcal{S} = S^1 \times S^2$, and also $\tau_2(0) = 0$ so that the portion of the metric involving τ_2 may be treated as error to produce $|\mathring{\nabla}^l(b - b_{AF})|_{b_{AF}} = O(r^{-1-l})$ where

$$b_{AF} = dr^2 + r^2\left(d\theta^2 + \sin^2\theta(d\psi^1 + \beta\ell d\psi^2)^2\right) + \ell^2(d\psi^2)^2; \tag{2.20}$$

this yields case (iii) of the proposition. If $\mathbf{v}_{N+1} \neq (-1, 0)$, then as in Remark 2.2 regularity demands that $p + \tau_2(0)(p\beta\ell - q) = 0$ and $1 + p\beta\ell = q$. It follows that $\tau_2(0) = p$. Hence, treating terms involving $\tau_2 - p\cos^2(\theta/2)$ as error produces $|\mathring{\nabla}^l(b - b_{ALF})|_{b_{ALF}} = O(r^{-1-l})$ where

$$b_{ALF} = dr^2 + r^2\left(d\theta^2 + \sin^2\theta(d\psi^1 + \beta\ell d\psi^2)^2\right) + \ell^2\left(d\psi^2 + p\cos^2(\theta/2)(d\psi^1 + \beta\ell d\psi^2)\right)^2; \tag{2.21}$$

this yields case (ii). □

2.2. **Brill coordinates.** Let (M, g) be a simply connected toric ALE or toric ALF manifold, possibly having conical singularities and corners along the axes, and let ϕ^i be a pair of independent 2π-periodic angles adapted to the Killing field generators of the toric action so that $\eta_i = \partial_{\phi^i}$. By [29] there exists a set of global (Brill) coordinates $(\rho, z, \phi^1, \phi^2)$ for M in which the metric may be expressed in submersion format

$$g = e^{2\alpha}\left(d\rho^2 + dz^2\right) + G_{ij}(d\phi^i + A^i_a dx^a)(d\phi^j + A^j_a dx^a), \tag{2.22}$$

where $(x^1, x^2) = (\rho, z)$. The first portion of (2.22) involving $e^{2\alpha}$ represents the metric on the orbit space M/T^2 which is parameterized by the half-plane $\{(\rho, z) \mid \rho \geq 0, z \in \mathbb{R}\}$, while $G = (G_{ij})$ yields the torus fiber metric, and the coefficients A^i_a measure the obstruction to local integrability of the distribution orthogonal to the fibers. All coefficients α, G_{ij}, and A^i_a are functions of (ρ, z) alone and satisfy the asymptotics as layed out in Section 7 for neighborhoods of corner points, axis points, and at infinity. This coordinate system gives rise to an advantageous expression for the scalar curvature, which makes contact with a certain harmonic map energy that is fundamental for mass comparison result. Lastly, we note that the logarithmic angle defect at points above the interior of an axis rod Γ_n with rod structure (v^1_n, v^2_n) may be expressed as

$$e^{\boldsymbol{\vartheta}} = \lim_{\rho \to 0} \frac{2\pi \cdot \text{Radius}}{\text{Circumference}} = \lim_{\rho \to 0} \sqrt{\frac{\rho^2 e^{2\alpha}}{G_{ij} v^i_n v^j_n}}. \tag{2.23}$$

The existence of this limit is a consequence of the asymptotics detailed in Section 7.

2.3. **Toric harmonic maps.** In the setting of simply connected toric ALE/ALF manifolds, the Ricci flat equations reduce to solving for an axisymmetric harmonic map ([35, Section 3], [40, Section 2], [42]) into the hyperbolic plane, $\Phi_o : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$. In fact, one may prescribe the desired rod structure and asymptotic type of the toric gravitational instanton, by solving for a harmonic map that is asymptotic to a given *model map* that realizes this structure. By *asymptotic*, we mean that the hyperbolic distance between the two maps stays bounded globally and converges to zero near infinity. The resulting instanton will most likely have concial singularities for generic rod data sets. The method to establish existence of such a harmonic map, asymptotic to a prescribed model map in this context, is based on an approach initiated by Weinstein [50] for 4-dimensional axisymmetric stationary vacuum black holes, and was later developed to incorporate rod structures by Khuri-Weinstein-Yamada [33]. The adaptation to the (Riemannian) setting of toric gravitational instantons was given by Kunduri-Lucietti [35, Theorem 1.2] for the AF case, and this was recently expanded and generalized by Li-Sun [40, Theorem 4.24]. The harmonic maps produced from this process are unique among those asymptotic to the given model. Although the two aforementioned results were carried out in the AF regime, the same technique holds for toric ALE/ALF instantons

[36, Theorem 1.1]. The only requirement is the ability to construct an appropriate model map, and this may be achieved in the same manner as [35, Theorem 1.2] except that at infinity we choose the model map to coincide with the harmonic map arising from the three Ricci flat model geometries of Section 2.1; note that in the ALF case this refers to the metric $\tilde{b}_{ALF}$ which is a modification of (2.8) using the function h. Thus, we obtain the following existence result.

Theorem 2.5. *Given a simply connected toric ALE or toric ALF manifold (M,g) with rod data set $\mathcal{R}$, possibly having conical singularities and corners along finite axes, there exists a corresponding toric gravitational instanton (M,g_o) potentially with conical singularities and corners sharing the same asymptotic ALE or ALF structure, and the same rod data set $\mathcal{R}$.*

Remark 2.6. Regularity of the harmonic maps associated with the instantons (M,g_o) was investigated in the vicinity of axis rods and corners in [40, Section 4.2]. The resulting asymptotics for the harmonic maps and Brill coordinate coefficients in these regions, as well as at infinity, are detailed in Section 7.

3. The Reduced Energy Functional

In the presence of a toric symmetry the scalar curvature naturally contains a harmonic map energy density arising from the torus fiber portion of the metric. This density, however, exhibits blow-up behavior at the axes and thus must be 'renormalized' in order to serve a useful role in the context of mass comparison. We begin with the basic expression for scalar curvature in this setting. This may be obtained from O'Neill's formulas for Riemannian submersions [45] although here we give a direct derivation.

Lemma 3.1. *Let (M,g) be a toric Riemannian 4-manifold with metric expressed in Brill coordinates (2.22). On $\mathbb{R}^3\setminus\Gamma$, define a function Z and a 2×2 symmetric matrix Φ with $\det\Phi=1$ by setting $G=\rho e^{Z}\Phi$, then the scalar curvature satisfies*

$$\begin{aligned} e^{2\alpha}R=&-2\Delta\alpha+2\nabla\alpha\cdot\nabla\log\rho-\frac{1}{4}\mathrm{Tr}\left(\Phi^{-1}\nabla\Phi\right)^2-\frac{1}{4}e^{-2\alpha}\delta_3^{ac}\delta_3^{bd}G_{ij}F^i_{ab}F^j_{cd}\\ &-2\Delta Z-\frac{3}{2}|\nabla Z|^2-\nabla Z\cdot\nabla\log\rho+\frac{1}{2}|\nabla\log\rho|^2 \end{aligned}\tag{3.1}$$

where $F^i_{ab}=\partial_aA^i_b-\partial_bA^i_a$ and $\delta_3=d\rho^2+dz^2+\rho^2d\varphi^2$ is the flat metric on $\mathbb{R}^3$ written in cylindrical coordinates, with ∇, Δ, and $\cdot$ denoting its covariant derivative, Laplacian, and inner product respetively.

Remark 3.2. In the Ricci flat setting Proposition A.2 shows that $Z=0$ and $A^i_a=0$ for all $i,a=1,2$.

Proof. According to appendix equation (A.6), a computation shows that the scalar curvature takes the form

$$\begin{aligned} e^{2\alpha}R=&-2\Delta\alpha+2\nabla\alpha\cdot\nabla\log\rho-\left(\Delta_2\log\det G+\frac{1}{4}\mathrm{Tr}\left(G^{-1}\nabla G\right)^2+\frac{1}{4}|\nabla\log\det G|^2\right)\\ &-\frac{1}{4}e^{-2\alpha}\delta_3^{ac}\delta_3^{bd}G_{ij}F^i_{ab}F^j_{cd} \end{aligned}\tag{3.2}$$

where Δ_2 is the Laplacian with respect to the 2-dimensional flat metric $\delta_2 = d\rho^2 + dz^2$. Define the symmetric unimodular matrix $\Phi := (\det G)^{-\frac{1}{2}} G$, let I_2 denote the identity matrix, and observe that

$$\begin{aligned}(G^{-1}\nabla G)^2 &= \left(\frac{1}{2}\nabla \log\det G\, I_2 + \Phi^{-1}\nabla\Phi\right)^2 \\ &= \frac{1}{4}|\nabla\log\det G|^2 I_2 + \nabla\log\det G\,\left(\Phi^{-1}\nabla\Phi\right) + \left(\Phi^{-1}\nabla\Phi\right)^2.\end{aligned} \tag{3.3}$$

Then combining this with $\mathrm{Tr}\left(\Phi^{-1}\nabla\Phi\right) = \nabla(\log\det\Phi) = 0$ produces

$$\mathrm{Tr}\left(G^{-1}\nabla G\right)^2 = \frac{1}{2}|\nabla\log\det G|^2 + \mathrm{Tr}\left(\Phi^{-1}\nabla\Phi\right)^2. \tag{3.4}$$

Next, define the function

$$Z := \frac{1}{2}\log\det G - \log\rho \tag{3.5}$$

and note that

$$|\nabla\log\det G|^2 = 4|\nabla Z|^2 + 8\nabla Z\cdot\nabla\log\rho + 4|\nabla\log\rho|^2, \tag{3.6}$$

as well as

$$\begin{aligned}\Delta_2\log\det G &= \Delta\log\det G - \nabla\log\det G\cdot\nabla\log\rho \\ &= \Delta\left(\log\det G - 2\log\rho\right) - 2\nabla\left(Z + \log\rho\right)\cdot\nabla\log\rho \\ &= 2\Delta Z - 2\nabla Z\cdot\nabla\log\rho - 2|\nabla\log\rho|^2.\end{aligned} \tag{3.7}$$

Inserting these expressions into (3.2) yields the desired result. □

We now seek an interpretation of the term involving Φ within the scalar curvature formula. To this end, define functions

$$\begin{aligned} &V := \frac{1}{2}\log\left(\frac{\Phi_{11}}{\Phi_{22}}\right), \qquad W := \sinh^{-1}\left(\Phi_{12}\right), \qquad \text{in the ALE, ALF, and AF}_0 \text{ cases,} \\ &V := \frac{1}{2}\log\left(\frac{\Phi_{11}}{\Phi_{22} - 2\beta\ell\Phi_{12} + \beta^2\ell^2\Phi_{11}}\right), \quad W := \sinh^{-1}\left(\Phi_{12} - \beta\ell\Phi_{11}\right), \qquad \text{in the AF}_{\beta\ell} \text{ case,}\end{aligned} \tag{3.8}$$

and observe that using $\det\Phi = 1$ the inverse relations are given by

$$\Phi_{11} = e^V\cosh W, \qquad \Phi_{12} = \sinh W, \qquad \Phi_{22} = e^{-V}\cosh W, \tag{3.9}$$

and

$$\Phi_{11} = e^V\cosh W, \quad \Phi_{12} = \sinh W + \beta\ell e^V\cosh W, \quad \Phi_{22} = (e^{-V} + \beta^2\ell^2 e^V)\cosh W + 2\beta\ell\sinh W, \tag{3.10}$$

respectively. It follows that

$$\frac{1}{2}\mathrm{Tr}\left(\Phi^{-1}\nabla\Phi\right)^2 = \cosh^2 W|\nabla V|^2 + |\nabla W|^2, \tag{3.11}$$

showing that this expression is a harmonic map energy density for the map $\Phi : \mathbb{R}^3\setminus\Gamma \to \mathbb{H}^2$, where the hyperbolic plane is parameterized by Fermi coordinates.

The asymptotics of Φ at the axes will typically produce an infinite energy, which motivates the following renormalization. Let (M, g) be a simply connected toric ALE or toric ALF manifold, and let (M, g_o) be the corresponding toric gravitational instanton having the same rod data set given by Theorem 2.5. Consider the maps $\Psi = (V, W)$ and $\Psi_o = (V_o, W_o)$ associated with g and g_o, respectively. Using

$$\begin{aligned} |\nabla V|^2 &= |\nabla(V - V_o)|^2 + 2\nabla(V - V_o)\cdot\nabla V_o + |\nabla V_o|^2, \\ |\nabla W|^2 &= |\nabla(W - W_o)|^2 + 2\nabla(W - W_o)\cdot\nabla W_o + |\nabla W_o|^2, \end{aligned} \tag{3.12}$$

and the harmonic property of $\log\rho$ with respect to δ_3, produces the difference of the g and g_o-scalar curvatures

$$\begin{aligned} e^{2\alpha}R =& \mathrm{div}_{\delta_3} X - \frac{3}{2}|\nabla Z|^2 - \frac{1}{4}e^{-2\alpha}\delta_3^{ac}\delta_3^{bd}G_{ij}F^i_{ab}F^j_{cd} + (V - V_o)\Delta V_o + (W - W_o)\Delta W_o \\ & - \frac{1}{2}\left(\sinh^2 W|\nabla V|^2 - \sinh^2 W_o|\nabla V_o|^2 + |\nabla(V - V_0)|^2 + |\nabla(W - W_o)|^2\right), \end{aligned} \tag{3.13}$$

where

$$X := -2\nabla(\alpha - \alpha_o + Z) + (2\alpha - 2\alpha_o - Z)\nabla\log\rho - (V - V_o)\nabla V_o - (W - W_o)\nabla W_o. \tag{3.14}$$

Let $\boldsymbol{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3)$ be a collection of small positive parameters. We may decompose the open ball $B_{2/\varsigma_3} \subset \mathbb{R}^3$ centered at the origin that includes part of the semi-infinite rods, into three types of pairwise disjoint regions $B_{2/\varsigma_3} = \Omega_{\boldsymbol{\varsigma}} \cup \mathcal{A}_{\boldsymbol{\varsigma}} \cup \left(\cup_{n=1}^N \overline{B_{\varsigma_2}(z_n)}\right)$ where

$$\Omega_{\boldsymbol{\varsigma}} = \{r < 2/\varsigma_3,\ \rho > \varsigma_1,\ \varsigma_2 < r_n \text{ for } n = 1, \ldots, N\}, \quad \mathcal{A}_{\boldsymbol{\varsigma}} = B_{2/\varsigma_3} \setminus \left(\Omega_{\boldsymbol{\varsigma}} \cup \left(\cup_{n=1}^N \overline{B_{\varsigma_2}(z_n)}\right)\right). \tag{3.15}$$

Here r is adapted to the 4-dimensional model geometries and is given by (7.1), (7.16) in the ALE, ALF cases respectively, whereas r_n is a radial coordinate defined by (7.56) which is centered at the nth corner point on the z-axis located at height z_n, and $B_{\varsigma_2}(z_n)$ is the open ball centered at this point of radius ς_2. Integrating (3.13) over $\Omega_{\boldsymbol{\varsigma}}$ and using the divergence theorem yields

$$\mathcal{B}^{\boldsymbol{\varsigma}}_{\text{axis}} + \mathcal{B}^{\boldsymbol{\varsigma}}_{\text{corner}} + \mathcal{B}^{\boldsymbol{\varsigma}}_{\infty} = \mathcal{I}_{\Omega_{\boldsymbol{\varsigma}}}(\Psi) + \int_{\Omega_{\boldsymbol{\varsigma}}}\left(e^{2\alpha}R + \frac{3}{2}|\nabla Z|^2 + \frac{1}{4}e^{-2\alpha}\delta_3^{ac}\delta_3^{bd}G_{ij}F^i_{ab}F^j_{cd}\right)dx \tag{3.16}$$

where

$$\begin{aligned} \mathcal{I}_{\Omega_{\boldsymbol{\varsigma}}}(\Psi) =& \frac{1}{2}\int_{\Omega_{\boldsymbol{\varsigma}}}\left(\sinh^2 W|\nabla V|^2 - \sinh^2 W_o|\nabla V_o|^2 + |\nabla(V - V_0)|^2 + |\nabla(W - W_o)|^2\right)dx \\ & - \int_{\Omega_{\boldsymbol{\varsigma}}}\left((V - V_o)\Delta V_o + (W - W_o)\Delta W_o\right)dx, \end{aligned} \tag{3.17}$$

and

$$\mathcal{B}^{\boldsymbol{\varsigma}}_{\text{axis}} = \int_{\partial\mathcal{A}_{\boldsymbol{\varsigma}}\cap\partial\Omega_{\boldsymbol{\varsigma}}} X(\nu)dA, \quad \mathcal{B}^{\boldsymbol{\varsigma}}_{\text{corner}} = \sum_{n=1}^N \int_{\partial\mathcal{B}_{\varsigma_2}(z_n)\cap\partial\Omega_{\boldsymbol{\varsigma}}} X(\nu)dA, \quad \mathcal{B}^{\boldsymbol{\varsigma}}_{\infty} = \int_{\partial B_{2/\varsigma_3}\cap\partial\Omega_{\boldsymbol{\varsigma}}} X(\nu)dA, \tag{3.18}$$

with ν denoting the unit outer normal. We define the *reduced energy* to be the following limit

$$\mathcal{I}(\Psi) := \lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\mathcal{I}_{\Omega_{\boldsymbol{\varsigma}}}(\Psi). \tag{3.19}$$

The corresponding limits for the boundary integrals (3.18) exist and are finite by Lemmas 5.1, 5.2, and 5.6, and the same will now be shown for the reduced energy.

