THE MASS-ANGULAR MOMENTUM INEQUALITY
FOR MULTIPLE BLACK HOLES

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ABSTRACT. This is the second in a series of two papers to establish the conjectured mass-angular momentum inequality for multiple black holes, modulo the extreme black hole 'no hair theorem'. More precisely it is shown that either there is a counterexample to black hole uniqueness, in the form of a regular axisymmetric stationary vacuum spacetime with an asymptotically flat end and multiple degenerate horizons which is 'ADM minimizing', or the following statement holds. Complete, simply connected, maximal initial data sets for the Einstein equations with multiple ends that are either asymptotically flat or asymptotically cylindrical, admit an ADM mass lower bound given by the square root of total angular momentum, under the assumption of nonnegative energy density and axisymmetry. Moreover, equality is achieved in the mass lower bound only for a constant time slice of an extreme Kerr spacetime. The proof is based on a novel flow of singular harmonic maps with hyperbolic plane target, under which the renormalized harmonic map energy is monotonically nonincreasing. Relevant properties of the flow are achieved through a refined asymptotic analysis of solutions to the harmonic map equations and their linearization.

1. INTRODUCTION

In [22] Penrose outlined heuristic arguments that give rise to the Penrose inequality [2, 14], as well as a conjectured lower bound [20, Section 8] for the ADM mass/energy m of a spacetime in terms of ADM angular momentum $\mathcal{J}$, which takes the form

$$(1.1) \qquad m \geq \sqrt{|\mathcal{J}|}.$$

The reasoning that leads to (1.1) requires conservation of angular momentum, and to achieve this it is typically assumed that the spacetime is axisymmetric and satisfies certain conditions on matter fields. In fact, counterexamples have been constructed by Huang-Schoen-Wang [13] when the axisymmetric assumption is removed. The original motivation for the Penrose inequality holds equally well for the mass-angular momentum inequality, namely they both serve as necessary conditions for the weak cosmic censorship [21, 23] and the final state [18] conjectures. Therefore, while a counterexample to the inequality would be detrimental for at least one of these latter two conjectures, confirmation of (1.1) only adds to the prevailing belief in their generic validity. Furthermore, inequality (1.1) may be interpreted as a refinement of the positive mass theorem [24, 31], in which a precise contribution to the total mass is expressed via the rotation of black holes. This manuscript is the second in a series of two papers to establish

the mass-angular momentum inequality for multiple black holes; the first paper in this series is [12].

The appropriate mathematical setting in which to study this inequality is that of an initial data set (M, g, k) for the Einstein equations, consisting of a smooth connected 3-manifold M with Riemannian metric g and a symmetric 2-tensor k representing the second fundamental form of an embedding into spacetime. These quantities satisfy the constraint equations

$$(1.2) \qquad 16\pi\mu = R + (\mathrm{Tr}_g k)^2 - |k|_g^2, \qquad 8\pi J = \mathrm{div}_g(k - (\mathrm{Tr}_g k)g),$$

where R is the scalar curvature of g, and μ, J denote the matter energy and momentum densities respectively. A natural hypothesis related to the dominant energy condition is the assumption of nonnegative energy density $\mu \geq 0$, which when combined with the *maximal slice* condition $\mathrm{Tr}_g k = 0$ guarantees nonnegative scalar curvature. Moreover, it will be assumed throughout that the data are *axially symmetric*. This means that a subgroup isomorphic to $U(1)$ is present within the isometry group of the Riemannian manifold, and that the extrinsic curvature is invariant under the $U(1)$ action. In particular, if η is the associated Killing field generating the symmetry and $\mathfrak{L}_\eta$ denotes Lie differentiation, then

$$(1.3) \qquad \mathfrak{L}_\eta g = \mathfrak{L}_\eta k = 0.$$

The manifold M will be *asymptotically flat with multiple ends*, in that there exists a compact set $\mathcal{C}$ and an integer $N \geq 2$ such that $M \setminus \mathcal{C} = \cup_{i=0}^N M_i$, where the pairwise disjoint ends M_i are either asymptotically flat or asymptotically cylindrical, with $i = 0$ designating an asymptotically flat end. The additional ends for $1 \leq i \leq N$ represent individual black holes. Recall that an ith end is referred to as asymptotically flat if it is diffeomorphic to $\mathbb{R}^3 \setminus B_1$, and in the resulting Cartesian coordinates with $\boldsymbol{\delta}$ denoting the Euclidean metric we have

$$(1.4) \qquad |\partial^\ell(g - \boldsymbol{\delta})|_{\boldsymbol{\delta}} = O(r^{-q-\ell}), \quad \ell \leq 6, \qquad |k|_{\boldsymbol{\delta}} = O(r^{-q-1}),$$

for some $q > \frac{1}{2}$ where r is the Euclidean radial coordinate; additionally the scalar curvature should be integrable $R \in L^1(M_i)$. We note that the condition $\ell \leq 6$ is stronger than the typical requirement, and that this is due to the need for Brill coordinates [6, Remark 3.2], (see also [15, 26]). Each asymptotically flat end comes equipped with the ADM energy, which is well-defined [1, 5] and given by

$$(1.5) \qquad m = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \star_{\boldsymbol{\delta}} \left(\mathrm{div}_{\boldsymbol{\delta}} g - d\mathrm{Tr}_{\boldsymbol{\delta}} g \right),$$

where $\star_{\boldsymbol{\delta}}$ is the Euclidean Hodge star operator, and S_r is the coordinate sphere of radius r. Furthermore, axial symmetry implies that the ADM angular momentum of the end is aligned along the symmetry axis and thus may be described by a single number

$$(1.6) \qquad \mathcal{J} = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (k - (\mathrm{Tr}_g k)g)(\eta, \nu) dA,$$

where ν and dA are the unit outer normal and area element of S_r. Notice that the integral is invariant over any surface homologous to a coordinate sphere when $J(\eta) = 0$. This latter condition will be assumed in order to obtain a twist potential, and therefore make contact with harmonic maps. Moreover, it also shows that the total angular momentum is a sum of the individual angular momenta $\mathcal{J}_i$ evaluated at the auxiliary ends, that is $\mathcal{J} = \sum_{i=1}^N \mathcal{J}_i$.

Consider the conformally cylindrical geometry $(\mathbb{R}_+ \times S^2, \bar{g})$ with model metric $\bar{g} = h \left(d\bar{t}^2 + \bar{g}_{S^2} \right)$, where h is a positive smooth function independent of the radial coordinate $\bar{t}$, and $\bar{g}_{S^2}$ is a metric on the 2-sphere. It is assumed further that $\bar{g}$ admits a Killing vector field that is tangent to S^2 and has periodic orbits. An end (M_i, g) is then referred to as *asymptotically cylindrical* if it is diffeomorphic to $\mathbb{R}_+ \times S^2$, and in the coordinates provided by the diffeomorphism

$$(1.7) \qquad |\bar{\nabla}^\ell (g - \bar{g})|_{\bar{g}} = O(e^{-q\bar{t}}), \quad \ell \leq 7,$$

for some $q > 0$ where $\bar{\nabla}$ denotes covariant differentiation with respect to $\bar{g}$. Again the large number of derivatives $\ell \leq 7$ is included for the existence of Brill coordinates [15, 26]. We note that extreme Kerr black holes possess an asymptotically cylindrical end in the sense described above.

In this setting, with the additional hypotheses of completeness, simple connectivity, and the absence of asymptotically cylindrical ends, the mass-angular momentum inequality (1.1) was initial studied by Dain and established for a single black hole through the combined work of Chruściel, Costa, Dain, Schoen, and Zhou in [6, 7, 9, 10, 25]. The survey [11] details many of these developments, and [4] addresses situations where completeness and simple connectivity may be removed. The case of multiple black holes was taken up by Chruściel-Li-Weinstein [8] (see also Khuri-Weinstein [17] for the inclusion of charge) who proved the lower bound

$$(1.8) \qquad m \geq \mathcal{F}(\mathcal{J}_1, \ldots, \mathcal{J}_N, z_1, \ldots, z_N),$$

where the function $\mathcal{F}$ is proportional to a renormalized harmonic map energy depending on the angular momenta $\mathcal{J}_i$ and positions z_i associated with the N black holes. Our main result shows that this function is bounded below by the desired quantity $\sqrt{|\mathcal{J}|}$, or else there is a counterexample to a form of extreme black hole uniqueness. Recall that a version of the *extreme black hole uniqueness conjecture* states that there do not exist regular asymptotically flat multiple degenerate black hole solutions to the axisymmetric stationary vacuum Einstein equations; see Conjecture 3.3 below for an equivalent PDE statement. We will refer to a regular axisymmetric stationary vacuum spacetime with an asymptotically flat end and multiple degenerate horizons as *ADM minimizing*, if it achieves the infimum of the ADM mass for solutions in this class which preserve the angular momentum of each black hole. The proof is based on a novel flow of singular harmonic maps with hyperbolic plane target, under which the renormalized harmonic map energy is monotonically nonincreasing. The relevant properties of the flow are achieved through a refined asymptotic analysis of solutions to the harmonic map equations and their linearization.