Proposition 3.3. *Let $\Psi = (V, W)$ and $\Psi_o = (V_o, W_o)$ be maps as described above. Then the reduced energy functional $\mathcal{I}(\Psi)$ is well-defined and finite.*

Proof. Since the limits of boundary integrals in (3.16) exist and are finite, it suffices to show the same for the bulk integral expression in this equation. To see this, observe that the asymptotics of Section 7 imply

$$|\nabla Z|^2 = O(r^{-6-2\kappa}) \quad \text{for ALE}, \qquad |\nabla Z|^2 = O(r^{-3-2\kappa}) \quad \text{for ALF/ AF}_{\beta\ell}, \tag{3.20}$$

$$|\nabla Z|^2 = O(1) \quad \text{in } \mathcal{A}_{\varsigma}, \qquad |\nabla Z|^2 = O(1) \quad \text{in } B_{\varsigma_2}(z_n), \tag{3.21}$$

showing that the second integrand is integrable. Moreover, since $R \in L^1(M)$ and $dx_g = e^{2\alpha}e^{Z}dx$, it follows that $e^{2\alpha}R \in L^1(\mathbb{R}^3)$. Furthermore, the asymptotics of Section 7 also produce

$$\delta_3^{ac}\delta_3^{bd}G_{ij}F^i_{ab}F^j_{cd} = O(r^{-8-2\kappa}) \quad \text{for ALE}, \quad \delta_3^{ac}\delta_3^{bd}G_{ij}F^i_{ab}F^j_{cd} = O(r^{-3-2\kappa}) \quad \text{for ALF/AF}_{\beta\ell}, \tag{3.22}$$

$$\delta_3^{ac}\delta_3^{bd}G_{ij}F^i_{ab}F^j_{cd} = O(\rho^{2\zeta}) \quad \text{in } \mathcal{A}_{\varsigma}, \qquad \delta_3^{ac}\delta_3^{bd}G_{ij}F^i_{ab}F^j_{cd} = O(r_n^{-2}) \quad \text{in } B_{\varsigma_2}(z_n). \tag{3.23}$$

Thus, the last integrand is integrable. □

4. Convexity of Reduced Energy Functional

Consider the hyperbolic plane $\mathbb{H}^2$ with metric expressed in Fermi coordinate (V, W) as follows

$$g_{\mathbb{H}^2} = \cosh^2 W dV^2 + dW^2. \tag{4.1}$$

Let $\Omega \subset \mathbb{R}^3 \setminus \Gamma$ be a domain, then the harmonic energy of a map $\Psi = (V, W) : \Omega \to \mathbb{H}^2$ is given by

$$E_\Omega(\Psi) = \frac{1}{4}\int_\Omega \operatorname{Tr}\left(\Phi^{-1}\nabla\Phi\right)^2 dx = \frac{1}{2}\int_\Omega \left(\cosh^2 W|\nabla V|^2 + |\nabla W|^2\right) dx. \tag{4.2}$$

Critical points $\Psi_o = (V_o, W_0)$ of this energy satisfy the harmonic map equations

$$\operatorname{div}\left(\cosh^2 W_o \nabla V_o\right) = 0, \qquad \Delta W_o - \sinh W_o \cosh W_o |\nabla V_o|^2 = 0\,. \tag{4.3}$$

Moreover, the relation between the harmonic energy E and reduced energy $\mathcal{I}$ takes the form

$$\mathcal{I}_\Omega(\Psi) = E_\Omega(\Psi) - E_\Omega(\Psi_o) - \int_{\partial\Omega} \left(\nu(V_o)\,(V - V_o) + \nu(W_o)\,(W - W_o)\right) dA, \tag{4.4}$$

where ν is the unit outward normal on $\partial\Omega$. The main goal of this section is to establish a gap lower bound for the reduced energy.

Theorem 4.1. *Suppose that the map $\Psi = (V, W)$ and related harmonic map $\Psi_o = (V_o, W_o)$ are smooth on $\mathbb{R}^3 \setminus \Gamma$, and satisfy the asymptotics of Section 7. Then there exists a constant $C > 0$ such that*

$$\mathcal{I}(\Psi) \geq C\left(\int_{\mathbb{R}^3} \operatorname{dist}^6_{\mathbb{H}^2}(\Psi, \Psi_o)\, dx\right)^{1/3}. \tag{4.5}$$

Since the target space is negatively curved, the harmonic energy is convex on bounded regions that exclude the axis and corner singularities. The singular behavior of the maps Ψ and Ψ_o near the axis, however, prevents this convexity from extending directly to the reduced energy on the whole of $\mathbb{R}^3$. It is therefore necessary to analyze the boundary behavior of the reduced energy separately near the axis, at the corners, and at infinity. Proving that the boundary terms make no contribution to the convexity argument requires a cut-and-paste construction in which Ψ is replaced by Ψ_o near the axis.

Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be a collection of small positive parameters such that $\varsigma_i < \varepsilon_i < 1$, where $\boldsymbol{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3)$ is given in Section 3. Consider the following cut-off function

$$\varphi_{\varepsilon_1} = \begin{cases} 0 & \text{if } \rho \leq \varepsilon_1 \\ \frac{\log(\rho/\varepsilon_1)}{\log(\sqrt{\varepsilon_1}/\varepsilon_1)} & \text{if } \varepsilon_1 < \rho < \sqrt{\varepsilon_1} \\ 1 & \text{if } \rho \geq \sqrt{\varepsilon_1} \end{cases}. \tag{4.6}$$

Recall the region $\mathcal{A}_{\boldsymbol{\varepsilon}}$ and define an additional annular cylindrical region about the z-axis by

$$\begin{aligned} \tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}} &= \{\varepsilon_1 \leq \rho \leq \sqrt{\varepsilon_1}\} \cap \{r < 2/\varepsilon_3,\ \varepsilon_2 < r_n \text{ for } n = 1, \dots, N\}, \\ \mathcal{A}_{\boldsymbol{\varepsilon}} &= \{\rho \leq \varepsilon_1\} \cap \{r < 2/\varepsilon_3,\ \varepsilon_2 < r_n \text{ for } n = 1, \dots, N\}. \end{aligned} \tag{4.7}$$

Furthermore, set $\Psi_{\boldsymbol{\varepsilon}} = (V_{\varepsilon_1}, W_{\varepsilon_1}) := (V_o + \varphi_{\varepsilon_1}(V - V_o), W_o + \varphi_{\varepsilon_1}(W - W_o))$ so that

$$(V_{\varepsilon_1}, W_{\varepsilon_1}) = \begin{cases} (V, W) & \text{in } \tilde{B}_{2/\varepsilon_3} \setminus \left(\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}} \cup \mathcal{A}_{\boldsymbol{\varepsilon}}\right) \\ (V_o + \varphi_{\varepsilon_1}(V - V_o), W_o + \varphi_{\varepsilon_1}(W - W_o)) & \text{in } \tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}} \\ (V_o, W_o) & \text{in } \mathcal{A}_{\boldsymbol{\varepsilon}} \end{cases}, \tag{4.8}$$

where $\tilde{B}_{2/\varepsilon_3} = B_{2/\varepsilon_3} \setminus \cup_{n=1}^N B_{\varepsilon_2}(z_n)$.

Lemma 4.2. *For fixed $\varepsilon_2, \varepsilon_3 > 0$ it holds that*

$$\lim_{\varepsilon_1 \to 0} \mathcal{I}_{\tilde{B}_{2/\varepsilon_3}}(\Psi_{\boldsymbol{\varepsilon}}) = \mathcal{I}_{\tilde{B}_{2/\varepsilon_3}}(\Psi). \tag{4.9}$$

Proof. Write

$$\mathcal{I}_{\tilde{B}_{2/\varepsilon_3}}(\Psi_{\boldsymbol{\varepsilon}}) = \mathcal{I}_{\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}}}(\Psi_{\boldsymbol{\varepsilon}}) + \mathcal{I}_{\mathcal{A}_{\boldsymbol{\varepsilon}}}(\Psi_{\boldsymbol{\varepsilon}}) + \mathcal{I}_{\tilde{B}_{2/\varepsilon_3} \setminus (\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}} \cup \mathcal{A}_{\boldsymbol{\varepsilon}})}(\Psi_{\boldsymbol{\varepsilon}}), \tag{4.10}$$

and observe that

$$\mathcal{I}_{\tilde{B}_{2/\varepsilon_3} \setminus (\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}} \cup \mathcal{A}_{\boldsymbol{\varepsilon}})}(\Psi_{\boldsymbol{\varepsilon}}) = \mathcal{I}_{\tilde{B}_{2/\varepsilon_3} \setminus (\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}} \cup \mathcal{A}_{\boldsymbol{\varepsilon}})}(\Psi). \tag{4.11}$$

Moreover $\Psi_{\boldsymbol{\varepsilon}} = \Psi_o$ on $\mathcal{A}_{\boldsymbol{\varepsilon}}$, so the reduced energy vanishes when restricted to $\mathcal{A}_{\boldsymbol{\varepsilon}}$. On the remaining region we have

$$\begin{aligned} 2\mathcal{I}_{\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}}}(\Psi_{\boldsymbol{\varepsilon}}) = &\underbrace{\int_{\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}}} |\nabla(V_{\varepsilon_1} - V_o)|^2}_{I_1} + \underbrace{\int_{\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}}} |\nabla(W_{\varepsilon_1} - W_o)|^2}_{I_2} \\ &+ \underbrace{\int_{\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}}} \left(\sinh^2 W_{\varepsilon_1} |\nabla V_{\varepsilon_1}|^2 - \sinh^2 W_o |\nabla V_o|^2\right) - 2(V_{\varepsilon_1} - V_o)\Delta V_o - 2(W_{\varepsilon_1} - W_o)\Delta W_o}_{I_3}. \end{aligned} \tag{4.12}$$

To estimate these expressions, it is helpful to decompose the region into connected components $\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}} = \cup_{n=1}^{N+1} \tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}}^n$, where the annular cylinder $\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}}^n$ is associated with the rod Γ_n. In what follows, we will analyze each integral according to the asymptotics of Section 7.2 for the three different types of rod structure on Γ_n, namely: (I) $\mathbf{v}_n = (1, 0)$, (II) $\mathbf{v}_n = (0, 1)$, and (III) $\mathbf{v}_n = (v_n^1, v_n^2) \in (\mathbb{Z} \setminus \{0\})^2$. In all cases it holds that

$$|I_1| \leq \int_{\tilde{\mathcal{A}}_{\boldsymbol{\varepsilon}}^n} \left(\underbrace{|\nabla(V - V_o)|^2}_{O(1)} + \underbrace{|V - V_o|^2}_{O(1)} \underbrace{|\nabla \varphi_{\varepsilon_1}|^2}_{O((\rho \log \varepsilon_1)^{-2})} \right) \rho d\rho dz \to 0, \tag{4.13}$$

and similarly

$$|I_2| \leq \int_{\tilde{\mathcal{A}}^n_{\varepsilon}} \left(\underbrace{|\nabla(W-W_o)|^2}_{O(1)} + \underbrace{|W-W_o|^2}_{O(1)} \underbrace{|\nabla\varphi_{\varepsilon_1}|^2}_{O((\rho\log\varepsilon_1)^{-2})} \right) \rho d\rho dz \to 0. \tag{4.14}$$

Next consider I_3. For this integral, we will further decompose case II into two subcases: II_0 in which $\beta = 0$, and II_β in which $\beta \neq 0$. From Section 7.2 it follows that

$$\sinh^2 W_{\varepsilon_1} - \sinh^2 W_o = \sinh(W_{\varepsilon_1}+W_o)\sinh(W_{\varepsilon_1}-W_o) = \begin{cases} O(\rho^2) & \text{case I and } \mathrm{II}_0 \\ O(\rho^{-2}) & \text{case } \mathrm{II}_\beta \text{ and III} \end{cases}, \tag{4.15}$$

and with the help of the harmonic map equations (4.3) we find

$$\begin{aligned} \Delta W_o = \sinh W_o \cosh W_o |\nabla V_o|^2 &= \begin{cases} O(\rho^{-1}) & \text{case I and } \mathrm{II}_0 \\ O(\rho^{2\zeta-2}) & \text{case } \mathrm{II}_\beta \\ O(1) & \text{case III} \end{cases}, \\ \Delta V_o = -2\tanh W_o \nabla W_o \cdot \nabla V_o &= \begin{cases} O(1) & \text{case I and } \mathrm{II}_0 \\ O(\rho^{\zeta-1}) & \text{case } \mathrm{II}_\beta \\ O(1) & \text{case III} \end{cases}, \end{aligned} \tag{4.16}$$

where $\zeta > 0$. Therefore, in all cases

$$\begin{aligned} |I_3| \leq & \int_{\tilde{\mathcal{A}}^n_{\varepsilon}} \underbrace{\left|\sinh^2 W_{\varepsilon_1} - \sinh^2 W_o\right| |\nabla V_o|^2}_{O(\rho^{2\zeta-2})} \rho d\rho dz \\ & + \int_{\tilde{\mathcal{A}}^n_{\varepsilon}} \underbrace{\sinh^2 W_{\varepsilon_1} |\nabla(V_{\varepsilon_1}-V_o)||\nabla(V_{\varepsilon_1}+V_o)|}_{O(\rho^{2\zeta-2})} \rho d\rho dz \\ & + 2\int_{\tilde{\mathcal{A}}^n_{\varepsilon}} \left(\underbrace{|V-V_o||\Delta V_o|}_{O(1)} + \underbrace{|W-W_o||\Delta W_o|}_{O(\rho^{2\zeta-2})} \right) \rho d\rho dz. \end{aligned} \tag{4.17}$$

Hence, $\mathcal{I}_{\tilde{\mathcal{A}}_{\varepsilon}}(\Psi_{\varepsilon}) \to 0$ and the desired is obtained. □

Proof of Theorem 4.1. Let $\Psi, \Psi_o : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$ be as in the statement of this theorem, and consider the cut-and-paste map Ψ_{ε} for $\varepsilon > 0$. Let $\Psi^t_{\varepsilon} = (V^t_{\varepsilon}, W^t_{\varepsilon})$ be the geodesic deformation from Ψ_o to Ψ_{ε} in $\mathbb{H}^2$. In particular, for each $x \in \mathbb{R}^3 \setminus \Gamma$ we have that $\Psi^{\bullet}_{\varepsilon}(x) : [0,1] \to \mathbb{H}^2$ is the geodesic with $\Psi^1_{\varepsilon}(x) = \Psi_{\varepsilon}(x)$ and $\Psi^0_{\varepsilon}(x) = \Psi_o(x)$. Let $\varepsilon > \varsigma > 0$ and observe that the second variation of energy [48, (2.4)] yields

$$\frac{d^2}{dt^2} E_{\Omega_\varsigma}(\Psi^t_{\varepsilon}) \geq 2\int_{\Omega_\varsigma} |\nabla \operatorname{dist}_{\mathbb{H}^2}(\Psi_{\varepsilon}, \Psi_0)|^2 dx. \tag{4.18}$$

Combining this with (4.4) produces

$$\begin{aligned} \frac{d^2}{dt^2}\mathcal{I}_{\Omega_\varsigma}(\Psi^t_{\varepsilon}) &= \frac{d^2}{dt^2}E_{\Omega_\varsigma}(\Psi^t_{\varepsilon}) - \frac{d^2}{dt^2}\int_{\partial\Omega_\varsigma} \left(\nu(V_o)\left(V^t_{\varepsilon} - V_o\right) + \nu(W_o)\left(W^t_{\varepsilon} - W_o\right)\right) dA \\ &\geq 2\int_{\Omega_\varsigma} |\nabla \operatorname{dist}_{\mathbb{H}^2}(\Psi_{\varepsilon}, \Psi_0)|^2 dx - \int_{\partial\Omega_\varsigma} \left(\nu(V_o)\ddot{V}^t_{\varepsilon} + \nu(W_o)\ddot{W}^t_{\varepsilon}\right) dA, \end{aligned} \tag{4.19}$$

where the 'dot' derivatives are with respect to t. Now integrate from 0 to t to find

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_{\Omega_\varsigma}(\Psi^t_\varepsilon) - \frac{d}{dt}\mathcal{I}_{\Omega_\varsigma}(\Psi^t_\varepsilon)|_{t=0} \geq 2t\int_{\Omega_\varsigma} |\nabla \operatorname{dist}_{\mathbb{H}^2}(\Psi_\varepsilon, \Psi_0)|^2 dx \\ - \int_{\partial\Omega_\varsigma} \Big(\nu(V_o)(\dot{V}^t_\varepsilon - \dot{V}^0_\varepsilon) + \nu(W_o)(\dot{W}^t_\varepsilon - \dot{W}^0_\varepsilon)\Big)\, dA.\end{aligned} \tag{4.20}$$

On the other hand, from the first variation of (4.4) we have

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_{\Omega_\varsigma}(\Psi^t_\varepsilon) = &-\int_{\Omega_\varsigma} \dot{V}^t_\varepsilon \operatorname{div}\left(\cosh^2 W^t_\varepsilon \nabla V^t_\varepsilon\right)\, dx \\ &- \int_{\Omega_\varsigma} \dot{W}^t_\varepsilon \left(\Delta W^t_\varepsilon - \sinh W^t_\varepsilon \cosh W^t_\varepsilon |\nabla V^t_\varepsilon|^2\right)\, dx \\ &+ \int_{\partial\Omega_\varsigma} \Big(\dot{W}^t_\varepsilon \left(\nu(W^t_\varepsilon) - \nu(W_o)\right) + \dot{V}^t_\varepsilon \left(\nu(V^t_\varepsilon)\cosh^2 W^t_\varepsilon - \nu(V_o)\right)\Big)\, dA.\end{aligned} \tag{4.21}$$

Since Ψ^t_ε at $t=0$ is a harmonic map and satisfies (4.3), it follows that

$$\frac{d}{dt}\mathcal{I}_{\Omega_\varsigma}(\Psi^t_\varepsilon)|_{t=0} = \int_{\partial\Omega_\varsigma} \dot{V}^0_\varepsilon \nu(V_o)\sinh^2 W_o dA. \tag{4.22}$$

Putting this together with (4.20) gives rise to

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_{\Omega_\varsigma}(\Psi^t_\varepsilon) \geq 2t\int_{\Omega_\varsigma} |\nabla \operatorname{dist}_{\mathbb{H}^2}(\Psi_\varepsilon, \Psi_0)|^2 dx \\ - \int_{\partial\Omega_\varsigma} \Big(\nu(V_o)(\dot{V}^t_\varepsilon - \dot{V}^0_\varepsilon - \dot{V}^0_\varepsilon \sinh^2 W_o) + \nu(W_o)(\dot{W}^t_\varepsilon - \dot{W}^0_\varepsilon)\Big)\, dA.\end{aligned} \tag{4.23}$$

Now integrate again from 0 to 1 and use the fact that $\mathcal{I}_{\Omega_\varsigma}(\Psi^0_\varepsilon) = 0$ to obtain

$$\begin{aligned}\mathcal{I}_{\Omega_\varsigma}(\Psi_\varepsilon) \geq \int_{\Omega_\varsigma} |\nabla \operatorname{dist}_{\mathbb{H}^2}(\Psi_\varepsilon, \Psi_0)|^2 dx \\ \underbrace{- \int_{\partial\Omega_\varsigma} \Big(\nu(V_o)(V_\varepsilon - V_o - \dot{V}^0_\varepsilon \cosh^2 W_o) + \nu(W_o)(W_\varepsilon - W_o - \dot{W}^0_\varepsilon)\Big)\, dA}_{:= I_{\partial\Omega_\varsigma}}\,.\end{aligned} \tag{4.24}$$

We will use the geodesic equations and distance function in the hyperbolic plane to estimate each term within $I_{\partial\Omega_\varsigma}$. Applying a Taylor expansion about $\Psi^0_\varepsilon = \Psi_o$ yields

$$\Psi^t_\varepsilon = \Psi_o + \dot{\Psi}^0_\varepsilon + \frac{1}{2}\ddot{\Psi}^{t'}_\varepsilon, \qquad \text{for some } t' \in (0,1)\,, \tag{4.25}$$

while the geodesic equations may be written explicitly as

$$\ddot{V}^t_\varepsilon + 2\tanh W^t_\varepsilon\, \dot{V}^t_\varepsilon \dot{W}^t_\varepsilon = 0, \qquad \ddot{W}^t_\varepsilon - \sinh W^t_\varepsilon \cosh W^t_\varepsilon\, \left(\dot{V}^t_\varepsilon\right)^2 = 0\,. \tag{4.26}$$

Moreover, since for each $x \in \mathbb{R}^3 \setminus \Gamma$ the geodesic $\Psi^t_\varepsilon(x)$ has constant velocity and there is a unique minimizing geodesic between any two points of $\mathbb{H}^2$, it holds that

$$\mathbf{d}^2 := \operatorname{dist}^2_{\mathbb{H}^2}(\Psi_\varepsilon, \Psi_o) = \left(\dot{W}^t_\varepsilon\right)^2 + \cosh^2 W^t_\varepsilon \left(\dot{V}^t_\varepsilon\right)^2 \qquad \text{for all } t \in [0,1]\,. \tag{4.27}$$

Then the boundary integral of (4.24) may be rewritten with the Taylor expansion (4.25) and geodesic equations (4.26) by

$$I_{\partial\Omega_\varsigma} = \int_{\partial\Omega_\varsigma}\left(\nu(V_o)\left(\dot{V}^0_{\varepsilon}\sinh^2 W_o + \tanh W^{t'}_{\varepsilon}\dot{V}^{t'}_{\varepsilon}\dot{W}^{t'}_{\varepsilon}\right) - \frac{1}{2}\nu(W_o)\sinh W^{t'}_{\varepsilon}\cosh W^{t'}_{\varepsilon}\left(\dot{V}^{t'}_{\varepsilon}\right)^2\right) dA, \tag{4.28}$$

and therefore

$$|I_{\partial\Omega_\varsigma}| \leq \int_{\partial\Omega_\varsigma}\left(|\nu(V_o)|\left(|\sinh W_o|\mathbf{d} + \mathbf{d}^2\right) + \frac{1}{2}|\nu(W_0)|\mathbf{d}^2\right) dA. \tag{4.29}$$

Recall the distance function in the hyperbolic plane

$$\cosh\mathbf{d} = \cosh(W_{\varepsilon} - W_o)\cosh(V_{\varepsilon} - V_o) + \sinh W_{\varepsilon}\sinh W_o\left(\cosh(V_{\varepsilon} - V_o) - 1\right). \tag{4.30}$$

Using the asymptotics of Section 7.1 for the asymptotic end, we then obtain decay rates for the distance function

$$\mathbf{d} = \begin{cases} O(r^{-1-\kappa}) & \text{ALE} \\ O(r^{-\frac{1}{2}-\kappa}) & \text{ALF} \\ O(r^{-\frac{1}{2}-\kappa}) & \text{AF}_{\beta\ell} \end{cases}. \tag{4.31}$$

Similarly, using Section 7.3 near each corner $z_n \in \Gamma$ we find

$$\mathbf{d} = O(|\log\sin 2\theta|), \tag{4.32}$$

as $r_n \to 0$. Next, observe that $\partial\Omega_{\boldsymbol{\varsigma}}$ may be decomposed into disjoint portions contained in the three regions: $\partial\mathcal{A}_{\boldsymbol{\varsigma}}$, $\cup_{n=1}^N \partial B_{\varsigma_2}(z_n)$, and $\partial B_{2/\varsigma_3}$. Since $\varsigma_1 < \varepsilon_1$ we have that $\Psi_{\varepsilon} = \Psi_o$ on $\partial\mathcal{A}_{\boldsymbol{\varsigma}} \cap \partial\Omega_{\boldsymbol{\varsigma}}$, and thus this portion of the integral (4.28) vanishes. Moreover, applying (4.29) and the distance function estimates, along with the asymptotics of Sections 7.1 and 7.3, we find that over the remaining two regions the integral tends to zero as $\varsigma_2, \varsigma_3 \to 0$. Therefore, with the aid of the Sobolev inequality it follows that

$$\mathcal{I}_{\Omega_\varsigma}(\Psi_{\varepsilon}) \geq C\left(\int_{\Omega_\varsigma}\mathrm{dist}^6_{\mathbb{H}^2}(\Psi_{\varepsilon}, \Psi_o)dx\right)^{1/3} + o(1), \tag{4.33}$$

for some constant $C > 0$ independent of $\boldsymbol{\varsigma}$ and $\boldsymbol{\varepsilon}$. It should be noted that error $o(1)$ is independent of ε_1 as well as ς_1 and converges to zero when $\varsigma_2, \varsigma_3 \to 0$.

We will now take a series of limits to arrive at the desired conclusion. First note that the integrand on the right-hand side of (4.33) vanishes on $\mathcal{A}_{\boldsymbol{\varsigma}}$, so that this integral may be taken over $\tilde{B}_{2/\varsigma_3}$. Moreover, on the left-hand side, by Proposition 3.3 we may take the limit as $\varsigma_1 \to 0$ to obtain the reduced energy over this same domain. Next take the liminf on both sides as $\varepsilon_1 \to 0$, and apply Lemma 4.2 as well as Fatou's lemma to find

$$\mathcal{I}_{\tilde{B}_{2/\varsigma_3}}(\Psi) \geq C\left(\int_{\tilde{B}_{2/\varsigma_3}}\mathrm{dist}^6_{\mathbb{H}^2}(\Psi, \Psi_o)dx\right)^{1/3} + o(1), \tag{4.34}$$

where we have used that the error terms are uniform in ε_1 and ς_1. Now take the liminf on both sides, first as $\varsigma_2 \to 0$ and then as $\varsigma_3 \to 0$, utilizing Proposition 3.3 on the left-hand side and Fatou's lemma again on the right-hand side to obtain

$$\mathcal{I}(\Psi) \geq C\left(\int_{\mathbb{R}^3}\mathrm{dist}^6_{\mathbb{H}^2}(\Psi, \Psi_o)dx\right)^{1/3}. \tag{4.35}$$

□

5. The Boundary Integrals

In this section, we investigate the boundary integrals appearing in (3.18). First, we use the asymptotics in Section 7.2 and express the boundary term on the axis (3.18) in terms of logarithmic angle defects on the axis.

Lemma 5.1. *Let $\boldsymbol{\vartheta}^n$ and $\boldsymbol{\vartheta}^n_o$, for $n = 0, \cdots, N+1$, be logarithmic angle defects on axis rod Γ_n for Riemannian manifolds (M, g) and (M_o, g_o), respectively. Then the boundary integral at the axis is*

$$\lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\mathcal{B}^{\boldsymbol{\varsigma}}_{axis} = -4\pi\sum_{n=1}^{N+1}\int_{\Gamma_n}(\boldsymbol{\vartheta}^n - \boldsymbol{\vartheta}^n_o)dz \tag{5.1}$$

Proof. First, observe that from (3.14),

$$X(\nu) = 2\partial_\rho(\alpha - \alpha_o + Z) - (2\alpha - 2\alpha_o - Z)\partial_\rho \log\rho + (V - V_o)\,\partial_\rho V_o + (W - W_o)\,\partial_\rho W_o \tag{5.2}$$

where we used $\nu = -\partial_\rho$. The functions in $X(\nu)$ has different asymptotics related to rod structure of the axis. In particular, we have three cases for rod structure: (I) $\mathbf{v}_n = (1, 0)$ on Γ_n, (II) $\mathbf{v}_n = (0, 1)$ on Γ_n, (III) $\mathbf{v}_n = (v^1_n, v^2_n) \in (\mathbb{Z}\backslash\{0\})^2$ on Γ_n. Using asymptotics in Section 7.2, we have

$$\text{Case I:}\qquad X(\nu)\rho = -2\alpha + 2\alpha_o + Z + (V - V_o) + O(\rho^\zeta) \tag{5.3}$$

$$\text{Case II}_0\text{:}\qquad X(\nu)\rho = -2\alpha + 2\alpha_o + Z - (V - V_o) + O(\rho^\zeta) \tag{5.4}$$

$$\text{Case II}_{\beta\ell}\text{:}\qquad X(\nu)\rho = -2\alpha + 2\alpha_o + Z + (W - W_o) + O(\rho^\zeta) \tag{5.5}$$

$$\text{Case III:}\qquad X(\nu)\rho = -2\alpha + 2\alpha_o + Z - (W - W_o) + O(\rho) \tag{5.6}$$

where the leading terms are $O(1)$ which we will denote by X_{axis}. Therefore, we have

$$\begin{aligned}\lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\mathcal{B}^{\boldsymbol{\varsigma}}_{\text{axis}} &= \lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\int_{\partial\mathcal{A}_{\boldsymbol{\varsigma}}\cap\partial\Omega_{\boldsymbol{\varsigma}}} X(\nu)\,dA \\ &= 2\pi\sum_{n=1}^{N+1}\int_{\Gamma_n} X_{\text{axis}}\,dz\end{aligned} \tag{5.7}$$

Consider the rod structure $\mathbf{v}_n = (v^1_n, v^2_n) \in \mathbb{Z}^2$ on rod Γ_n. Then the definition of regularity (2.23) on this rod yields

$$e^{\boldsymbol{\vartheta}^n} = \lim_{\rho\to 0}\frac{\int_0^\rho e^\alpha\,dz}{\sqrt{G_{ij}v^i_n v^j_n}} = \lim_{\rho\to 0}\frac{\int_0^\rho e^\alpha\,dz}{\rho e^{\frac{1}{2}Z - \frac{1}{2}\log\rho + \frac{1}{2}\log(\Phi_{ij}v^i_n v^j_n)}} = e^{\alpha - \frac{1}{2}Z + \frac{1}{2}\log\rho - \frac{1}{2}\log(\Phi_{ij}v^i_n v^j_n)}\Big|_{\Gamma_n} \tag{5.8}$$

This implies

$$-2\alpha + Z = \log\rho - \log(\Phi_{ij}v^i_n v^j_n) - 2\boldsymbol{\vartheta}^n, \quad \text{on} \quad \Gamma_n \tag{5.9}$$

For case I, we have

$$-2\alpha + Z = \log\rho - V - 2\boldsymbol{\vartheta}^n, \quad \text{on} \quad \Gamma_n \tag{5.10}$$

whereas for case II_0, we have

$$-2\alpha + Z = \log\rho + V - 2\boldsymbol{\vartheta}^n, \quad \text{on} \quad \Gamma_n \tag{5.11}$$

and case $\text{II}_{\beta\ell}$ yields

$$-2\alpha + Z = \log\rho - W - \log(2\beta\ell) - 2\boldsymbol{\vartheta}^n, \quad \text{on} \quad \Gamma_n \tag{5.12}$$

To get the regularity for case III, we need to investigate it more. Since $\mathbf{v}_n \in \mathrm{Ker}G$, we obtain

$$v_n^1\Phi_{11} + v_n^2\Phi_{12} = 0, \qquad v_n^1\Phi_{12} + v_n^2\Phi_{22} = 0, \quad \text{on} \quad \Gamma_n \tag{5.13}$$

Combining with $\mathbf{v}_n = (v_n^1, v_n^2) \in (\mathbb{Z}\backslash\{0\})^2$ on Γ_n, we obtain

$$v_n^1 = -e^{-V}\tanh W v_n^2, \quad \text{on} \quad \Gamma_n. \tag{5.14}$$

Together with (5.9), we have

$$\begin{aligned}\log\left(\Phi_{ij}v_n^i v_n^j\right) &= \log\left(\Phi_{11}e^{-2V}\tanh^2 W + \Phi_{22} - 2e^{-V}\tanh W\Phi_{12}\right) + 2\log v_n^2\\ &= \log\left(e^{-V}\frac{\sinh^2 W}{\cosh W} + e^{-V}\cosh W - 2e^{-V}\frac{\sinh^2 W}{\cosh W}\right) + 2\log v_n^2\\ &= -V - \log\cosh W + 2\log v_n^2\end{aligned} \tag{5.15}$$

Therefore, the regularity on the axis for case III is

$$-2\alpha + Z = \log\rho + V + \log\cosh W - 2\log v_n^2 - 2\boldsymbol{\vartheta}^n, \quad \text{on} \quad \Gamma_n \tag{5.16}$$

A similar statement holds for the harmonic map $\Psi_o = (V_o, W_o)$, logarithmic angle defect $\boldsymbol{\vartheta}_o^n$, and function α_o in these three cases. Combining the regularity at the axis and the definition of X_{axis} we have

$$X_{\text{axis}} = -2\left(\boldsymbol{\vartheta}^n - \boldsymbol{\vartheta}_o^n\right) \tag{5.17}$$

Together with (5.7) we have the result. $\square$

The second boundary term is related to corners.

Lemma 5.2. *The boundary integral about corners vanishes, that is*

$$\lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\mathcal{B}^{\boldsymbol{\varsigma}}_{corner} = 0. \tag{5.18}$$

Proof. Without loss of generality, we can assume that the rod structures on sides of z_n at the axis are $\mathbf{v}_n = (1,0)$ and $\mathbf{v}_{n+1} = (0,1)$. Applying the asymptotics in Section 7.3, we get

$$\begin{aligned} r_n X(\nu) = &\underbrace{2\partial_{r_n}(\alpha - \alpha_o + Z)}_{O(r_n)} - \underbrace{(2\alpha - 2\alpha_o - Z)}_{O(1)}\underbrace{\partial_{r_n}\log\rho}_{O(r_n^{-1})}\\ &+ \underbrace{(V - V_o)}_{O(1)}\underbrace{\partial_{r_n}V_o}_{O(r_n)} + \underbrace{(W - W_o)}_{O(1)}\underbrace{\partial_{r_n}W_o}_{O(r_n)}\end{aligned} \tag{5.19}$$

where $\nu = -\frac{1}{r_n}\partial_{r_n}$. Since the area form in $\mathbb{R}^3$ expressed in terms of the coordinates (7.56) is $\frac{1}{2}r_n^4\sin 2\theta d\theta d\phi$, as $\boldsymbol{\varsigma}\to 0$ or $r_n \to 0$, we have the result. $\square$

Before investigating the boundary integral at infinity, we determine the mass (1.4) for the metric (2.22). First, we investigate the integrand.

Lemma 5.3. *Let $b = b_{rr}dr^2 + b_{\theta\theta}d\theta^2 + G_{bij}d\phi^i d\phi^j$ be the asymptotic model metric with $b_{rr} = r^{-2}b_{\theta\theta}$. Then, we have*

$$\begin{aligned}\left(\mathrm{div}_b e - d\mathrm{Tr}_b e\right)(\partial_r) = &\frac{1}{2}b^{rr}b^{\theta\theta}\partial_r b_{\theta\theta}g_{rr} + \frac{1}{2}\left(b^{\theta\theta}\right)^2\partial_r b_{\theta\theta}g_{\theta\theta} - b^{\theta\theta}\partial_r g_{\theta\theta}\\ &+ b^{rr}\partial_r\log\rho\, g_{rr} - \mathrm{Tr}\left(G_b^{-1}\partial_r G\right) - \frac{1}{2}\mathrm{Tr}\left(G\partial_r G_b^{-1}\right)\\ &+ b^{\theta\theta}\partial_\theta g_{\theta r} + b^{\theta\theta}\partial_\theta\log\rho\, g_{\theta r}\end{aligned} \tag{5.20}$$

Proof. Consider the metric (2.22). Its components are given by

$$g_{ac} = e^{2\alpha}\delta_{2ac} + G_{ij}A^i_aA^i_c, \qquad g_{ia} = G_{ij}A^j_a, \qquad g_{ij} = G_{ij} \tag{5.21}$$

where $\delta_2 = d\rho^2 + dz^2$, for $a, c, d = \rho, z$ and $i, j, k, l = 1, 2$. The inverse metric is

$$g^{ab} = e^{-2\alpha}\delta_2^{ac}, \qquad g^{ij} = G^{ij} + e^{-2\alpha}\delta_2^{ac}A^i_aA^j_c, \qquad g^{ia} = -e^{-2\alpha}\delta_2^{ac}A^i_c \tag{5.22}$$

The 2-dimensional flat metric in polar coordinates is given explicitly by

$$\delta_2 = e^{2\alpha}\ell^2\left(dr^2 + r^2d\theta^2\right) \quad \text{ALF and ALF}_{\beta\ell}, \qquad \delta_2 = e^{2\alpha}\frac{r^2}{p^2}\left(dr^2 + r^2d\theta^2\right) \quad \text{ALE.} \tag{5.23}$$

First, we compute the Christoffel symbols associated to the metric b:

$$\Gamma^r_{rr} = \frac{1}{2}b^{rr}\partial_r b_{rr}, \qquad \Gamma^r_{\theta r} = \frac{1}{2}b^{rr}\partial_\theta b_{rr}, \qquad \Gamma^r_{\theta\theta} = -\frac{1}{2}b^{rr}\partial_r b_{\theta\theta}, \qquad \Gamma^r_{ij} = -\frac{1}{2}b^{rr}\partial_r G_{bij}, \tag{5.24}$$

$$\Gamma^\theta_{rr} = -\frac{1}{2}b^{\theta\theta}\partial_\theta b_{rr}, \qquad \Gamma^\theta_{\theta\theta} = \frac{1}{2}b^{\theta\theta}\partial_\theta b_{\theta\theta}, \qquad \Gamma^\theta_{\theta r} = \frac{1}{2}b^{\theta\theta}\partial_r b_{\theta\theta}, \qquad \Gamma^\theta_{ij} = -\frac{1}{2}b^{\theta\theta}\partial_\theta G_{bij}, \tag{5.25}$$

$$\Gamma^r_{ri} = \Gamma^r_{\theta i} = \Gamma^\theta_{\theta i} = \Gamma^\theta_{ri} = \Gamma^k_{rr} = \Gamma^k_{\theta\theta} = \Gamma^k_{r\theta} = \Gamma^k_{ij} = 0, \tag{5.26}$$

$$\Gamma^i_{rj} = \frac{1}{2}G^{ik}_b\partial_r G_{bkj}, \qquad \Gamma^i_{\theta j} = \frac{1}{2}G^{ik}_b\partial_\theta G_{bkj}. \tag{5.27}$$