Theorem 1.1. *Let (M, g, k) be a complete, simply connected, axially symmetric, maximal initial data set for the Einstein equations with one designated asymptotically flat end and finitely many other asymptotically flat or asymptotically cylindrical ends. Assume further that the nonnegative energy density condition is satisfied $\mu \geq 0$, and the momentum density vanishes in the direction of rotation $J(\eta) = 0$. Then either there exists an ADM minimizing counterexample to the extreme black hole uniqueness conjecture, or $m \geq \sqrt{|\mathcal{J}|}$ for all such initial data with equality if and only if the data arise from an extreme Kerr black hole.*

Remark 1.2. This result could be rephrased in the following alternative way, which emphasizes the role of potentially exotic multi-black hole solutions to the stationary vacuum Einstein

equations. If there is a counterexample to the mass-angular momentum inequality, then there exists an (ADM minimizing) asymptotically flat multiple degenerate axisymmetric stationary vacuum black hole solution in equilibrium.

As mentioned above, the proof is based on a flow of singular harmonic maps. More precisely, we observe that the first variation of renormalized harmonic energy under perturbations of the black hole singularities within the maps, is determined by the conical singularities along the axis in the associated stationary vacuum spacetimes. From this we are able to define a flow of the singular harmonic maps, along with the corresponding stationary vacuum black holes solutions, which moves the black holes guided by the conical singularities and results in a monotone nonincreasing renormalized energy. It is shown that the flow exists up to a maximal time, at which point three different phenomena can occur: the black holes either collide, scatter to infinity, or stagnate. In the first two cases, the limiting renormalized energy exists and is shown to be greater than or equal to the renormalized energy of the collision and scattering configurations. Thus, the flow may be restarted at the new collision/scattering configurations while preserving monotonicity through the singular time. In the case of a stagnating flow, we find that the limiting harmonic map gives rise to a regular multi-Kerr spacetime, and then appeal to the black hole uniqueness statement to show that its renormalized energy must not be less than the square root of total angular momentum. With these observations, an induction argument on the number of black holes is then used to establish the main result.

This paper is organized as follows. In the next section, background material and the set up relating a preliminary mass lower bound to the harmonic map energy are described. A flow of harmonic maps via movement of singular points is introduced in Section 3, and statements of the co-main results concerning families of singular harmonic maps and their linearization are also discussed. The flow is studied in more detail in Section 4, where it is shown that the renormalized energy is monotonically nonincreasing along the evolution. Moreover, collisions and scatterings of the singular points along the flow are analyzed in Sections 5 and 6 respectively, where it is proven that energy monotonicity is preserved in these situations. The proof of the main theorem is provided in Section 7. Section 8 is dedicated to an asymptotic analysis of the linearized local harmonic map system at horizons. In Section 9 differentiability properties of a multi-parameter family of singular harmonic maps are established. Finally, an appendix is included to collect miscellaneous auxiliary lemmas.

2. Background and Set Up

Let (M, g) be a complete, simply connected, axially symmetric Riemannian 3-manifold which is asymptotically flat with $N + 1$ ends. Then it is shown in [5, Theorem 2.9, Remark 3.2] (only asymptotically flat ends) and [26, Theorem 1.0.5] (including asymptotically cylindrical ends) that $M \cong \mathbb{R}^3 \setminus \{p_1, \ldots, p_N\}$, and that there exists a global (cylindrical) Brill coordinate system (ρ, z, ϕ) on M with $\rho \geq 0$ and $z \in \mathbb{R}$ parameterizing a half-plane orbit space, and where $\phi \in [0, 2\pi)$ is associated with the rotational Killing field so that $\eta = \partial_\phi$. Moreover, the points p_i (referred to as punctures) represent individual black holes and all lie on the z-axis Γ, where

$|\eta| = 0$. In these coordinates the metric may be expressed to exhibit the Riemannian submersion structure

$$(2.1) \qquad g = e^{-2\mathbf{U}+2\sigma}(d\rho^2 + dz^2) + \rho^2 e^{-2\mathbf{U}}(d\phi + A_\rho d\rho + A_z dz)^2,$$

where all coefficient functions are independent of ϕ and $\rho e^{-\mathbf{U}}(d\phi + A_\rho d\rho + A_z dz)$ is the dual 1-form to fiber directions $|\eta|^{-1}\eta$. In the designated asymptotically flat end M_0, as Euclidean distance $r = \sqrt{\rho^2 + z^2} \to \infty$ the coefficient functions satisfy the decay[1]

$$(2.2) \qquad \mathbf{U} = O_{\ell-3}(r^{-q}), \quad \sigma = O_{\ell-4}(r^{-q}), \quad A_\rho = \rho O_{\ell-3}(r^{-q-2}), \quad A_z = O_{\ell-3}(r^{-q-1}).$$

In the remaining ends associated with punctures we have $r_i \to 0$, where r_i is the Euclidean distance to p_i, and the coefficients satisfy the following decay in the asymptotically flat case

$$(2.3) \qquad \mathbf{U} = 2\log r_i + O_{\ell-4}(r_i^q), \quad \sigma = O_{\ell-4}(r_i^q), \quad A_\rho = \rho O_{\ell-3}(r_i^q), \quad A_z = O_{\ell-3}(r_i^{q+1}),$$

whereas in the asymptotically cylindrical case

$$(2.4) \quad \mathbf{U} = \log r_i + \hat{\mathbf{U}}(\theta_i) + O_{\ell-4}(r_i^q), \quad \sigma = \hat{\sigma}(\theta_i) + O_{\ell-4}(r_i^q), \quad A_\rho = \rho O_{\ell-3}(r_i^q), \quad A_z = O_{\ell-3}(r_i^{q+1}),$$

for some functions $\hat{\mathbf{U}}$ and $\hat{\sigma}$ of the polar angle θ_i centered at p_i.

To obtain a twist potential observe that a calculation produces

$$(2.5) \qquad d \star (k(\eta) \wedge \eta) = -\iota_\eta \star J(\eta) = 0,$$

where $\star$ is with respect to (M, g), interior product is denoted by ι, and we are using the same notation η for both the vector field and its dual 1-form. Since M is simply connected, there then exists a smooth potential function $\mathbf{v}$ such that

$$(2.6) \qquad d\mathbf{v} = \star(k(\eta) \wedge \eta).$$

Notice that the z-axis without punctures may be decomposed into a set of pairwise disjoint open intervals $\Gamma \setminus \{p_1, \ldots, p_N\} = \cup_{j=1}^{N+1} \Gamma_j$, for which $\mathbf{v} = c_j$ is constant on each Γ_j. The intervals are called *axis rods*, whereas the corresponding values of $\mathbf{v}$ are referred to as *potential constants*, and they determine the angular momentum of each black hole. More precisely in the maximal case

$$(2.7) \qquad \mathcal{J}_i = \lim_{r_i \to 0} \frac{1}{8\pi} \int_{S_{r_i}} k(\eta, \nu) dA = \frac{1}{4}(c_{i+1} - c_i),$$

where S_{r_i} is a coordinate sphere centered at p_i and ν is the unit normal pointing towards the designated asymptotically flat end. Here the labeling of intervals is such that Γ_{i+1}, Γ_i lie directly above and below p_i, respectively. Furthermore, using the orthonormal frame

$$(2.8) \qquad e_1 = e^{\mathbf{U}-\sigma}(\partial_\rho - A_\rho\partial_\phi), \quad e_2 = e^{\mathbf{U}-\sigma}(\partial_z - A_z\partial_\phi), \quad e_3 = \rho^{-1}e^{\mathbf{U}}\partial_\phi,$$

we find

$$(2.9) \qquad k(e_1, e_3) = -|\eta|^{-2}e_2(\mathbf{v}), \qquad k(e_2, e_3) = |\eta|^{-2}e_1(\mathbf{v}).$$

It follows that

$$(2.10) \qquad |k|_g^2 \geq 2\left(k(e_1, e_3)^2 + k(e_2, e_3)^2\right) = 2\frac{e^{6\mathbf{U}-2\sigma}}{\rho^4}|\nabla\mathbf{v}|^2,$$

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[1] The notation $\mathbf{h} = O_\ell(r^{-q})$ asserts that $|\partial_r^l \mathbf{h}| \leq Cr^{-q-l}$ for all $l \leq \ell$.

where the norm $|\cdot|$ is with respect to the Euclidean metric.

2.1. Relation between mass and harmonic map energy.
Recall that in Brill coordinates the scalar curvature may be expressed ([3], [10]) as

$$(2.11) \qquad 2e^{-2\mathbf{U}+2\sigma}R = 8\Delta\mathbf{U} - 4\Delta_{\rho,z}\sigma - 4|\nabla\mathbf{U}|^2 - \rho^2 e^{-2\sigma}(A_{\rho,z} - A_{z,\rho})^2,$$

where Δ is the Euclidean Laplacian on $\mathbb{R}^3$ and $\Delta_{\rho,z} = \partial_\rho^2 + \partial_z^2$. Integrating this formula by parts yields the following mass formula [6, (3.9)] ([26, (4.5.15)] including asymptotically cylindrical ends)

$$(2.12) \qquad m = \frac{1}{32\pi}\int_{\mathbb{R}^3}\left(2e^{-2\mathbf{U}+2\sigma}R + 4|\nabla\mathbf{U}|^2 + \rho^2 e^{-2\sigma}(A_{\rho,z} - A_{z,\rho})^2\right)d\mathbf{x}.$$

Observe that the nonnegative energy density and maximality conditions together with (2.10) imply that

$$(2.13) \qquad R = 16\pi\mu + |k|_g^2 \geq 2\frac{e^{6\mathbf{U}-2\sigma}}{\rho^4}|\nabla\mathbf{v}|^2,$$

and therefore

$$(2.14) \qquad m \geq \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(|\nabla\mathbf{U}|^2 + \frac{e^{4\mathbf{U}}}{\rho^4}|\nabla\mathbf{v}|^2\right)d\mathbf{x} =: \frac{1}{8\pi}\mathcal{E}(\bar{\Phi}).$$

Notice that with the substitutions $\mathbf{U} = \mathbf{u} + \ln\rho$, and using that $\ln\rho$ is harmonic on $\mathbb{R}^3\setminus\Gamma$, the functional on the right-hand side is up to boundary terms the harmonic map energy of a map $\bar{\Phi} = (\mathbf{u}, \mathbf{v}) : \mathbb{R}^3\setminus\Gamma \to \mathbb{H}^2$, where the metric $d\mathbf{u}^2 + e^{4\mathbf{u}}d\mathbf{v}^2$ of the hyperbolic plane (with curvature -4) is expressed in horospherical coordinates. We refer to $\mathcal{E}(\bar{\Phi})$ as the *renormalized energy* of the map $\bar{\Phi}$, since the infinite terms involving $\nabla\ln\rho$ have been removed.

In [8, Proposition 2.1] (see also [17, Corollary 3.2]) it is shown that there exists a unique singular harmonic map $\Phi = (u, v) : \mathbb{R}^3\setminus\Gamma \to \mathbb{H}^2$ having the same potential constants as $\bar{\Phi}$, which is asymptotic to the relevant extreme Kerr harmonic map near each puncture p_i as well as at infinity, and for which $u = -\ln\rho + O(1)$ upon approach to the rods Γ_i. The harmonic map equations are given by

$$(2.15) \qquad \Delta u - 2e^{4u}|\nabla v|^2 = 0, \qquad \Delta v + 4\nabla u \cdot \nabla v = 0.$$

Furthermore, it is proven in [8] and [17, Theorem 4.1] that this map minimizes[2] the renormalized energy so that $\mathcal{E}(\bar{\Phi}) \geq \mathcal{E}(\Phi)$, and hence

$$(2.16) \qquad m \geq \frac{1}{8\pi}\mathcal{E}(\Phi) =: \mathcal{F}(\mathcal{J}_1, \ldots, \mathcal{J}_N, z_1, \ldots, z_N).$$

This relies on the observation that asymptotically cylindrical ends carry less energy than asymptotically flat ends.

The remainder of this manuscript is dedicated to proving that the function $\mathcal{F}$ is bounded below by $\sqrt{|\mathcal{J}|}$ modulo extreme black hole uniqueness, in order to establish Theorem 1.1. This entails two separate analyses to obtain asymptotics near punctures and other regimes, namely one for the nonlinear harmonic map system (2.15) carried out in [12] and another for the linearized equations carried out in Section 9 below. Moreover, a flow of harmonic maps Φ_t, for $t \in \mathbb{R}$,

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[2]The assumption [17, (1.5)] indicates derivative decay of k, however this is not used. Therefore (1.4) is sufficient.

is developed based on the movement of punctures $p_i(t)$ in the direction of conical singularity angle defects of an associated stationary vacuum spacetime. The renormalized energy $\mathcal{E}(\Phi_t)$ will be shown to monotonically nonincrease along the flow and to converge, after possible jumps when punctures collide or scatter, to a value not less than the desired lower bound of the main theorem.

3. THE FLOW OF PUNCTURES AND FURTHER HARMONIC MAP ANALYSIS

Fix potential constants $c_1, \dots, c_{N+1}$ that give rise to the nonzero angular momenta $\mathcal{J}_i$ as given by (2.7), and let $\mathbf{z} = (z_1, \dots, z_N)$ be the z-coordinates for N distinct punctures p_i on the z-axis $\Gamma \subset \mathbb{R}^3$. As discussed above, for each such data set there is a unique singular harmonic map $\Phi_{\mathbf{z}} = (u_{\mathbf{z}}, v_{\mathbf{z}}) \colon \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$ with prescribed blow-up at punctures designated by $\mathbf{z}$, and having these given potential constants. The quantity $\mathbf{z}$ may be viewed as a multivariate parameter for the harmonic maps $\Phi_{\mathbf{z}}$. Let $\mathbf{z}_0 = (z_{0,1}, \dots, z_{0,N})$ with $z_{0,1} < z_{0,2} < \cdots < z_{0,N}$, and consider a ball $B_\varepsilon^N(\mathbf{z}_0) \subset \mathbb{R}^N$ with this center, having radius $\varepsilon > 0$ small enough to ensure that the components of each point in the ball remain distinct. We will denote the punctures associated with $\mathbf{z}_0$ by $p_i^0 = (0, 0, z_{0,i}) \in \mathbb{R}^3$. According to [12, Theorems 2.1 and 2.2] we have the following expansions for the harmonic maps $\Phi_{\mathbf{z}}$ in a neighborhood of each puncture with z-coordinate z_i, using polar coordinates (r_i, θ_i, ϕ) centered at this puncture, namely

$$(3.1) \qquad U_{\mathbf{z}} = u_{\mathbf{z}} + \ln \rho = \ln r_i + \bar{U}_{\mathbf{z},i}(\theta_i) + \tilde{U}_{\mathbf{z},i}, \qquad v_{\mathbf{z}} = \bar{v}_{\mathbf{z},i}(\theta_i) + \tilde{v}_{\mathbf{z},i},$$

where there exist constants $b_i \in (-1, 1)$ and $a_i = 2\mathcal{J}_i > 0$ such that the barred functions yield the renormalized tangent maps

$$(3.2) \qquad \begin{aligned} \bar{U}_{\mathbf{z},i} &= \bar{U}(\theta_i, b_i) = -\frac{1}{2} \ln \left(\frac{2a_i \sqrt{1 - b_i^2}}{1 + \cos^2 \theta_i + 2b_i \cos \theta_i} \right), \\ \bar{v}_{\mathbf{z},i} &= \bar{v}(\theta_i, b_i) = a_i \left(\frac{b_i + b_i \cos^2 \theta_i + 2 \cos \theta_i}{1 + \cos^2 \theta_i + 2b_i \cos \theta_i} \right). \end{aligned}$$

Although a_i and the angular momentum are assumed positive here, no generality is lost as the generic case of these local expansions may be obtained from this one by replacing $v_{\mathbf{z}}$ with $\pm v_{\mathbf{z}} + c$, for some appropriately chosen constant c; this transformation yields an isometry of the target hyperbolic space, and hence does not affect the harmonic map equations. Notice that each $(\bar{u}_{\mathbf{z},i} = \bar{U}_{\mathbf{z},i} - \ln \sin \theta_i, \bar{v}_{\mathbf{z},i})$ is harmonic from $\mathbb{S}^2 \setminus \{N, S\} \to \mathbb{H}^2$ where N, S represent north and south poles, and thus provides the tangent map at the puncture. The constants b_i will be referred to as *tangent map parameters*. Furthermore the error terms satisfy

$$(3.3) \qquad \begin{aligned} |\tilde{U}_{\mathbf{z},i}| + r_i |\partial_{r_i} \tilde{U}_{\mathbf{z},i}| + |\partial_{\theta_i} \tilde{U}_{\mathbf{z},i}| &= O(r_i^{\beta_i}), \\ (\sin \theta_i)^{-(3+\varsigma)} \left(|\tilde{v}_{\mathbf{z},i}| + r_i |\partial_{r_i} \tilde{v}_{\mathbf{z},i}| \right) + (\sin \theta_i)^{-(2+\varsigma)} |\partial_{\theta_i} \tilde{v}_{\mathbf{z},i}| &= O(r_i^{\beta_i}), \end{aligned}$$

for some $\beta_i \in (0, 1)$ where $\varsigma \in (0, 1)$ is arbitrary. Similar estimates hold for higher derivatives up to and including order three. It should be pointed out that while θ_i and b_i are functions of $\mathbf{z}$, the a_i are independent of $\mathbf{z}$. Furthermore, it follows from [12, Theorem 2.3] that the relevant

expansions near infinity (as $r \to \infty$) are given by

$$(3.4) \qquad u_{\mathbf{z}} = -\ln \rho + \frac{c_1}{r} + O(r^{-2}), \qquad v_{\mathbf{z}} = a_1 \cos \theta (3 - \cos^2 \theta) + a_2 + O\left(\frac{(\sin \theta)^{3+\varsigma}}{r}\right),$$

for some constants c_1, a_1, a_2 with corresponding fall-off for derivatives. Note that there is no constant term in the expansion of $u_{\mathbf{z}}$ due to asymptotic flatness.

The following result addresses the smooth dependence problem for the singular harmonic maps and their corresponding tangent maps. The notation ∂_{z_i} will be used to denote differentiation with respect to the components of the parameter $\mathbf{z}$, which should be distinguished from ∂_z which represents domain differentiation with respect to the z-coordinate of $\mathbb{R}^3$.