The r-component of $\mathrm{div}_b e$ is

$$(\mathrm{div}_b e)(\partial_r) = \underbrace{b^{ac}\left(\partial_a g_{cr} - \Gamma^d_{ac}g_{dr} - \Gamma^d_{ar}g_{cd}\right)}_{I_1} + \underbrace{b^{ij}\left(-\Gamma^c_{ij}g_{cr} - \Gamma^c_{ir}g_{jc} - \Gamma^k_{ir}g_{jk}\right)}_{I_2} \tag{5.28}$$

where $b^{ij} = G^{ij}_b$. We investigate terms separately as follows.

$$I_1 = b^{rr}\left(\partial_r g_{rr} - b^{rr}\partial_r b_{rr}g_{rr}\right) + b^{\theta\theta}\left(\partial_\theta g_{\theta r} + \frac{1}{2}b^{rr}g_{rr}\partial_r b_{\theta\theta} - \frac{1}{2}b^{\theta\theta}\partial_r b_{\theta\theta}g_{\theta\theta}\right) \tag{5.29}$$

and

$$I_2 = \frac{1}{2}b^{rr}g_{rr}\mathrm{Tr}(G_b^{-1}\partial_r G_b) + \frac{1}{2}b^{\theta\theta}g_{\theta r}\mathrm{Tr}(G_b^{-1}\partial_\theta G_b) + \frac{1}{2}\mathrm{Tr}(G\partial_r G_b^{-1}) \tag{5.30}$$

Adding I_1 and I_2, we obtain

$$\begin{aligned}(\mathrm{div}_b e)(\partial_r) = {} & b^{rr}\partial_r g_{rr} + b^{\theta\theta}\partial_\theta g_{\theta r} - \frac{b^{rr}}{2}\left(b^{rr}\partial_r b_{rr} - b^{\theta\theta}\partial_r b_{\theta\theta} - \mathrm{Tr}\left(G_b^{-1}\partial_r G_b\right)\right)g_{rr} \\ & + \frac{1}{2}b^{\theta\theta}\mathrm{Tr}\left(G_b^{-1}\partial_\theta G_b\right)g_{\theta r} - \frac{1}{2}\left(b^{rr}\right)^2\partial_r b_{rr}g_{rr} - \frac{1}{2}\left(b^{\theta\theta}\right)^2\partial_r b_{\theta\theta}g_{\theta\theta} \\ & + \frac{1}{2}\mathrm{Tr}\left(G\partial_r G_b^{-1}\right)\end{aligned} \tag{5.31}$$

On the other hand, the r-component of $d\mathrm{Tr}_b e$ is

$$\begin{aligned}(d\mathrm{Tr}_b e)(\partial_r) &= b^{ac}\left(\partial_r g_{ac} - 2\Gamma^d_{ra}g_{cd} - 2\Gamma^k_{ra}g_{ck}\right) + b^{ij}\left(\partial_r g_{ij} - 2\Gamma^d_{ri}g_{jd} - 2\Gamma^k_{ri}g_{jk}\right) \\ &= \underbrace{b^{ac}\left(\partial_r g_{ac} - 2\Gamma^d_{ra}g_{cd}\right)}_{I_3} + \underbrace{b^{ij}\left(\partial_r g_{ij} - 2\Gamma^k_{ri}g_{jk}\right)}_{I_4}\end{aligned} \tag{5.32}$$

Then we have

$$I_3 = b^{rr}\left(\partial_r g_{rr} - b^{rr}\partial_r b_{rr} g_{rr} + b^{\theta\theta}\partial_\theta b_{rr} g_{r\theta}\right) + b^{\theta\theta}\left(\partial_r g_{\theta\theta} - b^{rr}\partial_\theta b_{rr} g_{\theta r} - b^{\theta\theta}\partial_r b_{\theta\theta} g_{\theta\theta}\right) \tag{5.33}$$

and

$$I_4 = \mathrm{Tr}(G_b^{-1}\partial_r G) + \mathrm{Tr}(G\partial_r G_b^{-1}) \tag{5.34}$$

Adding I_3 and I_4 leads to

$$\begin{aligned}(d\mathrm{Tr}_b e)(\partial_r) = &\left(b^{rr}\partial_r g_{rr} + b^{\theta\theta}\partial_r g_{\theta\theta} + \mathrm{Tr}\left(G_b^{-1}\partial_r G\right)\right) - \left(b^{rr}\right)^2 \partial_r b_{rr} g_{rr} \\ &- \left(b^{\theta\theta}\right)^2 \partial_r b_{\theta\theta} g_{\theta\theta} + \mathrm{Tr}\left(G\partial_r G_b^{-1}\right)\end{aligned} \tag{5.35}$$

Subtracting this from (5.31), we get the result. □

Lemma 5.4. *Let $G = \rho e^Z \Phi$ and $G_b = \rho\Phi_b$ denote the matrices of the metrics g and b restricted to the torus, respectively. Then we have*

$$\begin{aligned}Tr\left(G_b^{-1}\partial_r G\right) = &\left(\partial_r \log\rho + \partial_r Z\right) e^Z \left(2\cosh(V - V_b)\cosh W_b \cosh W - 2\sinh W_b \sinh W\right) \\ &+ e^Z \left(2\sinh(V - V_b)\cosh W_b \cosh W \partial_r V - 2\cosh W \sinh W_b \partial_r W\right. \\ &\left. + 2\cosh(V - V_b)\sinh W \cosh W_b \partial_r W\right)\end{aligned} \tag{5.36}$$

and

$$\begin{aligned}Tr\left(G\partial_r G_b^{-1}\right) = &-\partial_r \log\rho e^Z \left(2\cosh(V - V_b)\cosh W_b \cosh W - 2\sinh W_b \sinh W\right) \\ &+ e^Z \left(2\sinh(V_b - V)\cosh W_b \cosh W \partial_r V_b - 2\cosh W_b \sinh W \partial_r W_b\right. \\ &\left. + 2\cosh(V - V_b)\sinh W_b \cosh W \partial_r W_b\right)\end{aligned} \tag{5.37}$$

Proof. First, consider the $\beta = 0$ case.

$$\Phi = \begin{pmatrix} e^V \cosh W & \sinh W \\ \sinh W & e^{-V}\cosh W \end{pmatrix}, \qquad \Phi_b = \begin{pmatrix} e^{V_b}\cosh W_b & \sinh W_b \\ \sinh W_b & e^{-V_b}\cosh W_b \end{pmatrix} \tag{5.38}$$

with similar expressions for $\beta > 0$. Observe that

$$\partial_r \Phi = \begin{pmatrix} e^V\left(\cosh W \partial_r V + \sinh W \partial_r W\right) & \cosh W \partial_r W \\ \cosh W \partial_r W & e^{-V}\left(-\cosh W \partial_r V + \sinh W \partial_r W\right). \end{pmatrix} \tag{5.39}$$

We obtain (for $\beta \geq 0$)

$$\begin{aligned}\mathrm{Tr}\left(\Phi_b^{-1}\Phi\right) &= e^{V - V_b}\cosh W_b \cosh W - 2\sinh W \sinh W_b + e^{-V + V_b}\cosh W_b \cosh W \\ &= 2\cosh(V - V_b)\cosh W_b \cosh W - 2\sinh W \sinh W_b.\end{aligned} \tag{5.40}$$

Furthermore, we have

$$\begin{aligned}\mathrm{Tr}\left(\Phi_b^{-1}\partial_r\Phi\right) &= e^{V - V_b}\left(\cosh W_b \cosh W \partial_r V + \cosh W_b \sinh W \partial_r W\right) - 2\cosh W \sinh W_b \partial_r W \\ &\quad + e^{-V + V_b}\left(-\cosh W_b \cosh W \partial_r V + \cosh W_b \sinh W \partial_r W\right) \\ &= 2\sinh(V - V_b)\cosh W_b \cosh W \partial_r V + 2\cosh(V - V_b)\cosh W_b \sinh W \partial_r W \\ &\quad - 2\cosh W \sinh W_b \partial_r W.\end{aligned} \tag{5.41}$$

Now we compute the derivative of G as follows.

$$\partial_r G = \partial_r\left(\rho e^Z \Phi\right) = e^Z \Phi \partial_r \rho + e^Z \rho \Phi \partial_r Z + e^Z \rho \partial_r \Phi. \tag{5.42}$$

Multiply this with $G_b^{-1} = \rho^{-1}\Phi_b^{-1}$, we have

$$\mathrm{Tr}\left(G_b^{-1}\partial_r G\right) = e^Z\left(\partial_r \log\rho + \partial_r Z\right)\mathrm{Tr}\left(\Phi_b^{-1}\Phi\right) + e^Z \mathrm{Tr}\left(\Phi_b^{-1}\partial_r\Phi\right). \tag{5.43}$$

Combining this with (5.40) and (5.41) lead to (5.36). Next we show equation (5.37) as follows. Observe that for $\beta = 0$

$$\partial_r \Phi_b^{-1} = \begin{pmatrix} e^{-V_b}\left(-\cosh W_b \partial_r V_b + \sinh W_b \partial_r W_b\right) & -\cosh W_b \partial_r W_b \\ -\cosh W_b \partial_r W_b & e^{V_b}\left(\cosh W_b \partial_r V_b + \sinh W_b \partial_r W_b\right)\end{pmatrix}, \tag{5.44}$$

with a similar expression for $\beta > 0$. We find for all $\beta \geq 0$

$$\begin{aligned}\mathrm{Tr}\left(\Phi\partial_r\Phi_b^{-1}\right) &= e^{V-V_b}\left(-\cosh W\cosh W_b\partial_r V_b + \cosh W \sinh W_b \partial_r W_b\right)\\ &\quad + e^{-V+V_b}\left(\cosh W\cosh W_b\partial_r V_b + \cosh W\sinh W_b\partial_r W_b\right)\\ &\quad - 2\cosh W_b \sinh W \partial_r W_b\\ &= -2\sinh(V-V_b)\cosh W\cosh W_b\partial_r V_b + 2\cosh(V-V_b)\cosh W\sinh W_b\partial_r W_b\\ &\quad - 2\cosh W_b\sinh W\partial_r W_b.\end{aligned} \tag{5.45}$$

Finally, we have

$$\mathrm{Tr}\left(G\partial_r G_b^{-1}\right) = -e^Z\mathrm{Tr}\left(\Phi\Phi_b^{-1}\right)\partial_r\log\rho + e^Z\mathrm{Tr}\left(\Phi\partial_r\Phi_b^{-1}\right). \tag{5.46}$$

Combining this with (5.40) and (5.45) leads to (5.36) and (5.37). □

Lemma 5.5. *Let (M, g) be a toric Riemanian manifold. If it is ALE and $b = b_{ALE}$, the mass is*

$$mass_b(M,g) = \frac{1}{4\pi}\lim_{r\to\infty}\int_{L(p,q)}\left(-2\partial_r\left(\alpha - \alpha_b + Z\right) + \left(2\alpha - 2\alpha_b - Z\right)\partial_r\log\rho\right)dA_r \tag{5.47}$$

where $dA_r = \frac{r^3}{2p}\sin 2\theta d\theta d\phi^1 d\phi^2$. If (M, g) is ALF with $b = b_{ALF}$ and $\Sigma^3 = L(p,q)$ or $AF_{\beta\ell}$ for $\beta\in\mathbb{R}$ with $b = b_{AF}$, and $\Sigma^3 = S^1\times S^2$, then the mass is

$$mass_b(M,g) = \frac{1}{4\pi}\lim_{r\to\infty}\int_{\Sigma^3}\left(-2\partial_r\left(\alpha-\alpha_b+Z\right) + \left(2\alpha - 2\alpha_b - Z\right)\partial_r\log\rho - (V-V_b)\partial_r V_b\right)dA_r \tag{5.48}$$

where $dA_r = \ell r^2\sin\theta d\theta d\phi^1 d\phi^2$.

Proof. We use the asymptotics in Section 7.1 to determine the mass. We compute the leading term in the expansion of $\mathrm{Tr}\left(G_b^{-1}\partial_r G\right)$ in (5.36) for each of these asymptotes.

$$\begin{aligned}\mathrm{Tr}\left(G_b^{-1}\partial_r G\right) &= \underbrace{\left(\partial_r\log\rho + \partial_r Z\right)e^Z}_{I_0}\underbrace{\left(2\cosh(V-V_b)\cosh W_b\cosh W - 2\sinh W_b\sinh W\right)}_{I_1}\\ &\quad + e^Z\left(\underbrace{2\sinh(V-V_b)\cosh W_b\cosh W\partial_r V}_{I_2} - \underbrace{2\cosh W\sinh W_b\partial_r W}_{I_3}\right.\\ &\quad \left. + \underbrace{2\cosh(V-V_b)\sinh W\cosh W_b\partial_r W}_{I_4}\right)\end{aligned} \tag{5.49}$$

The first term is

$$I_0 = (\partial_r \log\rho + \partial_r Z)\, e^Z = \begin{cases} (\partial_r \log\rho)(1+Z) + \partial_r Z + O(r^{-2-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ (\partial_r \log\rho)(1+Z) + \partial_r Z + O(r^{-3-2\kappa}) & \text{for ALE} \end{cases} \tag{5.50}$$

The second therm is

$$I_1 = 2\cosh\left(\mathrm{dist}_{\mathbb{H}^2}(\Psi, \Psi_b)\right) = \begin{cases} 2 + O(r^{-1-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ 2 + O(r^{-2-2\kappa}) & \text{for ALE} \end{cases} \tag{5.51}$$

and the third term is

$$I_2 = \begin{cases} 2(V - V_b)\partial_r V + O(r^{-\frac{5}{2}-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ O(r^{-3-2\kappa}) & \text{for ALE} \end{cases} \tag{5.52}$$

and the remaining terms have the decay

$$I_4 - I_3 = \begin{cases} O(r^{-2-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ O(r^{-3-2\kappa}) & \text{for ALE} \end{cases}. \tag{5.53}$$

Therefore, we have

$$\mathrm{Tr}\left(G_b^{-1}\partial_r G\right) = \begin{cases} 2\partial_r \log\rho(1+Z) + 2\partial_r Z + (V - V_b)\partial_r V + O(r^{-2-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ 2\partial_r \log\rho(1+Z) + 2\partial_r Z + O(r^{-3-2\kappa}) & \text{for ALE} \end{cases} \tag{5.54}$$

Next, consider the term

$$\begin{aligned} \mathrm{Tr}\left(G\partial_r G_b^{-1}\right) = &-\partial_r \log\rho e^Z \left(2\cosh(V - V_b)\cosh W_b \cosh W - 2\sinh W_b \sinh W\right) \\ &+ e^Z \left(2\sinh(V_b - V)\cosh W_b \cosh W \partial_r V_b - 2\cosh W_b \sinh W \partial_r W_b \right. \\ &\left. +2\cosh(V - V_b)\sinh W_b \cosh W \partial_r W_b\right) \end{aligned} \tag{5.55}$$

Since

$$\partial_r \log\rho e^Z = \begin{cases} \partial_r \log\rho(1+Z) + O(r^{-2-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ \partial_r \log\rho(1+Z) + O(r^{-3-2\kappa}) & \text{for ALE} \end{cases} \tag{5.56}$$

and following a similar computation to that above produces

$$\mathrm{Tr}\left(G\partial_r G_b^{-1}\right) = \begin{cases} -2\partial_r \log\rho(1+Z) - 2(V - V_b)\partial_r V_b + O(r^{-2-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ -2\partial_r \log\rho(1+Z) + O(r^{-3-2\kappa}) & \text{for ALE} \end{cases}. \tag{5.57}$$

Finally, we invetigate the last term in the definition of mass as follows.

$$g_{\theta r} = G_{ij} A_r^j A_\theta^i = \begin{cases} O(r^{-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ O(r^{-1-2\kappa}) & \text{for ALE} \end{cases} \tag{5.58}$$

Therefore, we have

$$b^{\theta\theta}\partial_\theta g_{\theta r} + b^{\theta\theta}\partial_\theta \log\rho g_{\theta r} = \begin{cases} O(r^{-2-2\kappa}) & \text{for AF}_{\beta\ell} \text{ and ALF} \\ O(r^{-3-2\kappa}) & \text{for ALE} \end{cases} \tag{5.59}$$

Consider first the ALE case. We combine $b_{rr} = r^{-2}b_{\theta\theta} = 1$, $g_{rr} = r^{-2}g_{\theta\theta} = e^{2\alpha}\frac{r^2}{p} = e^{2\alpha-2\alpha_b}$, (5.54), (5.57), (5.59) and (5.20) to obtain the mass density

$$(\mathrm{div}_b e - d\,\mathrm{tr}_b\, e)\,(\partial_r) = -\partial_r\,(2\alpha - 2\alpha_b + 2Z) + (2\alpha - 2\alpha_b - Z)\,\partial_r \log\rho + O_1(r^{-3-\kappa}) \tag{5.60}$$

This leads to (5.47).

We next consider the $\mathrm{AF}_{\beta\ell}$ and ALF cases. Note that $b_{rr} = r^{-2}b_{\theta\theta} = 1$, $g_{rr} = r^{-2}g_{\theta\theta} = e^{2\alpha}\ell^2 = e^{2\alpha-2\alpha_b}$. A computation yields

$$\begin{aligned}(\mathrm{div}_b e - d\mathrm{Tr}_b e)\,(\partial_r) = -\partial_r\,(2\alpha - 2\alpha_b + 2Z) + (2\alpha - 2\alpha_b - Z)\,\partial_r \log\rho \\ - (V - V_b)\partial V_b + O_1(r^{-2-\kappa})\end{aligned} \tag{5.61}$$

which leads to (5.48). □

Now we can show the relation of the boundary integral at infinity and the mass.