Theorem 3.1. *For any $\varsigma \in (0, 1)$ the map $B_\varepsilon^N(\mathbf{z}_0) \to C_{loc}^{3,\varsigma}(\mathbb{R}^3 \setminus \{p_1, \ldots, p_N\}, \mathbb{R}^2)$ given by $\mathbf{z} \mapsto (U_{\mathbf{z}}, v_{\mathbf{z}})$ is smooth, and the map $B_\varepsilon^N(\mathbf{z}_0) \to C_{loc}^{3,\varsigma}(\mathbb{S}^2, \mathbb{R}^2)$ given by $\mathbf{z} \mapsto (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i})$ is smooth for each $i = 1, \ldots, N$. Moreover, for any $i, j \in \{1, \ldots, N\}$ and $l + \kappa \leq 3$ the following estimates hold*

$$|\nabla^\kappa(\partial_{z_j} u_{\mathbf{z}} + \boldsymbol{\delta}_{ij}\partial_z u_{\mathbf{z}})| + (\sin \theta_i)^{\kappa-3-\varsigma}|\nabla^\kappa(\partial_{z_j} v_{\mathbf{z}} + \boldsymbol{\delta}_{ij}\partial_z v_{\mathbf{z}})| \leq C r_i^{-\kappa} \quad in \ B_{\varepsilon/2}(p_i) \setminus \Gamma,$$

$$(3.5) \qquad |\nabla_{\mathbb{S}^2}^\kappa \partial_{z_j} \bar{u}_{\mathbf{z},i}| + (\sin \theta_i)^{\kappa-3-\varsigma}|\nabla_{\mathbb{S}^2}^\kappa \partial_{z_j} \bar{v}_{\mathbf{z},i}| \leq C \quad on \ \mathbb{S}^2 \setminus \{N, S\},$$

$$|\nabla^\kappa(\partial_{z_j} u_{\mathbf{z}})| + (\sin \theta)^{\kappa-3-\varsigma}|\nabla^\kappa(\partial_{z_j} v_{\mathbf{z}})| \leq C r^{-(\kappa+1)} \quad in \ \mathbb{R}^3 \setminus \left(\Gamma \cup_{i=1}^N B_{\varepsilon/2}(p_i)\right),$$

and

$$(3.6) \qquad \begin{aligned} &|(r_i\partial_{r_i})^l \nabla_{\mathbb{S}^2}^\kappa[(\partial_{z_j} u_{\mathbf{z}} + \boldsymbol{\delta}_{ij}\partial_z u_{\mathbf{z}}) - \partial_{z_j} \bar{u}_{\mathbf{z},i}]| \\ &+ e^{(3+\varsigma-\kappa)u_{\mathbf{z}}}|(r_i\partial_{r_i})^l \nabla_{\mathbb{S}^2}^\kappa[(\partial_{z_j} v_{\mathbf{z}} + \boldsymbol{\delta}_{ij}\partial_z v_{\mathbf{z}}) - \partial_{z_j} \bar{v}_{\mathbf{z},i}]| \leq C r_i^{\bar{\beta}} \quad in \ B_{\varepsilon/2}(p_i) \setminus \Gamma, \end{aligned}$$

for some constants C and $\bar{\beta} > 0$ depending on ε, $c_1, \ldots, c_{N+1}$, and $\sum_{i=1}^N |z_i|$, with C depending also on ς.

The proof of this result is presented in Section 9, and relies on two primary elements. These include convexity estimates of the renormalized energy, derived by Schoen-Zhou [25] and Khuri-Weinstein [17], which provide the starting point $C^{1/2}$ regularity of $\mathcal{E}(\Phi_{\mathbf{z}})$. Additionally, a refined asymptotic analysis based on [12] is employed, which is detailed in Section 8.

Let $\varepsilon_0 > 0$ be sufficiently small and consider a smooth curve of punctures $\mathbf{z}(t) \subset B_\varepsilon^N(\mathbf{z}_0)$ for $t \in (-\varepsilon_0, \varepsilon_0)$, with $\mathbf{z}(0) = \mathbf{z}_0$. Then Theorem 3.1 implies that the family of harmonic maps $\Phi_t = (u_t, v_t) := (u_{\mathbf{z}(t)}, v_{\mathbf{z}(t)})$ is smooth in t away from the axis. We emphasize that θ_i and b_i are functions of t, whereas a_i are independent of t. The next result essentially follows from Theorem 3.1, and its proof is also given in Section 9.

Corollary 3.2. *Let $(\dot{U}_t, \dot{v}_t)$ be the derivative with respect to t of the renormalized family of harmonic maps. In a neighborhood of each puncture $p_i(t)$, as $r_i(t) \to 0$, we have the expansions*

$$(3.7) \qquad \begin{aligned} \dot{U}_t &= \left(-\frac{z - z_i}{r_i^2} + \frac{\rho}{r_i^2}\partial_{\theta_i}\bar{U}\right)\dot{z}_i + \dot{b}_i\partial_{b_i}\bar{U} + O(r_i^{\beta_i - 1}), \\ \dot{v}_t &= \frac{\rho}{r_i^2}(\partial_{\theta_i}\bar{v})\dot{z}_i + \dot{b}_i\partial_{b_i}\bar{v} + O(r_i^{\beta_i - 1}\sin\theta_i). \end{aligned}$$

Furthermore, there exists a constant C such that

$$(3.8) \qquad |\dot{U}_t| + \frac{e^{2U_t}}{\rho^2}|\dot{v}_t| \leq C \quad as \ \rho \to 0 \ away \ from \ punctures,$$

and

$$(3.9) \qquad |\dot{U}_t| + \frac{e^{2U_t}}{\rho^2}|\dot{v}_t| \leq Cr^{-1} \quad as \ r \to \infty.$$

The desired curve of punctures arises from a flow guided by the tangent map parameters, where the potential constants are held fixed according to the prescribed angular momenta. More precisely, we require the curve to satisfy the autonomous system

$$(3.10) \qquad \frac{dz_i}{dt} = -b_i(z_1, \ldots, z_N), \quad i = 1, \ldots, N,$$

which we will refer to as the *puncture flow*. In the next section it is shown that when an initial condition is given, there is a unique C^2 solution for a maximal time interval. Furthermore, it will also be established that the reduced energy is monotone along the flow, in fact when smooth the flow implies

$$(3.11) \qquad \frac{d}{dt}\mathcal{E}(\Phi_t) = -\sum_{i=1}^{N} f(b_i)b_i \leq 0$$

for some increasing function f with $f(0) = 0$. Monotonicity in the presence of collisions and scatterings is proven in Sections 5 and 6 respectively. The relation to extreme black hole uniqueness arises in the case that all b_i vanish. More precisely, with a harmonic map $\Phi = (u, v)$ as described, one may build a stationary vacuum spacetime on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{p_1, \ldots, p_N\})$ with metric in Weyl coordinates given by (see [12, Section 2])

$$(3.12) \qquad \mathbf{g} = -e^{2U}d\tau^2 + \rho^2 e^{-2U}(d\phi + wd\tau)^2 + e^{-2U+2\alpha}(d\rho^2 + dz^2),$$

where w is obtained from $dw = 2e^{4u}\iota_\eta \star_\delta dv$ and $\boldsymbol{\alpha}$ satisfies

$$(3.13) \quad \partial_\rho\boldsymbol{\alpha} = \rho\left[(\partial_\rho U)^2 - (\partial_z U)^2 + e^{4u}((\partial_\rho v)^2 - (\partial_z v)^2)\right], \quad \partial_z\boldsymbol{\alpha} = 2\rho\left(\partial_\rho U \partial_z U + e^{4u}\partial_\rho v \partial_z v\right).$$

According to [12, Theorem 2.4], the tangent map parameter at p_i is related to the logarithmic angle defects $\mathbf{b}_{i+1}$, $\mathbf{b}_i$ of the neighboring axis rods for the constructed spacetime, via the formula

$$(3.14) \qquad \mathbf{b}_{i+1} - \mathbf{b}_i = \ln\left(\frac{1 + b_i}{1 - b_i}\right).$$

Thus if all $b_i = 0$, which indicates a stationary point for the flow of punctures, then the associated spacetime would be devoid of conical singularities. This leads to a regular stationary vacuum solution with multiple degenerate black holes, which would be a counterexample to the extreme black hole uniqueness conjecture. Here we state a PDE version of the conjecture.

Conjecture 3.3. *Let $\Phi_{\mathbf{z}} : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$ be a singular harmonic map associated with a collection of $N > 1$ distinct punctures located on the z-axis at $\mathbf{z} = (z_1, \ldots, z_N)$, and having potential constants giving rise to nonzero angular momenta $\mathcal{J}_i$ at each such location. Then at least one of the tangent map parameters b_i, $i = 1, \ldots, N$ must be nonzero.*

4. Energy Monotonicity

In this section we will show that the renormalized harmonic map energy of harmonic maps moving according to the flow (3.10) is monotonically nonincreasing, whenever the flow is smooth. The first step is to establish a short time existence result in the next proposition. This follows from standard ODE theory, as the functions b_i used in the definition of the flow are well-behaved. In particular, they are smooth with respect to movement of punctures due to the tangent map statement of Theorem 3.1. We will also establish a version of this statement in the following result using a different method.

Proposition 4.1. *The tangent map parameters $b_i(\mathbf{z})$, $i = 1, \ldots, N$ are continuously differentiable in $\mathbf{z}$ as long as the z_i remain pairwise distinct. Consequently, given an initial condition $\mathbf{z}(0)$ with $z_i(0) < z_{i+1}(0)$ there exists a maximal time $T > 0$ (possibly infinite) and a unique $C^2\left([0,T), \mathbb{R}^N\right)$ solution $\mathbf{z}(t)$ of the initial value problem corresponding to the puncture flow (3.10). The solution satisfies $z_i(t) < z_{i+1}(t)$ for all i and $t \in [0,T)$, and the maximal time if finite is characterized by the property that either*

$$(4.1) \qquad \limsup_{t \to T}(z_{j+1}(t) - z_j(t)) = \infty, \quad or \quad \liminf_{t \to T}(z_{j+1}(t) - z_j(t)) = 0,$$

for some $j \in \{1, \ldots, N-1\}$.

Proof. Consider the harmonic map $\Phi_{\mathbf{z}} = (u_{\mathbf{z}}, v_{\mathbf{z}})$ with punctures p_i designated by $\mathbf{z}$, and having fixed potential constants. We may view $\mathbf{z}$ as a multivariate parameter for the harmonic maps $\Phi_{\mathbf{z}}$. Let $\mathbf{z}(0) = \mathbf{z}_0$ have distinct components, and take a ball $B_\varepsilon^N(\mathbf{z}_0) \subset \mathbb{R}^N$ where $\varepsilon > 0$ is small enough to ensure that the components of each point remain distinct. According to Theorem 3.1, the map $B_\varepsilon^N(\mathbf{z}_0) \to C^{3,\varsigma}(\mathbb{R}^3 \setminus \Gamma_\varepsilon, \mathbb{R}^2)$ given by $\mathbf{z} \mapsto (U_{\mathbf{z}}, v_{\mathbf{z}})$ is continuously differentiable, where $U_{\mathbf{z}} = u_{\mathbf{z}} + \ln \rho$ and $\Gamma_\varepsilon = \Gamma \cap \left(\cup_{i=1}^N B_\varepsilon(p_i^0)\right)$.

By [12, Theorem 2.4] we have the following relation between the tangent map parameter at the ith puncture and the difference of logarithmic angle defects associated with the two neighboring axis rods

$$(4.2) \qquad b_i(\mathbf{z}) = \tanh\left(\frac{\mathbf{b}_{i+1}(\mathbf{z}) - \mathbf{b}_i(\mathbf{z})}{2}\right),$$

which follows from (3.14). Moreover, [12, (2.12)] shows that the logarithmic angle defect of an axis rod agrees with the spacetime metric coefficient $\boldsymbol{\alpha}_{\mathbf{z}}$ of (3.13) restricted to the rod; this value is constant along the rod. Thus, if γ_i is a semi-circle in the ρz-half plane centered at puncture p_i^0 of radius 2ε and connecting points $\tilde{p}_i$, $\tilde{p}_{i+1}$ of the neighboring axes Γ_i, Γ_{i+1}, then

$$(4.3) \qquad \mathbf{b}_{i+1}(\mathbf{z}) - \mathbf{b}_i(\mathbf{z}) = \boldsymbol{\alpha}_{\mathbf{z}}(\tilde{p}_{i+1}) - \boldsymbol{\alpha}_{\mathbf{z}}(\tilde{p}_i) = \int_{\gamma_i} d\boldsymbol{\alpha}_{\mathbf{z}} =: \mathcal{I}_i(\mathbf{z}).$$

Note that ε may be chosen small enough so that the curves γ_i enclose no other initial puncture except p_i^0. It suffices then to show that $\mathcal{I}_i(\mathbf{z})$ is continuously differentiable for $\mathbf{z} \in B_\varepsilon^N(\mathbf{z}_0)$.

In what follows the dependence on $\mathbf{z}$ will be suppressed for clarity, and a dot will be used to denote partial differentiation with respect to the $\mathbf{z}$-parameters. Observe that by formally

differentiating the expression (4.3) we obtain

$$
\begin{aligned}
(4.4) \quad \frac{1}{2}\dot{\mathcal{I}}_i(\mathbf{z}) = & \int_{\gamma_i} \rho \left[\partial_\rho U \partial_\rho \dot{U} - \partial_z U \partial_z \dot{U} + 2\dot{U}e^{4u}\left((\partial_\rho v)^2 - (\partial_z v)^2 \right) + e^{4u}\left(\partial_\rho v \partial_\rho \dot{v} - \partial_z v \partial_z \dot{v} \right) \right] d\rho \\
& + \int_{\gamma_i} \rho \left[\partial_\rho \dot{U} \partial_z U + \partial_\rho U \partial_z \dot{U} + 4\dot{U}e^{4u}\partial_\rho v \partial_z v + e^{4u}\left(\partial_\rho \dot{v}\partial_z v + \partial_\rho v \partial_z \dot{v} \right) \right] dz,
\end{aligned}
$$

with the help of (3.13). Although e^{4u} blows-up upon approach to the axis Γ, this formula is valid. To see this, let $s \in [0,1]$ parameterize the curve γ_i and denote by $h_{\mathbf{z}}(s)$ the integrand of $\mathcal{I}_i(\mathbf{z})$, then the following properties are satisfied. First, it is clear that $h_{\mathbf{z}}$ is an integrable function of s for each $\mathbf{z} \in B^N_\varepsilon(p_0)$. Secondly, for almost all $s \in [0,1]$ the partial derivative $\dot{h}_{\mathbf{z}}$ exists for all $\mathbf{z} \in B^N_\varepsilon(\mathbf{z}_0)$, by virtue of Theorem 3.1; in fact, this holds for all s except at the end points $s = 0, 1$ and the expression for $\dot{h}_{\mathbf{z}}$ is as in (4.4). Thirdly, there is an integrable function $\bar{h}$ of s such that $|\dot{h}_{\mathbf{z}}(s)| \leq \bar{h}(s)$ for all $\mathbf{z} \in B^N_\varepsilon(\mathbf{z}_0)$ and $s \in (0,1)$. The function $\bar{h}$ may be taken to be an appropriate constant away from the endpoints, and near $s = 0, 1$ it suffices to choose $\bar{h}(s) \sim \rho(s)^{-1/2}$. The reason such an $\bar{h}$ dominates is due to Theorem 3.1 which implies that $\dot{U}$, $\nabla \dot{U}$, and $\nabla \dot{v}$ are continuous on $[0,1] \times B^N_\varepsilon(\mathbf{z}_0)$, as well as [19, Theorem 1.1] which yields the same for U, ∇U, and ∇v in addition to the asymptotics $v = c_i + O_1(\rho^{7/2})$ as $\rho \to 0$ near $\tilde{p}_i$ (similarly for $\tilde{p}_{i+1}$). In particular, the only terms within (4.4) which potentially blow-up upon approach to Γ, admit the estimate

$$
(4.5) \qquad \rho e^{4u} |\partial_\rho v \partial_\rho \dot{v} - \partial_z v \partial_z \dot{v}| \leq C\rho^{-1/2}
$$

where the constant C is independent of $\mathbf{z} \in B^N_\varepsilon(\mathbf{z}_0)$, showing that $\bar{h}$ bounds the integrand as desired. We may now differentiate under the integral and establish (4.4). Furthermore, similar arguments show that $\dot{\mathcal{I}}_i(\mathbf{z})$ is continuous on the ε-ball. It then follows from (4.2) and (4.3) that $b_i(\mathbf{z})$ is continuously differentiable as long as the components of $\mathbf{z}$ remain distinct.

Standard ODE theory now yields local in time existence and uniqueness of a C^2 solution $\mathbf{z}(t)$, in which the components remain pairwise distinct. If $T \in (0, \infty)$ denotes the maximal time of existence, and neither of the behaviors in (4.1) occur, then the functions $b_i(\mathbf{z}(t))$ are uniformly bounded on $[0,T]$. It follows that the solution may be smoothly extended to $t = T$, and local existence may be applied once more to contradict the maximality of T. Thus, we conclude that either the solution exists for all time or the maximal time of existence may be characterized by (4.1). □

We will now proceed to show that the renormalized energy $\mathcal{E}(\Phi_t)$ is monotonic along the flow (3.10). This will require several preliminary lemmas. Let $\epsilon, \delta, \lambda > 0$ be parameters with $\delta << \epsilon$ and define domains of the form

$$
(4.6) \qquad \Omega^t_{\epsilon,\delta,\lambda} = \{(\rho, z, \phi) \in \mathbb{R}^3 \mid \rho > \delta, \ r_i > \epsilon, \ i = 1, \dots, N\} \cap B_\lambda,
$$

where r_i is the Euclidean distance to the puncture $p_i(t)$. This domain is the region inside the ball of radius $r = \lambda$ centered at the origin, outside the cylinder of radius δ around the z-axis, and outside the balls of radius ϵ centered at the punctures $p_i(t)$. The boundary of this region is the union of a large sphere with discs removed at the north and south poles $S_{\lambda,\delta}$, a collection of cylinders $C^t_{\delta,\epsilon,\lambda,i}$, $i = 1, \dots, N+1$ in one-to-one correspondence with axis rods, and small spheres with discs removed at the north and south poles $S^t_{\epsilon,\delta,i}$, $i = 1, \dots, N$. Figure 1 provides

a pictorial representation that is projected onto the ρz-half plane and reflected across the axis. Note that $C^t_{\delta,\epsilon,\lambda,i}$ is independent of λ when $i = 2,\ldots,N$. Consider the renormalized harmonic map energy restricted to these domains, denoted $\mathcal{E}_{\Omega^t_{\epsilon,\delta,\lambda}}(\Phi_t)$. The strategy will be to differentiate this quantity with respect to t, and then integrate by parts thereby reducing the problem to an analysis of boundary terms. Although a special class of exhaustion domains is used for this purpose, which is chosen to simplify computations, the main result (Theorem 4.8) does not depend on this choice since the renormalized energy density is integrable on $\mathbb{R}^3$.