Lemma 5.6. *The boundary integral at infinity is*

$$\lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\mathcal{B}^{\varsigma}_{\infty} = 2\left(mass_b(M,g) - mass_b(M_o,g_o)\right)\,. \tag{5.62}$$

Proof. Recall from (3.18) that the boundary integral at infinity is

$$\lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\mathcal{B}^{\varsigma}_{\infty} = \lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\int_{\partial B_{2/\varsigma_3}\cap\partial\Omega_{\varsigma}} X(\nu)\,dA\,. \tag{5.63}$$

Using the asymptotics in Section 7.1, for ALE geometries we have

$$X(\nu)rp^{-1} = -2\partial_r\,(\alpha - \alpha_o + Z) + (2\alpha - 2\alpha_o - Z)\,\partial_r\log\rho + O(r^{-3-\kappa}), \tag{5.64}$$

with $dA = \frac{r^4}{2p^2}\sin 2\theta d\theta d\varphi$ and $\nu = pr\partial_r$. Moreover, for ALF and $\mathrm{AF}_{\beta\ell}$ geometries, we have

$$\ell X(\nu) = -2\partial_r\,(\alpha - \alpha_o + Z) + (2\alpha - 2\alpha_o - Z)\,\partial_r\log\rho - (V - V_o)\,\partial_r V_o + O(r^{-2-\kappa}), \tag{5.65}$$

with $dA = \ell^2 r^2 \sin\theta d\theta d\varphi$ and $\nu = \ell^{-1}\partial_r$. Consider the definition of mass for each asymptotic class in Lemma 5.5. Since all functions have toric symmetry, we can integrate over ϕ^1 and ϕ^2 to produce a factor of $4\pi^2$ (note that the calculation of $\mathcal{B}^{\varsigma}_{\infty}$ involves a trivial integration over the azimuthal angle φ in an auxiiliary $\mathbb{R}^3$). Combining these results completes the proof of the Lemma. □

6. Proof of the Theorem 1.7

In this section, we prove Theorem 1.7. By the scalar curvature equation (3.16) and Theorem 4.1, we have

$$\lim_{\varsigma_3\to 0}\lim_{\varsigma_2\to 0}\lim_{\varsigma_1\to 0}\left(\mathcal{B}^{\varsigma}_{\mathrm{axis}} + \mathcal{B}^{\varsigma}_{\mathrm{corner}} + \mathcal{B}^{\varsigma}_{\infty}\right) \geq \mathcal{I}(\Psi) \geq C\left(\int_{\mathbb{R}^3}\mathrm{dist}^6_{\mathbb{H}^2}(\Psi,\Psi_o)\,dx\right)^{\frac{1}{3}}\,. \tag{6.1}$$

Combining this with Lemma 5.1, Lemma 5.2, and Lemma 5.6 we achieve the following inequality

$$\mathrm{mass}_b\,(M,g) - \mathrm{mass}_b\,(M_o,g_o) \geq 2\pi\sum_{n=1}^{N+1}\int_{\Gamma_n}(\boldsymbol{\vartheta}^n - \boldsymbol{\vartheta}^n_o)dz\,. \tag{6.2}$$

If equality holds, then $Z = 0$, $R = 0$, $A^i_a = 0$, and $\Psi = \Psi_o$. Then the scalar curvature (3.13) implies that

$$\Delta_3\,(\alpha - \alpha_o) = 0 \tag{6.3}$$

From the asymptotics in Section 7.1, we have $\alpha - \alpha_o = o(1)$ as $r\to\infty$. By the maximum principle, we have $\alpha = \alpha_o$ on $\mathbb{R}^3$. Therefore, the metric (M,g) and (M_o,g_o) are isometric.

7. Asymptotics near infinity, axis, and corners

In this section, we record the asymptotic behavior of the metric and related functions of a Riemannian toric 4-manifold (M, g) in a neighborhood of the asymptotic region, the axes, corners.

7.1. **Near asymptotic end.** Consider the Riemannian toric 4-manifold (M, g). To determine the asymptotic behavior of the metric components and functions we use Definition 1.1, Definition 1.3, and the definition of the appropriate asymptotic model metric b in these settings given in Section 2. *ALE Asymptotics.* The ALE asymptotics are derived from Definition 1.1 and $b = b_{ALE}$ given in (2.7). The coordinate transformation from Brill to polar coordinate for b_{ALE} is

$$\rho = \frac{r^2}{2p}\sin 2\theta, \qquad z = \frac{r^2}{2p}\cos 2\theta. \tag{7.1}$$

As a simple example, consider the standard Euclidean metric on $\mathbb{R}^4$. In standard spherical coordinates $(r, \theta, \phi^1, \phi^2)$ the metric is

$$\delta_4 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta(d\phi^1)^2 + \cos^2\theta(d\phi^2)^2\right) \tag{7.2}$$

where $r > 0$, $\theta \in (0, \pi/2)$, and ϕ^i each generate $2\pi-$periodic rotations. Using (7.1) with $p = 1$ produces the Euclidean metric in Brill coordinates:

$$\delta_4 = \frac{d\rho^2 + dz^2}{2\sqrt{\rho^2+z^2}} + (\sqrt{\rho^2+z^2} - z)(d\phi^1)^2 + (\sqrt{\rho^2+z^2} + z)(d\phi^2)^2. \tag{7.3}$$

Note that the conformal factor of the orbit space is not a constant as in the ALF setting.

We now consider the genreal case. In the asymptotic region, we choose the coordinate (r, θ, ϕ^i) adapted to the toric symmetry obtained by pull back via the diffeomorphism to the model $((\mathbb{R}^4 \setminus \mathbb{B}_R)/\mathbb{Z}_p, b_{\mathrm{ALE}})$. Therefore, the asymptotic decay of g must satisfy

$$|g - b_{ALE}|_{b_{ALE}} = O_2(r^{-1-\kappa}) \tag{7.4}$$

for $\kappa > 0$. Combining with the asymptotics at the axis, this implies the following decay for metric components and functions.

$$\alpha = -\log\left(\frac{r}{p}\right) + O_1(r^{-1-\kappa}), \quad \frac{q}{p}A_a^1 + A_a^2 = \frac{1}{\cos\theta}O_1(r^{-3-\kappa}), \tag{7.5}$$

$$A_a^1 = \frac{1}{\sin\theta}O_1(r^{-3-\kappa}), \qquad G_{12} = \frac{q}{p}r^2\cos^2\theta\left(1 + \rho^2 O_1(r^{-5-\kappa})\right) + \rho^2 O_1(r^{-3-\kappa}), \tag{7.6}$$

$$G_{11} = r^2\left(\frac{1}{p^2}\sin^2\theta + \frac{q^2}{p^2}\cos^2\theta\right)\left(1 + \rho^2 O_1(r^{-5-\kappa})\right), \qquad G_{22} = r^2\cos^2\theta\left(1 + \rho^2 O_1(r^{-5-\kappa})\right). \tag{7.7}$$

Now using the definitions of V, W in (3.8) and Z in (3.5), we obtain the following decays.

$$V = \frac{1}{2}\log\left(\frac{\tan^2\theta + q^2}{p^2}\right) + \rho^2 O_1(r^{-5-\kappa}), \quad \sinh W = q\cot\theta\left(1 + \rho^2 O_1(r^{-5-\kappa})\right) + \rho O(r^{-3-\kappa}), \tag{7.8}$$

$$|\nabla V|^2 = \frac{p^2\tan^2\theta}{r^4\left(\sin^2\theta + q^2\cos^2\theta\right)^2} + \frac{p^2\sin^2\theta}{\left(\sin^2\theta + q^2\cos^2\theta\right)^2}O(r^{-5-\kappa}) + \sin^2\theta O(r^{-6-2\kappa}), \tag{7.9}$$

$$Z = O_1(r^{-1-\kappa}), \qquad |\nabla W|^2 = \frac{p^2q^2\left(r^{-4} + O(r^{-5-\kappa})\right)}{\sin^2\theta(\sin^2\theta + q^2\cos^2\theta)} + O(r^{-6-2\kappa}), \qquad |\nabla Z| = O(r^{-3-\kappa}). \tag{7.10}$$

By [7, Theorem, page 314], any ALE Ricci flat 4-manifold with L^2 Riemann tensor is of order four, that is

$$|g_o - b_{ALE}|_{b_{ALE}} = O_2(r^{-4}) \tag{7.11}$$

Therefore, we have

$$\alpha_o = -\log\left(\frac{r}{p}\right) + O_1(r^{-4}), \quad , \quad G_{o12} = \frac{q}{p} r^2 \cos^2\theta \left(1 + \rho^2 O_1(r^{-8})\right) + \rho^2 O_1(r^{-6}), \tag{7.12}$$

$$G_{o11} = r^2 \left(\frac{1}{p^2}\sin^2\theta + \frac{q^2}{p^2}\cos^2\theta\right)\left(1 + \rho^2 O_1(r^{-8})\right), \qquad G_{o22} = r^2\cos^2\theta\left(1 + \rho^2 O_1(r^{-8})\right). \tag{7.13}$$

Now using the definitions of V, W in (3.8) and Z in (3.5), we obtain the following decays.

$$V_o = \frac{1}{2}\log\left(\frac{\tan^2\theta + q^2}{p^2}\right) + \rho^2 O_1(r^{-8}), \qquad \sinh W_o = q\cot\theta\left(1 + \rho^2 O_1(r^{-8})\right) + \rho O(r^{-6}), \tag{7.14}$$

Moreover, we have

$$\alpha - \alpha_0 = O(r^{-1-\kappa}), \qquad V - V_o = \rho^2 O(r^{-5-\kappa}), \qquad W - W_o = \rho^2 O(r^{-5-\kappa}), \tag{7.15}$$

ALF Asymptotics. The ALF asymptotic behavior is derived from Definition 1.3 and $b = b_{ALE}$ as given in equation (2.8). The coordinate transformation from Brill to polar coordinates for b_{ALF} is

$$\rho = \ell r \sin\theta, \qquad z = \ell r\cos\theta. \tag{7.16}$$

In the asymptotic region, we choose the coordinate (r, θ, ϕ^i) adapted to the toric symmetry obtained by pull back via the diffeomorphism to the model $(\mathbb{R}_+ \times L(p,q), b_{\mathrm{ALF}})$. Therefore, the asymptotic expansion of the metric is

$$|g - b_{ALF}|_{b_{ALF}} = O_2(r^{-\frac{1}{2}-\kappa}) \tag{7.17}$$

for $\kappa > 0$ and the asymptotic of metric near the axis, implies the following decay for metric components and functions.

$$\alpha = -\log\ell + O_1(r^{-\frac{1}{2}-\kappa}), \qquad A_a^1 = \rho^{-1}O_1(r^{-\frac{1}{2}-\kappa}), \quad A_a^2 + \frac{p\cos^2\left(\frac{\theta}{2}\right)}{q}A_a^1 = O_1(r^{-\frac{1}{2}-\kappa}), \tag{7.18}$$

$$G_{12} = \frac{\ell^2 p}{q}\cos^2\left(\frac{\theta}{2}\right)\left(1 + \rho^2 O_1(r^{-\frac{5}{2}-\kappa})\right), \qquad G_{22} = \ell^2\left(1 + \rho^2 O_1(r^{-\frac{5}{2}-\kappa})\right), \tag{7.19}$$

$$G_{11} = \left(r^2\sin^2\theta + \frac{\ell^2 p^2}{q^2}\cos^4\left(\frac{\theta}{2}\right)\right)\left(1 + \rho^2 O_1(r^{-\frac{5}{2}-\kappa})\right), \tag{7.20}$$

which lead to the following asymptotics for V, W, and Z.

$$V = \frac{1}{2}\log\left[\frac{\rho^2}{\ell^4} + \frac{p^2}{4q^2}\left(1 + \frac{z}{\sqrt{\rho^2 + z^2}}\right)^2\right] + \rho^2 O_1(r^{-\frac{5}{2}-\kappa}), \qquad |\nabla V| = \tan\left(\frac{\theta}{2}\right)O(r^{-1}), \tag{7.21}$$

$$\sinh W = \frac{\ell^2 p}{q\rho}\cos^2\left(\frac{\theta}{2}\right)\left(1 + \rho^2 O_1(r^{-\frac{5}{2}-\kappa})\right), \qquad |\nabla W| = O(r^{-2}), \tag{7.22}$$

and

$$Z = O_1(r^{-\frac{1}{2}-\kappa}), \quad |\nabla Z| = O(r^{-\frac{3}{2}-\kappa}). \tag{7.23}$$

The harmonic map Ψ_o has similar asymptotics. In particular, we have

$$\alpha-\alpha_0=O(r^{-\frac{1}{2}-\kappa}),\qquad V-V_o=\rho^2O(r^{-\frac{5}{2}-\kappa}),\qquad W-W_o=\rho^2O(r^{-\frac{5}{2}-\kappa}) \tag{7.24}$$

$AF_{\beta\ell}$ & AF_0 Asymptotics. The end $\mathrm{AF}_{\beta\ell}$ is the special case of ALF, when $\mathcal{S}$ in Definition 1.3 is $S^1\times S^2$ and $b=b_{AF}$ as given in (2.10). In this case, choose the coordinate (r,θ,ϕ^i) adapted to the toric symmetry obtained by pull back via the diffeomorphism to the model $(\mathbb{R}_+\times S^1\times S^2, b_{\mathrm{AF}})$. Therefore, the asymptotic expansion of the metric

$$|g-b_{AF}|_{b_{AF}}=O_2(r^{-\frac{1}{2}-\kappa}) \tag{7.25}$$

for $\kappa>0$. This yields the following decay for metric components and functions if $\beta\neq 0$:

$$\alpha=-\log\ell+O_1(r^{-\frac{1}{2}-\kappa}),\qquad A^2_a=O_1(r^{-\frac{1}{2}-\kappa}),\quad A^1_a+\beta\ell A^2_a=\rho^{-1}O_1(r^{-\frac{1}{2}-\kappa}), \tag{7.26}$$

$$G_{12}=\beta\ell r^2\sin^2\theta\left(1+\rho^2O_1(r^{-\frac{5}{2}-\kappa})\right),\qquad G_{11}=r^2\sin^2\theta\left(1+\rho^2O_1(r^{-\frac{5}{2}-\kappa})\right), \tag{7.27}$$

$$G_{22}=\left(\ell^2+\beta^2\ell^2r^2\sin^2\theta\right)\left(1+\rho^2O_1(r^{-\frac{5}{2}-\kappa})\right), \tag{7.28}$$

or

$$G_{12}-\beta\ell G_{11}=\rho^2O_1(r^{-\frac{3}{2}-\kappa}),\qquad G_{11}=\ell^{-2}\rho^2\left(1+\rho^2O_1(r^{-\frac{5}{2}-\kappa})\right), \tag{7.29}$$

$$G_{22}-2\beta\ell G_{12}+\beta^2\ell^2G_{11}=\ell^2+\rho^2O_1(r^{-\frac{5}{2}-\kappa}). \tag{7.30}$$

In the AF_0 case we have

$$\alpha=-\log\ell+O_1(r^{-\frac{1}{2}-\kappa}),\qquad A^1_a=\rho^{-1}O_1(r^{-\frac{1}{2}-\kappa}),\quad A^2_a=O_1(r^{-\frac{1}{2}-\kappa}), \tag{7.31}$$

$$G_{11}=r^2\sin^2\theta\left(1+\rho^2O_1(r^{-\frac{5}{2}-\kappa})\right),\quad G_{22}=\ell^2+\rho^2O_1(r^{-\frac{5}{2}-\kappa}),\quad G_{12}=\rho^2O_1(r^{-\frac{3}{2}-\kappa}). \tag{7.32}$$

The decays of V, W, and Z are as follows for both the AF_0 and $\mathrm{AF}_{\beta\ell}$ cases

$$V=\log(\ell^{-2}\rho)+\rho^2O_1(r^{-\frac{5}{2}-\kappa}),\qquad W=\rho O_1(r^{-\frac{3}{2}-\kappa}),\qquad Z=O_1(r^{-\frac{1}{2}-\kappa}), \tag{7.33}$$

$$|\nabla V|=\frac{1}{\rho}+O(r^{-\frac{3}{2}-\kappa}),\qquad |\nabla W|=O(r^{-\frac{3}{2}-\kappa}),\qquad |\nabla Z|=O(r^{-\frac{3}{2}-\kappa}). \tag{7.34}$$

The harmonic map Ψ_o has similar asymptotics which implies

$$\alpha-\alpha_0=O(r^{-\frac{1}{2}-\kappa}),\qquad V-V_o=\rho^2O(r^{-\frac{5}{2}-\kappa}),\qquad W-W_o=\rho^3O(r^{-\frac{7}{2}-\kappa}). \tag{7.35}$$

7.2. **Near Axis.** To derive the asymptotic behavior near the axis, it is convenient to distinguish between three cases of rod structure: (I) $\mathbf{v}_n=(1,0)$ on Γ_n, (II) $\mathbf{v}_n=(0,1)$ on Γ_n, (III) $\mathbf{v}_n=(v^1_n,v^2_n)\in(\mathbb{Z}\backslash\{0\})^2$ on Γ_n.

Case I. If the rod structure is $\mathbf{v}_n=(1,0)$ on Γ_n, the leading term of the metric takes the form

$$g_{\mathrm{cone}}=e^{2\alpha_c(z)}\left(d\rho^2+dz^2\right)+c_{11}(z)\rho^2(d\phi^1+c_{12}(z)d\phi^2)^2+c_{22}(z)(d\phi^2)^2, \tag{7.36}$$

where $c_{ii}(z)>0$ for $i=1,2$. For fixed $z\in\Gamma_n$ the transformation to geodesic normal coordinates in Definition 2.1 is $s=e^{\alpha_c(z)}\rho$. The metric g should satisfy the following decay

$$|g-g_{\mathrm{cone}}|_{g_{\mathrm{cone}}}=O_1(\rho^{1+\zeta}) \tag{7.37}$$

where $\zeta > 0$. Therefore, the metric components are

$$\alpha = \alpha_c + O_1(\rho^{1+\zeta}), \qquad G_{11} = c_{11}(z)\rho^2 + O_1(\rho^{3+\zeta}), \qquad G_{22} = c_{22}(z) + O_1(\rho^{1+\zeta}), \tag{7.38}$$

$$G_{12} = c_{12}(z)c_{11}(z)\rho^2 + O_1(\rho^{2+\zeta}), \qquad A^1_a = O_1(\rho^{\zeta}), \qquad A^2_a = O_1(\rho^{1+\zeta}). \tag{7.39}$$

This asymptotic behavior is consistent with those of the harmonic maps given in [40, 44]. Using the definition of $\Phi = \rho^{-1}e^{-Z}G$ and the fact that $\det\Phi = 1$, we have

$$Z = \frac{1}{2}\log(c_{11}c_{22}) + O_1(\rho^{1+\zeta}), \qquad \Phi = \begin{pmatrix} \rho\sqrt{\frac{c_{11}}{c_{22}}} + O_1(\rho^{2+\zeta}) & c_{12}\sqrt{\frac{c_{11}}{c_{22}}}\rho + O_1(\rho^{1+\zeta}) \\ c_{12}\sqrt{\frac{c_{11}}{c_{22}}}\rho + O_1(\rho^{1+\zeta}) & \rho^{-1}\sqrt{\frac{c_{22}}{c_{11}}} + O_1(\rho^{\zeta}) \end{pmatrix}. \tag{7.40}$$

Combining with the definitions of V and W in (3.8), we obtain

$$V = \log\left(\rho\sqrt{\frac{c_{11}}{c_{22}}}\right) + O_1(\rho^{1+\zeta}), \quad W = (c_{12} - \beta\ell)\sqrt{\frac{c_{11}}{c_{22}}}\rho + O_1(\rho^{1+\zeta}), \tag{7.41}$$

Setting $\beta = 0$, we recover the asymptotics for the ALE, ALF, and AF_0 cases. Moreover,

$$|\nabla V|^2 = \rho^{-2} + \left(\partial_z \log\left(\sqrt{\frac{c_{11}}{c_{22}}}\right)\right)^2 + O(\rho^{\zeta-1}), \quad |\nabla W|^2 = \frac{c_{11}}{c_{22}}(c_{12} - \beta\ell)^2 + O(\rho^{\zeta}). \tag{7.42}$$

$$\nabla V \cdot \nabla W = \frac{1}{\rho}\sqrt{\frac{c_{11}}{c_{22}}}(c_{12} - \beta\ell) + O(\rho^{\zeta-1}) \tag{7.43}$$

The harmonic map Ψ_o has the same asymptotics (7.41) and (7.42) as shown by Li and Sun [40, Proposition 4.12]. Note that the functions α_c corresponding to g_o and g differ because the transformation to geodesic coordinates is different for each metric.