The movement of $\Omega^t_{\epsilon,\delta,\lambda}$ in time may be obtained from the action of a 1-parameter family of diffeomorphisms applied to a fixed initial domain, such that the associated flow vector field takes the form $Z = h\partial_z$ where h is a smooth function on $\mathbb{R}^3$ which vanishes except near the punctures $p_i(t)$, and in a sufficiently small neighborhood of these points $h = \dot{z}_i(t)$. According to Corollary 3.2 and Proposition 4.1, the 1-parameter family of harmonic maps $\Phi_t = (u_t, v_t)$ is continuously differentiable in t away from the axis, and $\dot{\Phi}_t \in C^2(\Omega^t_{\epsilon,\delta,\lambda}, \mathbb{R}^2)$. Since the domains are bounded and removed from the axis, we may differentiate under the integral using Leibniz's rule and utilize the harmonic map equations to find

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_{\Omega^t_{\epsilon,\delta,\lambda}}(\Phi_t) &= \int_{\Omega^t_{\epsilon,\delta,\lambda}} \left(2\nabla\dot{U}_t \cdot \nabla U_t + 4\frac{e^{4U_t}}{\rho^4}\dot{U}_t|\nabla v_t|^2 + 2\frac{e^{4U_t}}{\rho^4}\nabla\dot{v}_t \cdot \nabla v_t \right) d\mathbf{x} \\
&\quad + \int_{\partial\Omega^t_{\epsilon,\delta,\lambda}} \left(|\nabla U_t|^2 + \frac{e^{4U_t}}{\rho^4}|\nabla v_t|^2 \right) \iota_Z d\mathbf{x} \\
&= \int_{\partial\Omega^t_{\epsilon,\delta,\lambda}} 2\left(\dot{U}_t\nabla_\nu U_t + \frac{e^{4U_t}}{\rho^4}\dot{v}_t\nabla_\nu v_t \right) dA + \int_{\partial\Omega^t_{\epsilon,\delta,\lambda}} \left(|\nabla U_t|^2 + \frac{e^{4U_t}}{\rho^4}|\nabla v_t|^2 \right) \iota_Z d\mathbf{x} \\
&= \underbrace{\int_{S_{\lambda,\delta}}}_{\mathbf{I}_1(\lambda,\delta)} + \sum_{i=1}^{N+1}\underbrace{\int_{C^t_{\delta,\epsilon,\lambda,i}}}_{\mathbf{I}^t_2(\delta,\epsilon,\lambda,i)} + \sum_{i=1}^{N}\underbrace{\int_{S^t_{\epsilon,\delta,i}}}_{\mathbf{I}^t_3(\epsilon,\delta,i)} 2\left(\dot{U}_t\nabla_\nu U_t + \frac{e^{4U_t}}{\rho^4}\dot{v}_t\nabla_\nu v_t \right) dA \\
&\quad + \sum_{i=1}^{N}\underbrace{\int_{S^t_{\epsilon,\delta,i}}\left(|\nabla U_t|^2 + \frac{e^{4U_t}}{\rho^4}|\nabla v_t|^2 \right)\iota_Z d\mathbf{x}}_{\mathbf{I}^t_4(\epsilon,\delta,i)},
\end{aligned}
$$

(4.7)

where ν is the unit outer normal to the boundary, and ι_Z denotes interior product. In the next four lemmas, exhaustion limits for each of the integrals $\mathbf{I}_1$, $\mathbf{I}^t_2$, $\mathbf{I}^t_3$, and $\mathbf{I}^t_4$ will be evaluated.

Lemma 4.2. *For each $\delta > 0$ we have $\lim_{\lambda\to\infty}\mathbf{I}_1(\lambda,\delta) = 0$.*

Proof. It suffices to show that the integrand is $o(r^{-2})$. By [12, Theorem 2.3] and (3.9) of Corollary 3.2, there exists a constant C such that for all r large enough

$$
|U_t| \le C, \qquad |\nabla U_t| + \frac{|\nabla v_t|}{\rho^2} \le \frac{C}{r^2}, \qquad |\dot{U}_t| + \frac{|\dot{v}_t|}{\rho^2} \le \frac{C}{r},
$$
(4.8)

from which the desired result follows. $\qquad\square$

Lemma 4.3. *For each $\epsilon > 0$ and $i = 1,\ldots,N+1$ we have*

$$
\lim_{\delta\to 0}\lim_{\lambda\to\infty}\mathbf{I}^t_2(\delta,\epsilon,\lambda,i) = 0.
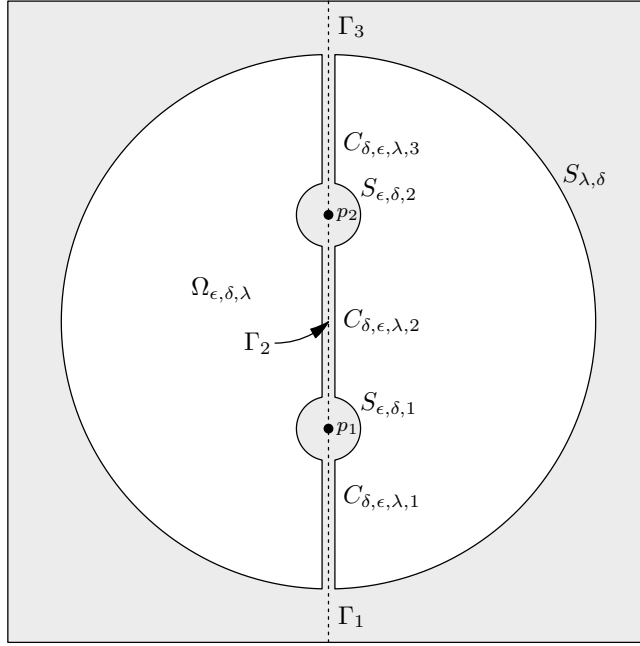$$
(4.9)

FIGURE 1. Domain of integration and boundary components.

Proof. First consider the case $i = 2, \ldots, N$, in which $\mathbf{I}_2^t(\delta, \epsilon, \lambda, i) = \mathbf{I}_2^t(\delta, \epsilon, i)$ is independent of λ. Since the areas of the relevant cylinders $C_{\delta,\epsilon,\lambda,i}^t$ are $O(\delta)$ as $\delta \to 0$, it suffices to show that the integrands are $O(1)$. For fixed $\epsilon > 0$, observe that [12, Theorem 2.1], [19, Theorem 1.1], and (3.8) of Corollary 3.2 imply the existence of a constant C such that

$$(4.10) \qquad |U_t| \le C, \qquad |\nabla U_t| + \frac{|\nabla v_t|}{\rho^2} \le C, \qquad |\dot{U}_t| + \frac{|\dot{v}_t|}{\rho^2} \le C,$$

yielding the desired result.

Consider now the case when $i = 1, N+1$. Since the areas of the relevant cylinders $C_{\delta,\epsilon,\lambda,i}^t$ are $O(\delta\lambda)$, it suffices to show that the integrands are $O(r^{-2})$. For fixed $\epsilon > 0$, observe that [12, Theorem 2.3] and (3.9) of Corollary 3.2 imply that for all r large enough

$$(4.11) \qquad |U_t| \le C, \qquad |\nabla U_t| + \frac{|\nabla v_t|}{\rho^2} \le \frac{C}{r^2}, \qquad |\dot{U}_t| + \frac{|\dot{v}_t|}{\rho^2} \le \frac{C}{r},$$

where C is independent of δ. The desired result now follows. □

Lemma 4.4. *For each $i = 1, \ldots, N$ we have*

$$(4.12) \qquad \lim_{\epsilon \to 0} \lim_{\delta \to 0} \mathbf{I}_3^t(\epsilon, \delta, i) = -f_3(b_i)b_i,$$

where $f_3(b)$ is defined by (4.23).

Proof. According to [12, Theorem 2.1 and 2.2], expansions for the harmonic map Φ_t are given by (3.1)-(3.3) in a neighborhood of each puncture. Next observe that by Proposition 4.1 we

have $\dot{b}_i \partial_{b_i} \bar{U}_t = O(1)$ and $\dot{b}_i \partial_{b_i} \bar{v}_t = O(1)$ as $r_i \to 0$. Thus, (3.7) of Corollary 3.2 implies that on $S^t_{\epsilon,\delta,i}$ the following asymptotic profiles hold

$$
\dot{U}_t \partial_{r_i} U_t = \left[\left(-\frac{(z-z_i)}{r_i^2} + \frac{\rho}{r_i^2} \partial_{\theta_i} \bar{U}_t \right) \dot{z}_i + \dot{b}_i \partial_{b_i} \bar{U}_t + O(r_i^{\beta_i - 1}) \right] \left[\frac{1}{r_i} + O(r_i^{\beta_i - 1}) \right]
$$

(4.13)

$$
= -\frac{(z-z_i)}{r_i^3} \dot{z}_i + \frac{\rho}{r_i^3} (\partial_{\theta_i} \bar{U}_t) \dot{z}_i + O(r_i^{\beta_i - 2}),
$$

$$
\frac{e^{4U_t}}{\rho^4} \dot{v}_t \partial_{r_i} v_t = \frac{1}{\sin^4 \theta_i} \left[\left(\frac{1 + \cos^2 \theta_i + 2b_i \cos \theta_i}{2a_i \sqrt{1 - b_i^2}} \right)^2 + O(r_i^{\beta_i}) \right]
$$

(4.14)

$$
\cdot \left[\frac{\sin \theta_i}{r_i} (\partial_{\theta_i} \bar{v}_t) \dot{z}_i + \dot{b}_i \partial_{b_i} \bar{v}_t + O(r_i^{\beta_i - 1} \sin \theta_i) \right] \underbrace{O(r_i^{\beta_i - 1} (\sin \theta_i)^{7/2})}_{\partial_{r_i} v_t}
$$

$$
= O((\sin \theta_i)^{-1/2} r_i^{\beta_i - 2}).
$$

We now let $\delta \to 0$ and integrate over the spheres $S^t_{\epsilon,i}$ on which $r_i = \epsilon$, and note that the first term on the right-hand side of (4.13) integrates to zero due to symmetry. Since

(4.15)
$$
\partial_{\theta_i} \bar{U} = -\frac{\sin \theta_i (\cos \theta_i + b_i)}{1 + \cos^2 \theta_i + 2b_i \cos \theta_i}
$$

by (3.2), it follows that

(4.16)
$$
\lim_{\delta \to 0} \int_{S^t_{\epsilon,\delta,i}} 2 \left(\dot{U}_t \nabla_\nu U_t + \frac{e^{4U_t}}{\rho^4} \dot{v}_t \nabla_\nu v_t \right) dA = -4\pi \dot{z}_i \int_0^\pi (\partial_{\theta_i} \bar{U}) \sin^2 \theta_i d\theta_i + O(\epsilon^{\beta_i})
$$

$$
= -f_3(b_i) b_i + O(\epsilon^{\beta_i}),
$$

where f_3 is defined in Lemma 4.7 and we have used (3.10). Letting $\epsilon \to 0$ now produces the desired result. □

Lemma 4.5. *For each $i = 1, \ldots, N$ we have*

(4.17)
$$
\lim_{\epsilon \to 0} \lim_{\delta \to 0} \mathbf{I}^t_4(\epsilon, \delta, i) = -f_4(b_i) b_i,
$$

where $f_4(b)$ is defined by (4.23).

Proof. Observe that on $S^t_{\epsilon,\delta,i}$ a direct computation produces

(4.18)
$$
\iota_Z d\mathbf{x} = \dot{z}_i dx \wedge dy = -\dot{z}_i r_i^2 \sin \theta_i \cos \theta_i d\phi_i \wedge d\theta_i.
$$

Furthermore, (3.1)-(3.3) imply that in a neighborhood of the ith puncture

$$
|\nabla U_t|^2 = (\partial_{r_i} U_t)^2 + r_i^{-2} (\partial_{\theta_i} U_t)^2
$$

(4.19)

$$
= \left(r_i^{-1} + O(r_i^{\beta_i - 1}) \right)^2 + r_i^{-2} \left(\partial_{\theta_i} \bar{U}_t + O(r_i^{\beta_i}) \right)^2
$$

$$
= r_i^{-2} \left(1 + (\partial_{\theta_i} \bar{U}_t)^2 \right) + O(r_i^{\beta_i - 2}),
$$

(4.20)
$$\frac{e^{4U_t}}{\rho^4}|\nabla v_t|^2 = e^{4u_t}\left((\partial_{r_i} v_t)^2 + r_i^{-2}(\partial_{\theta_i} v_t)^2\right)$$
$$= \left[e^{4\bar{u}_t} + O((\sin\theta_i)^{-4} r_i^{\beta_i})\right]$$
$$\cdot \left[O((\sin\theta_i)^{3+\varsigma} r_i^{\beta_i-1})^2 + r_i^{-2}\left(\partial_{\theta_i}\bar{v}_t + O((\sin\theta_i)^{2+\varsigma} r_i^{\beta_i})\right)^2\right]$$
$$= r_i^{-2} e^{4\bar{u}_t}(\partial_{\theta_i}\bar{v}_t)^2 + O(r_i^{\beta_i-2}).$$

Therefore, integrating over the surface with respect to the induced boundary orientation yields

(4.21)
$$\lim_{\delta\to 0}\int_{S^t_{\epsilon,\delta,i}}\left(|\nabla U_t|^2 + \frac{e^{4U_t}}{\rho^4}|\nabla v_t|^2\right)\iota_Z d\mathbf{x}$$
$$= -2\pi\dot{z}_i\int_0^\pi \left(1 + (\partial_{\theta_i}\bar{U})^2 + e^{4\bar{u}}(\partial_{\theta_i}\bar{v})^2\right)\sin\theta_i\cos\theta_i d\theta_i + O(\epsilon^{\beta_i})$$
$$= -f_4(b_i)b_i + O(\epsilon^{\beta_i}).$$

In the last step we used (4.15), together with the relation $e^{4\bar{u}}\partial_{\theta_i}\bar{v} = -(2a_i\sin\theta_i)^{-1}$ from [12, (4.5)], to obtain

(4.22)
$$1 + (\partial_{\theta_i}\bar{U})^2 + e^{4\bar{u}}(\partial_{\theta_i}\bar{v})^2 = \frac{2(1 + b_i\cos\theta_i)}{1 + \cos^2\theta_i + 2b_i\cos\theta_i}.$$

Letting $\epsilon \to 0$ now produces the desired result. $\qquad\square$

Remark 4.6. We point out that the estimates in the proofs of the preceding three lemmas are uniform in t, for t in a sufficiently small neighborhood of any given value t_0 for which $\mathbf{z}(t_0)$ has pairwise distinct components.

Lemma 4.7. *Let*

(4.23) $\quad f_3(b) = 4\pi\int_0^\pi \frac{\sin^3\theta(\cos\theta + b)}{1 + \cos^2\theta + 2b\cos\theta}d\theta, \qquad f_4(b) = -4\pi\int_0^\pi \frac{(1 + b\cos\theta)\sin\theta\cos\theta}{1 + \cos^2\theta + 2b\cos\theta}d\theta.$

Then $f_3(0) = f_4(0) = 0$ and $f_3'(b), f_4'(b) > 0$ for all $-1 < b < 1$.

Proof. When $b = 0$ the integrands, after translation by $\pi/2$ to the left, become odd functions and hence $f_3(0) = f_4(0) = 0$. Next observe that for each $b \in (-1, 1)$ a direct computation shows

(4.24) $\quad f_3'(b) = 4\pi\int_0^\pi \frac{\sin^5\theta}{(1 + \cos^2\theta + 2b\cos\theta)^2}d\theta, \qquad f_4'(b) = 4\pi\int_0^\pi \frac{\sin^3\theta\cos^2\theta}{(1 + \cos^2\theta + 2b\cos\theta)^2}d\theta$

from which the desired results follows. $\qquad\square$

We are now ready to establish the main result of this section. We will utilize an elementary fact concerning the interchange of derivatives and limits. Recall that if $\{h_n\}$ is a sequence of continuously differentiable functions on a bounded interval I such that $\{h_n(t_*)\}$ converges for some $t_* \in I$, and $\{h_n'\}$ is uniformly convergent on I, then $\{h_n\}$ converges uniformly on the same interval to a continuously differentiable function h with $h'(t) = \lim_{n\to\infty} h_n'(t)$ for all $t \in I$.

Theorem 4.8. *Let $\mathbf{z}(t)$ be a solution of the puncture flow (3.10) for $t \in [0, T)$ provided by Proposition 4.1, and consider the 1-parameter family of harmonic maps $\Phi_t = \Phi_{\mathbf{z}(t)}$. Then the flow of reduced energy $\mathcal{E}(\Phi_t)$ is continuously differentiable and monotonically nonincreasing on $[0, T)$.*

Proof. Set $\Omega_{\epsilon,\delta}^t = \Omega_{\epsilon,\delta,\infty}^t$, and observe that since the renormalized energy densities of the harmonic maps are globally integrable, we have that $\mathcal{E}_{\Omega_{\epsilon,\delta,\lambda}^t}(\Phi_t) \to \mathcal{E}_{\Omega_{\epsilon,\delta}^t}(\Phi_t)$ as $\lambda \to \infty$ for all $t \in [0, T)$. Then with the aid of Remark 4.6, upon letting $\lambda \to \infty$ in (4.7) we may interchange derivative and limit and apply Lemma 4.2 to find that $\mathcal{E}_{\Omega_{\epsilon,\delta}^t}(\Phi_t)$ is continuously differentiable in t with

$$(4.25) \quad \frac{d}{dt}\mathcal{E}_{\Omega_{\epsilon,\delta}^t}(\Phi_t) = \lim_{\lambda \to \infty} \frac{d}{dt}\mathcal{E}_{\Omega_{\epsilon,\delta,\lambda}^t}(\Phi_t) = \lim_{\lambda \to \infty} \sum_{i=1}^{N+1} \mathbf{I}_2^t(\delta, \epsilon, \lambda, i) + \sum_{i=1}^{N} \mathbf{I}_3^t(\epsilon, \delta, i) + \sum_{i=1}^{N} \mathbf{I}_4^t(\epsilon, \delta, i).$$