Case II. In this case the rod structure is $\mathbf{v}_n = (0, 1)$ on Γ_n, which is equivalent to Case I by exchanging the decays of G_{11} and G_{22}. In particular we may write

$$g_{\text{cone}} = e^{2\alpha_c(z)}\left(d\rho^2 + dz^2\right) + +\hat{c}_{11}(z)(d\phi^1)^2 + \hat{c}_{22}(z)(z)\rho^2(d\phi^2 + \hat{c}_{12}(z)(z)d\phi^1)^2. \tag{7.44}$$

with the same expansion for Z as in Case 1 and

$$\Phi = \begin{pmatrix} \rho^{-1}\sqrt{\frac{\hat{c}_{11}}{\hat{c}_{22}}} + O(\rho^{\zeta}) & \hat{c}_{12}\sqrt{\frac{\hat{c}_{22}}{\hat{c}_{11}}}\rho + O(\rho^{1+\zeta}) \\ \hat{c}_{12}\sqrt{\frac{\hat{c}_{22}}{\hat{c}_{11}}}\rho + O(\rho^{1+\zeta}) & \rho\sqrt{\frac{\hat{c}_{22}}{\hat{c}_{11}}} + O_1(\rho^{2+\zeta}) \end{pmatrix}. \tag{7.45}$$

For $\beta = 0$ we obtain

$$V = -\log\left(\rho\sqrt{\frac{\hat{c}_{22}}{\hat{c}_{11}}}\right) + O_1(\rho^{1+\zeta}), \quad W = \hat{c}_{12}\sqrt{\frac{\hat{c}_{22}}{\hat{c}_{11}}}\rho + O_1(\rho^{1+\zeta}), \tag{7.46}$$

whereas for $\beta \neq 0$,

$$V = -\log(\beta\ell) + O(\rho^{1+\zeta}), \qquad W = \log\rho - \log(2\beta\ell) + O(1) \tag{7.47}$$

We have for $\beta = 0$ that

$$|\nabla V|^2 = \frac{1}{\rho^2} + O(\rho^{-1+\zeta}), \quad |\nabla W|^2 = \frac{\hat{c}_{12}^2\hat{c}_{22}}{\hat{c}_{11}} + O(\rho^{\zeta}), \quad \nabla V \cdot \nabla W = -\frac{\hat{c}_{12}}{\rho}\sqrt{\frac{\hat{c}_{22}}{\hat{c}_{11}}} + O(\rho^{\zeta-1}), \tag{7.48}$$

and for $\beta \neq 0$,

$$|\nabla V|^2 = O(\rho^{2\zeta}), \qquad |\nabla W|^2 = \frac{1}{\rho^2} + O(\rho^{-1}), \quad \nabla V \cdot \nabla W = O(\rho^{\zeta-1}). \tag{7.49}$$

Case III. Let $B=(b_{kl})\in SL(2,\mathbb{Z})$ and $\mathbf{v}_n=(v_n^1,v_n^2)\in(\mathbb{Z}\backslash\{0\})^2$ be the rod structure on Γ_n, then Φ in Case III is related to Φ in Case I using a transformation with matrix B. Therefore, we have

$$\Phi=B^t\begin{pmatrix}\rho\sqrt{\frac{c_{11}}{c_{22}}}+O_1(\rho^{2+\zeta}) & c_{12}\sqrt{\frac{c_{11}}{c_{22}}}\rho+O(\rho^{1+\zeta})\\ c_{12}\sqrt{\frac{c_{11}}{c_{22}}}\rho+O(\rho^{1+\zeta}) & \rho^{-1}\sqrt{\frac{c_{22}}{c_{11}}}+O(\rho^{\zeta})\end{pmatrix}B\,. \tag{7.50}$$

Concretely, one may choose

$$B=\begin{pmatrix}w_n^2 & -w_n^1\\ -v_n^2 & v_n^1\end{pmatrix} \tag{7.51}$$

where $w_n^i\in\mathbb{Z}$ are chosen so that $\det B=1$. This implies that each component of Φ has the following expansion:

$$\Phi_{kl}=\rho^{-1}\sqrt{\frac{c_{22}}{c_{11}}}b_{2k}b_{2l}+\rho\sqrt{\frac{c_{11}}{c_{22}}}B_k^t\begin{pmatrix}1 & c_{12}\\ c_{12} & 0\end{pmatrix}B_l+O_1(\rho^{1+\zeta})\,, \tag{7.52}$$

where B_k is the kth columns of B. Since $\Phi_{kl}v_n^kv_n^l=0$, we obtain several conditions including $b_{21}v_n^1+b_{22}v_n^2=0$. We may identify $b_{21}=-v_n^2$ and $b_{22}=v_n^1$, and can assume that $v_n^1+\beta\ell v_n^2\neq0$ otherwise $\beta\ell\in\mathbb{Q}$ and we may redefine the torus generators to reduce back to the case when $\beta=0$. We then have

$$V=\log\left(\frac{|b_{21}|}{|b_{22}-\beta\ell b_{21}|}\right)+O_1(\rho^2),\quad \pm W=-\log\rho+\log\left(2|b_{21}(b_{22}-\beta\ell b_{21})|\sqrt{\frac{c_{22}}{c_{11}}}\right)+O_1(\rho^2), \tag{7.53}$$

where $\pm=\mathrm{sgn}(b_{21}(b_{22}-\beta\ell b_{21}))$, and

$$|\nabla V|=O(\rho),\qquad |\nabla W|=\frac{1}{\rho}+O(1)\,. \tag{7.54}$$

Furthermore

$$V+\log\cosh W=-\log\rho+\log\left(b_{21}^2\sqrt{\frac{c_{22}}{c_{11}}}\right)+O_1(\rho^2)\,, \tag{7.55}$$

which will be used in the proof of Lemma 5.1. The harmonic map Ψ_o has the same asymptotics (7.53) as shown by Li-Sun [40, Proposition 4.12].

7.3. **Near Corners.** Consider a corner point z_n, the metric should be asymptotic to the flat cone metric (2.4) on $\mathbb{R}^4$. Let r_n be the geodesic distance to the corner z_n with

$$\rho=\frac{r_n^2}{2}\sin2\theta,\qquad z-z_n=\frac{r_n^2}{2}\cos2\theta, \tag{7.56}$$

the Definition 2.1 implies that

$$\alpha=-\log(r_n)+O_1(r_n^2),\qquad G_{11}=c_1^2r^2\sin^2\theta\left(1+O_1(r_n^2)\right),\qquad G_{12}=\rho O(r_n^2)\,, \tag{7.57}$$

$$G_{22}=c_2^2r^2\cos^2\theta\left(1+O_1(r_n^2)\right),\qquad A_a^1=\frac{1}{\sin\theta}O_1(1),\qquad A_a^2=\frac{1}{\cos\theta}O_1(1). \tag{7.58}$$

Next, we use the definition of V,W, and Z to obtain the following estimates as $r_n\to0$

$$V=\frac12\log\left(\frac{c_1^2\tan\theta}{c_2^2\cot\theta+\beta^2\ell^2c_1^2\tan\theta}\right)+O_1(r_n^2),\qquad W=\sinh^{-1}\left(-\beta\ell\frac{c_1^2}{|c_1c_2|}\tan\theta\right)+O_1(r_n^2)\,, \tag{7.59}$$

$$|\nabla V|^2=\frac{c_2^4\cos^4\theta}{\rho^2\left(c_2^2\cos^2\theta+\beta^2\ell^2c_1^2\sin^2\theta\right)^2}+O(r_n^{-2}),\qquad Z=\log\left(c_1c_2\right)+O_1(r_n^2) \tag{7.60}$$

$$|\nabla W|^2 = \frac{\beta^2\ell^2 c_1^2 \sec^4\theta}{r_n^4\left(c_2^2+\beta^2\ell^2c_1^2\tan^2\theta\right)} + \beta O(r_n^{-2}) + O(1), \qquad |\nabla Z| = O(1). \tag{7.61}$$

Asymptotics of the harmonic map Ψ_o are similar to above.

8. Models and Examples

In this section, we provide computations of the mass for some well-known explicitly known families of gravitational instantons in the four asymptotic classes.

8.1. $\mathbf{AF}_{\beta\ell}$ and $\mathbf{AF}_0$ Riemannian Manifolds. We consider here two explicit examples of asymptotically flat (AF) gravitational instantons and determine their mass according to our definition.

8.1.1. *Kerr gravitational instanton.* The two-parameter family of Kerr gravitational instantons $(\mathbb{R}^2\times\mathbb{S}^2, g_K)$ are $\mathrm{AF}_{\beta\ell}$ with the smooth Ricci flat metric

$$\begin{aligned} g_K = {}& \frac{\ell^2 f}{\Sigma}(d\phi^2 + \frac{a}{\ell}\sin^2\theta(d\phi^1+\beta\ell d\phi^2))^2 + \frac{\sin^2\theta}{\Sigma}((r^2-a^2)(d\phi^1+\beta\ell d\phi^2) - a\ell d\phi^2)^2 \\ & + \Sigma\left(\frac{dr^2}{f} + d\theta^2\right). \end{aligned} \tag{8.1}$$

where (ϕ^1,ϕ^2) are independently $2\pi-$periodic coordinates, and $f = r^2 - 2Mr - a^2$ and $\Sigma = r^2 - a^2\cos^2\theta$. The solution is parametrized by (M,a) where without loss of generality we may arrange $a \geq 0$. The radial coordinate $r \in (r_+,\infty)$ where $r_+ := M + \sqrt{M^2+a^2}$ is the real, positive root of f and $\theta \in (0,\pi)$. It is convenient to eliminate the parameter M using

$$M = \frac{r_+^2 - a^2}{2r_+}. \tag{8.2}$$

Notice that positive-definiteness of the metric requires $r_+ > a$ to ensure $\Sigma > 0$. The asymptotic geometry is characterized by (ℓ,β) where

$$\beta = \frac{a}{r_+^2 - a^2}, \qquad \ell = \frac{2r_+(r_+^2-a^2)}{r_+^2+a^2}. \tag{8.3}$$

The canonical coordinates (ρ, z) are related to (r,θ) by

$$\rho = \ell\sqrt{f}\sin\theta, \qquad z = \ell(r-M)\cos\theta. \tag{8.4}$$

where $\gamma = L^2$ (c.f. (5.22)). There are three rods:

(1) a semi-infinite rod $(-\infty, z_1)$, $z_1 = -(r_+ - M)$ with rod vector $(\partial_{\phi^1},\partial_{\phi^2}) = (1,0)$;
(2) a finite rod (z_1,z_2) with $z_2 = r_+ - M$ (corresponding to $r = r_+, 0 < \theta < \pi$) with rod vector $(0,1)$; and
(3) a semi-infinite rod (z_2,∞) with rod vector $(1,0)$.

As $r\to\infty$, we can read off

$$\alpha_{Kerr} = \frac{1}{2}\log\left(\frac{\Sigma}{\ell^2(f+(M^2+a^2)\sin^2\theta)}\right) = -\log\ell + \frac{M}{r} + O(r^{-2}) \tag{8.5}$$

and

$$V_{Kerr} = \log\left(\frac{\rho}{\ell^2}\right) + \frac{2M}{r} + O(r^{-2}), \qquad W_{Kerr} = -\frac{2Ma\sin\theta}{r^2} + O(r^{-3}). \tag{8.6}$$

Therefore, $V_{Kerr} - V_b = 2M/r + O(r^{-2})$ and $\alpha_{Kerr} - \alpha_b = M/r + O(1/r^2), W_{Kerr} - W_b = O(r^{-2})$, and $\partial_r \log\rho = 1/r + M/r^2 + O(1/r^3)$. Using the formula (5.48) we find the mass to be

$$\text{mass}_b(M, g_K) = 4\pi M\ell = 4\pi\frac{(r_+^2 - a^2)^2}{(r_+^2 + a^2)} > 0. \tag{8.7}$$

The one-parameter family of AF_0 Schwarzschild instantons is recovered when $a = 0$. The asymptotic $\mathbb{S}^1$ has bounded circumference $2\pi\ell = 8\pi M$. The mass is given by (8.7) and is positive.

8.1.2. *The Chen-Teo instanton.* The two-parameter family of Chen-Teo gravitational instantons $(\mathbb{CP}^2\setminus\mathbb{S}^1, g_{CT})$ [15] are $\mathrm{AF}_{\beta\ell}$ with the smooth Ricci flat metric given explicitly in [35, Appendix B2]:

$$g_{CT} = \frac{F(x,y)}{(x-y)H(x,y)}\left(d\bar\tau + \frac{G(x,y)}{F(x,y)}d\bar\phi\right)^2 + \frac{\kappa H(x,y)}{(x-y)^3}\left(\frac{dx^2}{X(x)} - \frac{dy^2}{Y(y)} - \frac{X(x)Y(y)}{\kappa F(x,y)}d\bar\phi^2\right). \tag{8.8}$$

where the auxiliary angles $(\bar\tau, \bar\phi)$ are related to the canonical $2\pi-$periodic coordinates (ϕ^1, ϕ^2) by

$$\bar\tau = \frac{b_1}{k_1}\phi^2 + \frac{b_2}{k_2}\phi^1, \qquad \bar\phi = \frac{\phi^2}{k_1} + \frac{\phi^1}{k_2}. \tag{8.9}$$

where explicit expressions for the constants (b_i, k_i) are given in [35, Eqs 175-175] and the metric functions are given in [35, Eq 165]. The coordinates (y, x) parameterize the interior of a rectangle $x_1 < y < x_2 < x < x_3$ where x_i are the roots of a quartic $P(u)$ with $X(x) = P(x), Y(y) = P(y)$. There is a (twisted) AF end, not covered in this coordinate chart, which arises as $x \to x_2^+, y \to x_2^-$. The remaining functions $F(x,y), G(x,y), H(x,y)$ are bivariate polynomials of degree 6, 8, and 3 respectively. This is a two-parameter family characterized by an overall scale parameter $\kappa > 0$ and a parameter $\xi \in (1/2, 1/\sqrt{2})$. We may pass to the standard (r,θ) chart by setting

$$x = x_2 - \frac{x_2\sqrt{1-\nu^2)\kappa}\cos^2\left(\frac{\theta}{2}\right)}{r}, \qquad y = x_2 + \frac{x_2\sqrt{1-\nu^2)\kappa}\sin^2\left(\frac{\theta}{2}\right)}{r}. \tag{8.10}$$

In terms of these, the asymptotic moduli are

$$\ell = \frac{8\sqrt{\kappa}\xi^4}{\sqrt{1-4\xi^4}(2\xi^2 - 2\xi + 1)^2}, \qquad \beta = \frac{(1-\xi)^2\sqrt{1-4\xi^4}}{2\sqrt{\kappa}\xi^2}. \tag{8.11}$$

The canonical variables are (ρ, z) are then

$$\begin{aligned} \rho &= \left(\frac{b_2 - b_1}{k_1 k_2}\right)\cdot\frac{\sqrt{-X(x)Y(y)}}{(x-y)^2} \\ z &= \left(\frac{b_2 - b_1}{k_1 k_2}\right)\cdot\frac{2(a_0 + a_2 xy + a_4 x^2y^2) + (x+y)(a_1 + a_3 xy)}{2(x-y)^2}. \end{aligned} \tag{8.12}$$

where a_i are constants (see [35, pg. 28]). A computation yields the expansions

$$\begin{aligned} e^{2\alpha} &= \frac{1}{\ell^2}\left[1 + \frac{(1+2\xi^2)\sqrt{k(1-4\xi^4)}}{(1-2\xi^2)r} + O(r^{-2})\right], \\ V &= \log\left(\frac{\rho}{\ell^2}\right) + \frac{\sqrt{\kappa}(1+2\xi^2)^2}{\sqrt{1-4\xi^4}r} + O(r^{-2}), \end{aligned} \tag{8.13}$$

Using $\partial_r \log\rho = \frac{1}{r} + O(r^{-2})$ and the formula (5.48) the mass of the two-parameter family of Chen-Teo instantons is

$$\text{mass}_b(M, g_{CT}) = \frac{2\pi\ell(1+2\xi^2)^2\sqrt{\kappa}}{\sqrt{1-4\xi^4}} > 0. \tag{8.14}$$

8.1.3. *Reissner-Nordstrom instanton.* This is a two-parameter family of scalar-flat instantons $(\mathbb{R}^2 \times \mathbb{S}^2, g_{RN})$ with smooth metirc

$$g_{RN} = \ell^2 U(r)(d\phi^2)^2 + \frac{dr^2}{U(r)} + r^2\left(d\theta^2 + \sin^2\theta(d\phi^1)^2\right), \quad U(r) := 1 - \frac{2M}{r} + \frac{c_1}{r^2} \tag{8.15}$$

where (ϕ^1, ϕ^2) have 2π-period, $r > r_+ := M + \sqrt{M^2 - c_1}$, $\theta \in (0, \pi)$, and regularity requires

$$\ell = \frac{r_+^2}{\sqrt{M^2 - c_1}}. \tag{8.16}$$

Regularity of the metric merely requires $r_+ > 0$. Note that M, c_1 must satisfy $c_1 \leq M^2$ and if $c_1 > 0$, then we require $M > 0$. In practice, it is convenient to express the solution in terms of (r_+, c_1). By using

$$M = \frac{r_+^2 + c_1}{2r_+},$$

we have

$$U(r) = \frac{(r - r_+)(r - c_1 r_+^{-1})}{r^2 r_+^2}, \qquad \ell = \frac{2r_+^3}{r_+^2 - c_1}.$$

We have defined r_+ to be the largest (positive) root of $U(r)$. This means we must impose $r_+^2 > c_1$ (note that c_1 can have either sign). There is a curvature singularity in the metric at $r = 0$.

The canonical coordinates are obtained by setting

$$\rho = \ell\sqrt{r^2 - 2Mr + c_1}\sin\theta, \quad z = \ell\,(r - M)\cos\theta. \tag{8.17}$$

with

$$\alpha_{RN} = \frac{1}{2}\log\left[\frac{r^2}{\ell^2((r-M)^2 - (M^2 - c_1)\cos^2\theta)}\right], \qquad V_{RN} = \frac{1}{2}\log\left[\frac{\rho^2}{\ell^4 U(r)^2}\right]. \tag{8.18}$$

It is straightforward to read off the rod structure associated to the solution. Let $z_2 = -z_1 := \ell\sqrt{M^2 - c_1} > 0$. Then we find

1. a semi-infinite rod $\Gamma_1 = (-\infty, z_1)$ with rod vector $(\partial_{\phi^1}, \partial_{\phi^2}) = (1, 0)$,
2. a finite rod $\Gamma_2 = (z_1, z_2)$ with rod vector $(0, 1)$;
3. a semi-infinite rod $\Gamma_3 = (z_2, \infty)$ with rod vector $(1, 0)$

We compute the mass of the Reissner-Nordstrom metric. The definition of mass is

$$\begin{aligned}\operatorname{mass}(M, g) = \frac{1}{4\pi}\lim_{r\to\infty}\int_{\Sigma^3} &(-\partial_r\,(2\alpha - 2\alpha_b + 2Z) \\ &+ (2\alpha - 2\alpha_b - Z)\,\partial_r\log\rho - (V - V_b)\,\partial_r V_b)\,\ell r^2\sin\theta d\theta d\phi^1 d\phi^2\end{aligned} \tag{8.19}$$

Here $\alpha_b = -\log\ell$ and $V_b = \log(\rho) - 2\log\ell$. The asymptotic of α_{RN} and V_{RN} are as follows.

$$\partial_r\log\rho = \frac{1}{r} + \frac{M}{r^2} + O(r^{-3}), \quad \alpha_{RN} = \alpha_b + \frac{M}{r} + O(r^{-2}), \qquad V_{RN} = V_b + \frac{2M}{r} + O(r^{-2}). \tag{8.20}$$

This gives

$$\operatorname{mass}(M, g) = \frac{1}{4\pi}\lim_{r\to\infty}\int_{\Sigma^3}\left(\frac{2M}{r^2}\right)\ell r^2\sin\theta d\theta d\phi^1 d\phi^2 = 4\pi M\ell. \tag{8.21}$$

Observe that $\operatorname{mass}(M, g)$ can be negative provided $M < 0, c_1 < 0$.

8.1.4. *Checking Theorem 1.7 for Reissner-Nordstrom in Comparison to Schwarzschild.* Here we provide a simple illustration of our main Theorem by fixing the underlying manifold $\mathbb{R}^2\times\mathbb{S}^2$ and showing that the difference in the mass of a Reissner-Nordstrom and Schwarzschild instanton with the same rod structure satisfies the inequality (1.7). The metric of the Schwarzschild instanton is recovered from the Reissner-Nordstrom metric by setting $c_1 = 0$. Clearly, they have the same rod structure and the cone angles on semi-infinite rods I_1 and I_3 are zero. We have the following lemma for the finite rod.

Lemma 8.1. *The cone angle on finite rod Γ_2 for Reissner-Nordstrom and Schwarzschild instantons is as follows.*

$$\boldsymbol{\vartheta}^2_{RN} = \frac{1}{2}\log\left(\frac{(z_1+\sqrt{z_1^2+\ell^2 c_1})^4}{\ell^4 z_1^2}\right), \qquad \boldsymbol{\vartheta}^2_{S} = \frac{1}{2}\log\left(16\ell^{-4}z_1^2\right) \tag{8.22}$$

Proof. The metric of Reissner-Nordstrom is

$$g_{RN} = e^{2\alpha_{RN}}\left(d\rho^2+dz^2\right) + G_{RN11}(d\phi^1)^2 + G_{RN22}(d\phi^2)^2 \tag{8.23}$$

Define $R_{z_i} := \sqrt{\rho^2+(z-z_i)^2}$ and

$$\alpha_{RN} = \frac{1}{2}\log\left(\frac{\left(R_{z_1}+R_{z_2}+2\sqrt{z_2^2+\ell^2c_1}\right)^2}{4\ell^2 R_{z_1}R_{z_2}}\right) \tag{8.24}$$

$$G_{RN22} = \frac{\ell^2\left(R_{z_1}+R_{z_2}\right)^2 - 2\ell^2(z_2^2+z_1^2)}{(R_{z_1}+R_{z_2}+2\sqrt{z_2^2+\ell^2c_1})^2} = \rho\Phi_{RN22} \tag{8.25}$$

$$G_{RN11} = \rho^2\frac{\left(R_{z_1}+R_{z_2}+2\sqrt{z_2^2+\ell^2c_1}\right)^2}{\ell^2\left(R_{z_1}+R_{z_2}\right)^2 - 2\ell^2(z_2^2+z_1^2)} = \rho\Phi_{RN11} \tag{8.26}$$

We compute the right-hand side of the above equation.

$$\left(\alpha_{RN}+\frac{1}{2}\log\rho-\frac{1}{2}\log(\Phi_{RN22})\right)\Big|_{\Gamma_2} = \frac{1}{2}\log\left(\frac{\rho^2(R_{z_1}+R_{z_2}+2\sqrt{z_2^2+\ell^2c_1})^4}{4\ell^4R_{z_1}R_{z_2}\left(\left(R_{z_1}+R_{z_2}\right)^2-2(z_2^2+z_1^2)\right)}\right) \tag{8.27}$$

On axis rod I_2 we have

$$R_{z_1} = \sqrt{\rho^2+(z-z_1)^2} = (z-z_1) + \frac{1}{2(z-z_1)}\rho^2 + O_1(\rho^4), \tag{8.28}$$

$$R_{z_2} = \sqrt{\rho^2+(z-z_2)^2} = -(z-z_2) - \frac{1}{2(z-z_2)}\rho^2 + O_1(\rho^4). \tag{8.29}$$

Then

$$\left(R_{z_1}+R_{z_2}+2\sqrt{z_2^2+L^2c_1}\right)^4 = \left(z_2-z_1+2\sqrt{z_2^2+L^2c_1}\right)^4 + O_1(\rho^2) \tag{8.30}$$

$$(R_{z_1}+R_{z_2})^2 - 2(z_2^2+z_1^2) = \frac{(z_2-z_1)^2}{(z_2-z)(z-z_1)}\rho^2 + O_1(\rho^4), \qquad \text{since} \quad z_2=-z_1 \tag{8.31}$$

$$R_1R_2 = -4(z-z_1)(z-z_2) + O_1(\rho^2) \tag{8.32}$$

Therefore, we have

$$\boldsymbol{\vartheta}^2_{RN} = \frac{1}{2}\log\left(\frac{(z_2 - z_1 + 2\sqrt{z_2^2 + \ell^2 c_1})^4}{4\ell^4(z_2-z_1)^2}\right) = \frac{1}{2}\log\left(\frac{(z_1 + \sqrt{z_1^2+\ell^2 c_1})^4}{\ell^4 z_1^2}\right) \tag{8.33}$$

Setting $c_1 = 0$, we obtain the cone angle for Schwarzschild instanton. □

To compare the Reissner-Nordström and Schwarzschild families of instanton and verify Theorem 1.7, we need to have the same size for length of rod Γ_2. To achieve this, the mass parameter M of Schwarzschild instanton is $\sqrt{M^2 - c_1}$. Define the following quantity.

$$\mathcal{P}(M, c_1) := \text{mass}_b\,(M, g_{RN}) - \text{mass}_b\,(M, g_S) - 2\pi \sum_{n=1}^{N+1} \int_{\Gamma_n} (\boldsymbol{\vartheta}^n - \boldsymbol{\vartheta}^n_o) dz \tag{8.34}$$

Lemma 8.2. *For the Reissner-Nordström and Schwarzschild instantons we have $\mathcal{P}(M, c_1) \geq 0$ and equality holds if and only if $c_1 = 0$ in the Reissner-Nordström instanton*

Proof. Observe that $\text{mass}_b\,(M_{RN}, g_{RN}) = 4\pi M\ell$ and $\text{mass}_b\,(M_o, g_o) = 4\pi\ell\sqrt{M^2 - c_1}$. Moreover, $z_2 = -z_1 = \ell\sqrt{M^2 - c_1}$. Then

$$\begin{aligned}
(4\pi\ell)^{-1}\mathcal{P} &= \text{mass}\,(M, g) - \text{mass}\,(M_o, g_o) - \frac{1}{2\ell}(\boldsymbol{\vartheta}^n_{RN} - \boldsymbol{\vartheta}^n_S)\text{length}(I_2) \\
&= M - \sqrt{M^2 - c_1} + \frac{1}{4\ell}\log\left(\frac{16(\sqrt{M^2 - c_1})^4}{(\sqrt{M^2 - c_1} + M)^4}\right)(z_2 - z_1) \\
&= M - \sqrt{M^2 - c_1} + 2\sqrt{M^2 - c_1}\log\left(\frac{2\sqrt{M^2 - c_1}}{\sqrt{M^2 - c_1} + M}\right)
\end{aligned} \tag{8.35}$$

Define

$$x \equiv \log\left(\frac{2\sqrt{M^2 - c_1}}{\sqrt{M^2 - c_1} + M}\right) \tag{8.36}$$

Then we have

$$M = \frac{2 - e^x}{e^x}\sqrt{M^2 - c_1} \tag{8.37}$$

Therefore, we can rewrite (8.35) as

$$\mathcal{P}(M, c_1) = 2e^{-x}\sqrt{M^2 - c_1}\,(1 - e^x + xe^x) \tag{8.38}$$

Clearly, $\mathcal{P} \geq 0$ because $1 - e^x + xe^x$ is decreasing for $x < 0$ and increasing for $x > 0$ and zero at $x = 0$. Moreover, $x = 0$ is equivalent to $c_1 = 0$. □

8.2. **ALF Riemannian Manifolds.** We present here three explicit families of ALF geometries with $k = 1$ so that the boundary at infinity is S^3; the first two are Ricci flat and the third is scalar flat.

8.2.1. *Taub-NUT gravitational instantons.* The (Ricci-flat) Taub-NUT space $(\mathbb{R}^4, g_{RN})$ is a complete, ALF $k = 1$ gravitational instanton. In local coordaintes the metric is given by

$$\begin{aligned}
g_{TN} &= H^{-1}\ell^2(d\phi^2 + \cos^2\left(\frac{\theta}{2}\right)d\phi^1)^2 + H(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta(d\phi^1)^2) \\
H &= 1 + \frac{\ell}{2r}
\end{aligned} \tag{8.39}$$

where (ϕ^1, ϕ^2) are independently $2\pi-$periodic angles and $r > 0, \theta \in (0, \pi)$. This is a one-parameter family of metrics parametrrized by the radius ℓ of the S^1 at infinity. The associated $2\pi-$periodic generators are ∂_{ϕ^1} and ∂_{ϕ^2}. We then may select

$$\rho = \ell r \sin\theta, \qquad z = \ell r \cos\theta, \tag{8.40}$$

from which we read off

$$\alpha_{TN} = \frac{1}{2}\log\left[\frac{H}{\ell^2}\right]. \tag{8.41}$$

The rod structure for a space asymptotic to Taub-NUT would have a semi-infinite rod $(-\infty, z_1)$ with rod vector in the (ℓ_1, ℓ_2) basis is $(1, 0)$ (corresponding to $\theta = \pi$) and a semi infinite rod (z_N, ∞) with rod vector $(1, -1)$ (corresponding to $\theta = 0$). The asymptotic boundary is topologically $L(1, 1) = S^3$ with one direction (the S^1 fibre) having bounded size while the S^2 base grows to infinite size. Notice that $r = 0$ is a corner point where both generators degenerate. As is well known, although the local metric has a coordinate singularity at $r = 0$, the point $r = 0$ can be added so that g_{TN} extends to a smooth metric on $\mathbb{R}^4$. In our formalism, this can be seen by observing that the rod vectors satisfy the admissibility condition that the determinant of the matrix whose columns are the adjacent rod vectors has unit modulus. The topology of Taub-NUT is thus that of $\mathbb{R}^4$.

Then we can choose

$$\Phi = \frac{1}{\ell r \sin\theta}\begin{pmatrix} \frac{\ell^2}{H}\cos^4\left(\frac{\theta}{2}\right) + Hr^2\sin^2\theta & \frac{\ell^2}{H}\cos^2\left(\frac{\theta}{2}\right) \\ \frac{\ell^2}{H}\cos^2\left(\frac{\theta}{2}\right) & \frac{\ell^2}{H}\end{pmatrix} \tag{8.42}$$

The Taub-NUT space is itself Ricci-flat. One finds that

$$V - V_b = \frac{\ell}{2r} + O(r^{-2}), \qquad \alpha - \alpha_b = \frac{\ell}{4r} + O(r^{-2}). \tag{8.43}$$

Using the formula (5.48), one finds

$$\mathrm{mass}(M, g) = \pi\ell^2. \tag{8.44}$$

8.2.2. *Taub-Bolt gravitational instanton.* The one parameter family of Euclidean Taub-Bolt gravitational instantons are Ricci-flat and ALF with S^3 asymptotic boundary. In local coordinates,

$$\begin{aligned} g_{TB} &= U(r)\ell^2\left(d\phi^2 + \cos^2\left(\frac{\theta}{2}\right)d\phi^1\right)^2 + \frac{dr^2}{U(r)} + \left(r^2 - \frac{\ell^2}{16}\right)\left(d\theta^2 + \sin^2\theta(d\phi^1)^2\right) \\ U(r) &= \frac{(2r-\ell)(8r-\ell)}{16r^2 - \ell^2} \end{aligned} \tag{8.45}$$

where $r \in (\ell/2, \infty), \theta \in (0, \pi)$, and (ϕ^1, ϕ^2) are independently $2\pi-$periodic angles. The metric is parameterized by $\ell > 0$ which characterizes the radius of the S^1 in the asymptotic region. In this basis of generators $(\partial_{\phi^1}, \partial_{\phi^2})$, the rod structure consists of (1) a semi-infinite rod $r > \ell/2$, $\theta = \pi$ $(1, 0)$, (2) a finite rod $r = \ell/2$, $\theta \in (0, \pi)$ with rod vector $(0, 1)$, and (3) a semi-infinite rod $r > \ell/2$, $\theta = 0$ with rod vector $(1, -1)$. It therefore has the same asymptotic behaviour as the Taub-NUT instanton discussed above. Thus it can be considered as the result of adding a finite rod or S^2, to the Taub-NUT space. This is analogous to Euclidean Schwarzschild having an extra finite rod relative to the vacuum $S^1 \times \mathbb{R}^3$. Canonical coordinates are obtained by defining

$$\rho = \frac{\ell}{4}\sqrt{(2r-\ell)(8r-\ell)}\sin\theta, \qquad z = \ell\left(r - \frac{5\ell}{16}\right)\cos\theta. \tag{8.46}$$

and the conformal factor is

$$\alpha_{TB} = \frac{1}{2}\log\left[\frac{16(16r^2-\ell^2)}{\ell^2\left[(16r-5\ell)^2-9\ell^2\cos^2\theta\right]}\right]. \tag{8.47}$$

The associated harmonic map matrix is

$$\Phi = \rho^{-1}\begin{pmatrix} U(r)\ell^2\cos^4\left(\frac{\theta}{2}\right)+\left(r^2-\frac{\ell^2}{16}\right)\sin^2\theta & U(r)\ell^2\cos^2\left(\frac{\theta}{2}\right) \\ U(r)\ell^2\cos^2\left(\frac{\theta}{2}\right) & U(r)\ell^2 \end{pmatrix}. \tag{8.48}$$

To compute the mass, the appropriate reference space is the Taub-NUT geometry with the same L. We find

$$\alpha_{TB}-\alpha_b = \frac{5\ell}{16r}+O(r^{-2}), \qquad V_{TB}-V_b = \frac{5\ell}{8r}+O(r^{-2}). \tag{8.49}$$

Using the formula (5.48) yields

$$\mathrm{mass}(M,g) = \frac{5\pi\ell^2}{4}. \tag{8.50}$$

8.2.3. *Charged Taub-Bolt.* The following two-paramter familiy of complete, ALF scalar-flat metrics can be obtained by a suitable analytic continuation of a local family of Lorentzian metrics that satisfy the Einstein-Maxwell equations. It can be thought of as a one-parameter 'charged' generalization of the Ricci flat Taub-Bolt solution in the same way Reisner-Nordstrom contains the Schwarzschild instanton. In the standard coordinate system, the local metric is given by

$$g = \frac{\ell^2F(r)}{r^2-\frac{\ell^2}{16}}\left(d\phi^2+\cos^2\left(\frac{\theta}{2}\right)d\phi^1\right)^2+\left(r^2-\frac{\ell^2}{16}\right)\left[\frac{\mathrm{d}r^2}{F(r)}+\mathrm{d}\theta^2+\sin^2\theta(d\phi^1)^2\right] \tag{8.51}$$

where

$$F(r) = (r-r_+)\left(r-\left(r_++\frac{\ell}{8}-\frac{2r_+^2}{\ell}\right)\right) \tag{8.52}$$

The solution is parameterized by the positive parameters (r_+,ℓ) with $r_+>\ell/4$ and where the coordinate ranges $r>r_+$, $\theta\in(0,\pi)$, and (ϕ^1,ϕ^2) ane independently $2\pi-$periodic angles. The apparent singularity of the metric as $r\to r_+$ can be smoothly resolved by adding in a sphere S^2 'bolt' at $r=r_+$. Observe that r_+ is the largest root of $F(r)$ because $r_+>r_++\ell/8-2r_+^2/\ell$. It is straighgtforward to verify that the asymptotic geometry as $r\to\infty$ is ALF with asymptotic boundary S^3.

Note that the Taub-NUT and Taub-Bolt metrics can be recovered by setting $r_+=\ell/4$ and $r_+=\ell/2$ respectively (in the former case, the radial coordinate r must be shifted in order to recover the explicit metric g_{TN} (8.39)).

Observe that the function $\rho=\sqrt{\det G}$ is harmonic on the 2d orbit space (in this case, parameterized by (r,θ)). We can also integrate for the harmonic conjugate z to produce the caonical coordiantes

$$\rho = \sqrt{F}\ell\sin\theta, \qquad z = \left(r-\frac{\ell}{16}-r_++\frac{r_+^2}{\ell}\right)\ell\cos\theta. \tag{8.53}$$

The rod structure $(1,0)$, $(0,1)$, and $(1,-1)$ in the $(\partial_{\phi^1},\partial_{\phi^2})$ basis. A computation yields

$$\alpha = \frac{1}{2}\log\left[\left(r^2-\frac{\ell^2}{16}\right)\frac{256}{P^2-(\ell^2\cos\theta-16\cos\theta r_+^2)^2}\right], \tag{8.54}$$

where $P=\ell^2-16r\ell+16r_+\ell-16r_+^2$. Moreover,

$$V=\frac{1}{2}\log\left[\frac{\left(r^2-\frac{\ell^2}{16}\right)^2\sin^2\theta+\ell^2F(r)\cos^4\frac{\theta}{2}}{\ell^2F(r)}\right]. \tag{8.55}$$

we obtain

$$\alpha-\alpha_b=\frac{c}{r}+O((r^{-2}),\quad V-V_b=\frac{2c}{r}+O((r^{-2}),\qquad c=r_++\frac{\ell}{16}-\frac{r_+^2}{\ell}. \tag{8.56}$$

Using (5.48), this yields

$$\text{mass}(M,g)=4\pi\ell c. \tag{8.57}$$

Observe that for Taub-NUT ($c=\ell/4$) and Taub-Bolt ($c=5\ell/16$) (8.57) agrees with our previous computation of the mass in those cases.

8.3. **ALE geometries.**

8.3.1. *Eguchi-Hanson instanton.* This is a one-parameter family of hyperKähler metrics on $M=T^*\mathbb{S}^2$ with metric given by

$$\begin{aligned}g_{EH}&=\frac{dr^2}{f(r)}+r^2\left[d\theta^2+\frac{f(r)}{4}\left(d\phi^1+2\cos^2\theta d\phi^2\right)^2+\frac{\sin^2 2\theta}{4}(d\phi^2)^2\right]\\&=\frac{dr^2}{f(r)}+r^2\left[d\theta^2+\frac{\sin^2\theta(d\phi^1)^2}{4}+\cos^2\theta\left(d\phi^2+\frac{d\phi^1}{2}\right)^2-\frac{a^4}{4r^4}\left(d\phi^1+2\cos^2\theta d\phi^2\right)^2\right].\end{aligned} \tag{8.58}$$

where $f(r)=1-a^4/r^4$, $a>0$. The second form of the metric exhibits clearly that the Eguchi-Hanson metric is ALE with boundary at infinity $L(2,1)$. The harmonic map is

$$\Phi=\rho^{-1}\begin{pmatrix}\frac{r^2f(r)}{4} & \frac{r^2f(r)}{2}\cos^2\theta\\ \frac{r^2f(r)}{2}\cos^2\theta & r^2\cos^2\theta\left(f(r)\cos^2\theta+\sin^2\theta\right)\end{pmatrix} \tag{8.59}$$

where the Weyl coordinates are

$$\rho=\frac{1}{4}\sqrt{r^4-a^4}\sin 2\theta,\qquad z=\frac{r^2}{4}\cos 2\theta. \tag{8.60}$$

so that the orbit space metric is $e^{2\alpha}(\mathrm{d}\rho^2+\mathrm{d}z^2)$ where

$$e^{2\alpha_{EH}}=\frac{4r^2}{r^4-a^4\cos^2 2\theta},\qquad \alpha_{EH}=\frac{1}{2}\log\left[\frac{4r^2}{r^4-a^4\cos^2 2\theta}\right]. \tag{8.61}$$

We may also read off

$$V_{EH}=\frac{1}{2}\log\left[\frac{f(r)}{4\cos^2\theta(f(r)\cos^2\theta+\sin^2\theta)}\right],\qquad W_{EH}=\operatorname{arcsinh}\left[\frac{r^2f(r)}{2\rho}\cos^2\theta\right]. \tag{8.62}$$

We can read off the rod structure associated to the basis $(\partial_{\phi^1},\partial_{\phi^2})$; there is a semi-infinite rod I_1 with $z\in(-\infty,-a^2/4)$ with rod vector $(0,1)$; a finite rod $z\in(-a^2/4,a^2/4)$ and rod vector $(1,0)$, and a semi infinite rod $z\in(a^2/4,\infty)$ with rod vector $(2,-1)$.

To compute the mass, observe that

$$
\begin{aligned}
\alpha_{EH} - \alpha_b &= O(r^{-4}), \\
V_{EH} &= -\log(2\cos\theta) - \frac{a^4 \sin^2\theta}{2r^4} + O(r^{-6}) \\
W_{EH} &= \log(\cot\theta + \csc\theta) - \frac{a^4\cos\theta}{2r^4} + O(r^{-6})
\end{aligned} \tag{8.63}
$$

and hence from (5.47) we confirm that the Eguchi-Hanson instanton has vanishing mass.

Appendix A. Ricci and Scalar Curvature computations

Consider a Riemannian manifold (M, g) of dimension n whose metric takes the following local form (2.22)

$$
g = \hat{g}_{ab} dx^a dx^b + G_{ij}(d\phi^i + A^i_a dx^a)(d\phi^j + A^j_b dx^b) \tag{A.1}
$$

where $\partial/\partial\phi^i$, $i = 1, \dots n-2$ are Killing vector fields generating the Abelian isometry group $\mathbb{T}^{n-2}$ and x^a, $a = 1, 2$ are coordinates on the space transverse to the torus action, $\hat{g}_{ab} = g(\partial/\partial x^a, \partial/\partial x^b)$, G_{ij} is a matrix valued function and the one-forms $A^j = A^j_b dx^b$ measure the failure of the distribution orthogonal to the torus action to be locally integrable. The principal orbits of the torus action are two-dimensional and (A.1) describe a class of cohomogeneity-two metrics. All functions appearing in the metric depend only on $\{x^a\}$. The inverse metric is

$$
g^{ab} = \hat{g}^{ab}, \qquad g^{ij} = G^{ij} + \hat{g}^{ab} A^i_a A^j_b, \qquad g^{ia} = -\hat{g}^{ab} A^i_b \tag{A.2}
$$

The Ricci curvature is (see e.g., [25, Appendix A] or [42, (4.6)]):

$$
\begin{aligned}
R_{ij} &= -\frac{1}{2}\partial_a\partial^a G_{ij} - \frac{1}{4}\partial_a\left(\log\det G + \log\det\hat{g}\right)\partial^a G_{ij} \\
&\quad + \frac{1}{2}\partial^a G_{ik} G^{kl} \partial_b G_{lj} + \frac{1}{4} G_{ik} G_{jl} \hat{g}^{ac} g^{bd} F^k_{ab} F^l_{cd} \\
R_{ia} &= R_{ij} A^j_a + \frac{1}{2\sqrt{\det G \det\hat{g}}}\hat{g}_{ab}\partial_c\left(\sqrt{\det G\det\hat{g}}\, G_{ij}\hat{g}^{bd}\hat{g}^{ce} F^J_{de}\right) \\
R_{ab} &= -R_{ij} A^i_a A^j_b + R_{ia} A^i_b + R_{ib} A^i_a - \frac{1}{2}\hat{g}^{cd} G_{ij} F^i_{ac} F^j_{bd} + \hat{R}_{ab} \\
&\quad - \frac{1}{2}\hat{\nabla}_a\hat{\nabla}_b \log\det G - \frac{1}{4}\mathrm{Tr}\left(G^{-1}\partial_a G G^{-1}\partial_b G\right)
\end{aligned} \tag{A.3}
$$

where $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a$ and $\hat{\nabla}$ is the Levi-Civita connection with respect to $\hat{g}$. The scalar curvature of (M, g) is

$$
\begin{aligned}
R &= \hat{g}^{ab} R_{ab} + \left(G^{ij} + \hat{g}^{ab} A^i_a A^j_b\right) R_{ij} - 2\hat{g}^{ab} A^i_b R_{ai} \\
&= \hat{R} - \hat{g}^{ab} G^{ij}\hat{\nabla}_a\hat{\nabla}_b G_{ij} + \frac{3}{4}\hat{g}^{ab} G^{ij}\hat{\nabla}_a G_{jk} G^{kl}\hat{\nabla}_b G_{li} - \frac{1}{4}\hat{g}^{ab} G^{ij}\hat{\nabla}_a G_{ij} G^{kl}\hat{\nabla}_b G_{kl} \\
&\quad - \frac{1}{4}\hat{g}^{ac}\hat{g}^{bd} G_{ij} F^i_{ab} F^j_{cd}
\end{aligned} \tag{A.4}
$$

[[Add citation to John Lott's formula, equation 4.7 of his ”Dimensional reduction...” paper.]] At this point we have not chosen any coordinates on the two-dimensional space of orbits of the torus action. As shown in [29], there is a natural coordinate chart $x^a = (\rho, z)$ on the right half plane $\rho > 0, z \in \mathbb{R}$ with axis $\rho = 0$ parameterizing the boundary of the orbit space consisting of axes Γ_i and corner points z_i where G has rank 1 and rank 0 respectively. in which the transverse metric on the orbit space takes the conformally flat form

$$\hat{g} = e^{2\alpha}\left(d\rho^2 + dz^2\right) \tag{A.5}$$

for some smooth function α. In this chart the scalar curvature is

$$\begin{aligned} e^{2\alpha}R = &-2\Delta_2\alpha - \left(\frac{\Delta_2 \det G}{\det G} + \frac{1}{4}\mathrm{Tr}\left(G^{-1}\nabla G\right)^2 - \frac{3}{4}|\nabla \log\det G|^2\right) \\ &- \frac{1}{4}e^{-2\alpha}\delta_2^{ac}\delta_2^{bd}G_{ij}F^i_{ab}F^j_{cd}. \end{aligned} \tag{A.6}$$

where the gradient ∇ and Laplacian Δ_2 are with respect to the flat metric $\delta_2 = d\rho^2 + dz^2$.

Let us now consider the class of Ricci flat metrics. The condition $\mathrm{Ric}(g) = 0$ applied to (A.1) is equivalent to the following system on the orbit space:

$$\begin{aligned} &\partial_a\left(\sqrt{\det G \det \hat{g}}\,\hat{g}^{ab}G^{ik}\partial_b G_{kj}\right) = \frac{1}{2}\sqrt{\det G\det\hat{g}}\,F^i_{ab}G_{jk}\hat{g}^{ac}\hat{g}^{bd}F^k_{cd} \\ &\partial_a\left(\sqrt{\det G\det\hat{g}}\,G_{ij}\hat{g}^{ac}\hat{g}^{bd}F^j_{cd}\right) = 0 \\ &\hat{R}_{ab} = \frac{1}{4}\mathrm{Tr}\left(G^{-1}\partial_a G G^{-1}\partial_b G\right) + \frac{1}{2}\hat{\nabla}_a\hat{\nabla}_b \log\det G + \frac{1}{2}\hat{g}^{cd}G_{ij}F^i_{ac}F^j_{bc} \end{aligned} \tag{A.7}$$

A standard argument applying Froebenius' theorem along with Ricci flatness and the assumption that the axis set is non-empty implies that the distribution orthogonal to the torus action is locally integrable [35].

Proposition A.1. *In the Ricci flat setting, the two-plane distribution orthogonal to the torus generators is integrable, implying $A^i_a = 0$*

Proof. Next, let $\eta_i = \partial/\partial\phi^i$, $i = 1, 2, \dots n-2$ denote the set of mutually commuting Killing vector fields generating the torus symmetry. This implies $\mathcal{L}_{\eta_i} g = 0$ and $[\eta_i, \eta_j] = 0$. The fixed point sets of the torus action correspond to sets along which one or more linear combinations of the η_i degenerate. Define the *axis set* by

$$\Gamma := \{p \in M \mid \det G_{ij}|_p = 0\}. \tag{A.8}$$

where $G_{ij} = g(\eta_i, \eta_j)$. Define the smooth functions

$$\omega_i := \star(\eta_1 \wedge \eta_2 \wedge \dots \wedge \eta_n \wedge \mathrm{d}\eta_i) \tag{A.9}$$

where for convenience, we have used the same symbol η_i to denote the metric dual one-forms $g(\eta_i, \ )$. We now demonstrate that the functions ω_i are constant and moreover, if Γ is non-empty, then $\omega_i \equiv 0$ and the torus action is orthogonally transitive. The result $d\omega_i = 0$ follows using the identity $d \star d\eta = -2(-1)^n \star \mathrm{Ric}(\eta)$ valid for any Killing field η, Ricci-flatness, repeated use of Cartan's formula $\mathcal{L}_\eta \alpha = di_\eta\alpha + i_\eta d\alpha$ for any smooth $p-$form α, and the fact that the generators commute. Hence $\omega_i = c_i$ are constants (recall that M is connected). If $\Gamma \neq \emptyset$, each $c_i = 0$ since at least one linear combination of the $\{\eta_i\}$ degenerates in M. Then, by Frobenius' integrability theorem, since $\omega_i \equiv 0$, the torus action is orthogonally transitive, i.e. the 2-plane distribution orthogonal to $\mathrm{span}(\eta_1, \eta_2, \dots \eta_{n-2}) \subset TM$ is integrable at every point. This implies $g_{ai} = 0$ for $a = \{\rho, z\}$ and hence $A^i_a = 0$. □

Since $A^i = 0$, the second condition (A.7) is automatically satisfied. The trace of the first equation of (A.7) implies that the function $\sqrt{\det G}$ is harmonic with respect to $\hat{g}$, that is

$$\hat{\Delta}\sqrt{\det G} = \hat{\nabla}_a\hat{\nabla}^a\sqrt{\det G} = 0. \tag{A.10}$$

We may then identify ρ with $\sqrt{\det G}$.

Proposition A.2. *In the Ricci flat setting, $\sqrt{\det G} = \rho$ in the (ρ, z) coordinate chart.*

Proof. Both $\sqrt{\det G}$ and ρ are harmonic with respect to $\hat{g}$ and considered as functions on the right half plane, both vanish on the boundary (since the torus action degenerates on the boundary $\rho = 0$, $\operatorname{rank} G < 2$). Moreover for our choices of asymptotic behaviour, it is straightforward to check that $\sqrt{\det G} \sim \rho$. The claim then follows from the maximum principle. □

Using the identification $\rho^2 = \det G$, The Ricci flat equations (A.7) can then be rewritten as The equation $\mathrm{Ric}(g) = 0$ is equivalent to the following equations on the orbit space $(\hat{M}, \hat{g})$:

$$\hat{\nabla}_a \left(\rho G^{ik} \hat{\nabla}^a G_{kj}\right) = 0 \tag{A.11}$$

$$\hat{R}_{ab} = \frac{1}{4}\mathrm{Tr}\left(G^{-1}\partial_a G G^{-1}\partial_b G\right) + \hat{\nabla}_a\hat{\nabla}_b \log\rho \tag{A.12}$$

Proposition A.3. *$\hat{g}$ is entirely determined once a solution to (A.11) is known.*

Proof. The Ricci tensor of (A.5) is

$$\hat{R}_{ab} = -\Delta_2\alpha\delta_{ab} \tag{A.13}$$

where $\delta_{ab} = \mathrm{diag}(1,1)$. Therefore (A.12) reduces to

$$-\hat{\Delta}_2\alpha = -\frac{1}{\rho^2} - \frac{\partial_\rho\alpha}{\rho} - \frac{1}{4}\partial_\rho G^{ij}\partial_\rho G_{ij} \tag{A.14}$$

$$0 = -\frac{\partial_z\alpha}{\rho} - \frac{1}{4}\partial_\rho G^{ij}\partial_z G_{ij} \tag{A.15}$$

$$-\hat{\Delta}_2\alpha = \frac{\partial_\rho\alpha}{\rho} - \frac{1}{4}\partial_z G^{ij}\partial_z G_{ij} \tag{A.16}$$

which gives the conditions

$$\partial_z\alpha = -\frac{\rho}{4}\partial_\rho G^{ij}\partial_z G_{ij}, \qquad \partial_\rho\alpha = -\frac{1}{2\rho} + \frac{\rho}{8}\partial_z G^{ij}\partial_z G_{ij} - \frac{\rho}{8}\partial_\rho G^{ij}\partial_\rho G_{ij} \tag{A.17}$$

These can be written more compactly as

$$\partial_z\alpha = \frac{\rho}{4}\mathrm{Tr}(j_z j_\rho), \qquad \partial_\rho\alpha = -\frac{1}{2\rho} + \frac{\rho}{8}\mathrm{Tr}(j_\rho^2 - j_z^2) \tag{A.18}$$

where $j := G^{-1}\mathrm{d}G$ is a matrix valued one form. The integrability condition for the first order equations for α are equivalent to (A.11) which can be re-expressed in the form of a conservation law

$$\nabla_2 \cdot (\rho j) = 0. \tag{A.19}$$

□

Define the matrix $\Phi := (\det G)^{-\frac{1}{n-2}} G = \rho^{\frac{2}{2-n}} G$. Observe that $\det\Phi = 1$. We compute

$$\begin{aligned} 0 &= \hat{\nabla}_a\left(\rho G^{-1}\hat{\nabla}^a G\right) = \hat{\nabla}_a\left(\rho\rho^{\frac{2}{2-n}}\Phi^{-1}\hat{\nabla}^a(\rho^{\frac{2}{n-2}}\Phi)\right) \\ &= \hat{\nabla}_a\left(\rho\Phi^{-1}\hat{\nabla}^a\Phi\right) + \frac{2}{n-2}I_{n-2}\hat{\nabla}_a\hat{\nabla}^a\rho \\ &= \hat{\nabla}_a\left(\rho\Phi^{-1}\hat{\nabla}^a\Phi\right). \end{aligned} \tag{A.20}$$

Introduce an auxiliary angular coordinate $\varphi \sim \varphi + 2\pi$ and the three-dimensional Euclidean metric

$$\delta_3 = d\rho^2 + dz^2 + \rho^2 d\varphi^2. \tag{A.21}$$

The set of coordinates (ρ, z, φ) can then be identified with standard cylindrical coordinates on Euclidean $\mathbb{R}^3$. With respect to δ_3, (A.20) takes the form

$$\mathrm{div}_{\delta_3}\left(\Phi^{-1}\nabla\Phi\right) = 0 \tag{A.22}$$

where the divergence and ∇ operators are associated to δ_3. Thus once a solution for Φ is determined, we can obtain the matrix G and by the above arguments, α is determined up to an integration constant. Observing that Φ is a unimodular symmetric matrix of dimension $n-2$, we see the Ricci flat equations are equivalent to a harmonic map (A.20) given by $\Phi : \mathbb{R}^3 \to SL(n-2,\mathbb{R})/SO(n-2)$. Since $\det G = 0$ on the boundary of the orbit space (the $z-$ axis) Φ necessarily has singular behaviour on the axis that encodes the rod structure of M.

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