Next set $\Omega_\epsilon^t = \Omega_{\epsilon,0}^t$ and note that in a similar manner, using now Lemma 4.3, we have that $\mathcal{E}_{\Omega_\epsilon^t}(\Phi_t)$ is continuously differentiable in t with

$$(4.26) \quad \frac{d}{dt}\mathcal{E}_{\Omega_\epsilon^t}(\Phi_t) = \lim_{\delta \to 0} \frac{d}{dt}\mathcal{E}_{\Omega_{\epsilon,\delta}^t}(\Phi_t) = \lim_{\delta \to 0} \sum_{i=1}^{N} \left(\mathbf{I}_3^t(\epsilon, \delta, i) + \mathbf{I}_4^t(\epsilon, \delta, i) \right).$$

Finally, interchanging derivative and limit one more time while letting $\epsilon \to 0$ shows that $\mathcal{E}(\Phi_t)$ is continuously differentiable in t, and employing Lemmas 4.4 and 4.5 yields

$$(4.27) \quad \frac{d}{dt}\mathcal{E}(\Phi_t) = \lim_{\epsilon \to 0} \frac{d}{dt}\mathcal{E}_{\Omega_\epsilon^t}(\Phi_t) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \sum_{i=1}^{N} \left(\mathbf{I}_3^t(\epsilon, \delta, i) + \mathbf{I}_4^t(\epsilon, \delta, i) \right) = - \sum_{i=1}^{N} f(b_i)\, b_i \leq 0,$$

where $f = f_3 + f_4$ and the last inequality follows from Lemma 4.7. $\square$

5. Collision of Punctures

Consider the 1-parameter family of harmonic maps $\Phi_t = (u_t, v_t) := (u_{\mathbf{z}(t)}, v_{\mathbf{z}(t)})$ with fixed potential constants, defined by the flow of punctures (3.10). Let $z_i(t)$, $1 \leq i \leq N$ denote the z-coordinate of the i-th puncture at time t, and assume that $z_1(0) < \cdots < z_N(0)$. According to Theorem 3.1 and Proposition 4.1, this family of maps is continuously differentiable in time away from the axis for $t \in [0, T)$. At time T (which may be infinite), at least one of the possibilities from (4.1) occurs. In the latter case, namely when $\liminf_{t \to T} (z_{j+1}(t) - z_j(t)) = 0$ for some $j \in \{1, \ldots, N-1\}$, we say that two (or more) adjacent punctures $z_j < z_{j+1}$ *collide* at time T. Note that more than two punctures can collide at time T in a single location, or in multiple locations simultaneously. In this section we will treat the case in which no scattering occurs, that is, when the first possibility of (4.1) does not happen. With this setting, after possibly translating in the z-direction, we may assume without loss of generality that the set of punctures remains within a fixed compact set for all time. Hence, there is a sequence of times $t_n \to T$ such that $\lim_{n \to \infty} z_i(t_n) = z_i^*$ for all i, and $z_j(t_n)$ eventually collides with $z_{j+1}(t_n)$ for some j. Note that the neighboring jth punctures collide if and only if $z_j^* = z_{j+1}^*$, in which case we call z_j^* a *collision location*. The collection of values $\{z_1^*, \ldots, z_N^*\}$, which has at most $N-1$ distinct elements, is referred to as the *collision configuration*. Consider a harmonic map Φ_* having the set of punctures given by $\{z_1^*, \ldots, z_N^*\}$ and with the same potential constants on the rods that

remain; this will be referred to as the *collision configuration map*. We point out that different sequences $t_n \to T$ may result in different collision configurations and collision configuration maps. Nevertheless, the limiting renormalized energy is the same for all sequences, that is $\lim_{t \to T} \mathcal{E}(\Phi_t)$ exists in light of the monotonicity provided by Theorem 4.8. The proposition below contains the primary result of this section, which will allow the flow to 'pass through' the maximal time of existence when a collision happens, while maintaining the desired monotonicity property with the use of collision configuration maps.

Proposition 5.1. *Let Φ_t be the 1-parameter family of harmonic maps with fixed potential constants defined by the flow of punctures (3.10), in which a collision occurs at the limiting time T, with no scattering. If Φ_* is a collision configuration map described above, then*

$$\mathcal{E}(\Phi_*) \leq \lim_{t \to T} \mathcal{E}(\Phi_t). \tag{5.1}$$

Proof. Let $\{t_n\}$ be a sequence of times with $t_n \to T$, which leads to a collision configuration $\{z_1^*, \ldots, z_N^*\}$ and the associated collision configuration map Φ_*. For convenience denote $\Phi_n = \Phi_{t_n}$. We claim that

$$\Phi_n \to \Phi_* \text{ in } C^3 \text{ away from the axis,} \tag{5.2}$$

and that for any $\epsilon > 0$ there is an n_ϵ large enough such that the renormalized energies satisfy

$$\mathcal{E}(\Phi_*) < \mathcal{E}(\Phi_{n_\epsilon}) + \epsilon. \tag{5.3}$$

A simple limit as $\epsilon \to 0$ in (5.3) then yields (5.1). The rest of the proof is devoted to the proof of (5.2) and (5.3).

For each n, consider a harmonic map Ψ_n having the same punctures as Φ_n except for the colliding punctures which are replaced by the corresponding set of collision locations, and having the same potential constants on the rods that remain. The maps Ψ_n are called *auxiliary collision configuration maps*. The punctures at collision locations within Ψ_n have absorbed all the colliding punctures from Φ_n, and it follows from (2.7) that the resulting angular momentum at any collision location is the sum of all the angular momenta from punctures colliding at that point. With Φ_* given as the harmonic map associated with the collision configuration, then observe that Theorem 3.1 implies

$$\Psi_n \to \Phi_* \text{ in } C^3 \text{ away from the axis,} \tag{5.4}$$

since no collisions occur in the sequence Ψ_n.

We will construct a *model map* Ξ_n for Φ_n which coincides with Ψ_n outside a fixed compact set. This will lead to an upper bound on the hyperbolic distance $d_{\mathbb{H}^2}(\Phi_n, \Psi_n)$ which is independent of n, and will allow us to argue that $\Phi_n \to \Phi_*$ uniformly on compact subsets of $\mathbb{R}^3 \setminus \Gamma$. Recall that a model map [16, Definition 4.1] for Φ is a map $\Xi \colon \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$ which is *asymptotic* to Φ in the sense that $d_{\mathbb{H}^2}(\Phi, \Xi)$ is bounded and tends to zero at infinity, and such that its tension $\tau(\Xi)$ is bounded and satisfies $|\tau(\Xi)| \leq -\Delta w$, for some positive function $w \in C^2(\mathbb{R}^3)$ which tends to zero at infinity. In order not to encumber the notation, we will omit the subscript n until the L^∞-bound has been obtained.

The construction of the model map, and the estimation of its tension will proceed in three cases. We will first consider the case where there is only one collision, and only two colliding
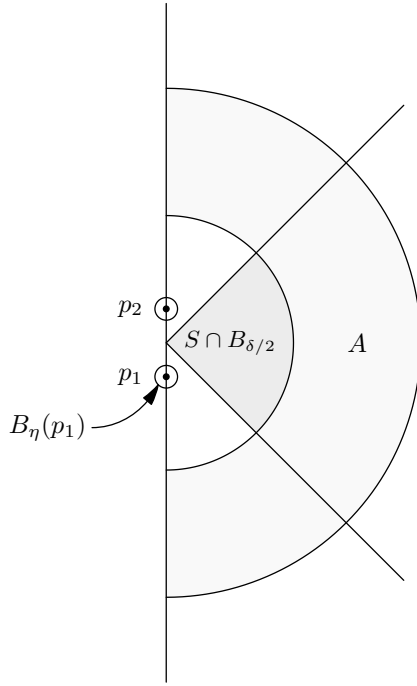
FIGURE 2. Case 1: collision of two punctures.

punctures. We will then consider the case where there is one collision location but possibly more than two colliding punctures. Finally, we will treat the general case.

Case 1. Consider the situation in which only two punctures collide, and denote for convenience the two colliding punctures by p_1, p_2 with z-coordinates $z_1 < z_2$. Without loss of generality it may be assumed that the midpoint between these two punctures occurs at the origin. Set $\eta = (z_2 - z_1)/4$ and let $B_i = B_\eta(p_i)$ be the ball of radius η centered at p_i; clearly $B_1 \cap B_2 = \emptyset$. For any $\delta > 0$, n may be taken large enough and hence η small enough to guarantee that the union of these two balls lies within the $\delta/4$-ball centered at the origin, and to guarantee that the collision location lies within $B_{\delta/4}$. We define the model map by

(5.5) $$\Xi = (u, v) = \chi_0 \Theta_0 + \chi_1 \Theta_1 + \chi_2 \Theta_2,$$

where $\Theta_i = (u_i, v_i)$, $i = 1, 2$ are the corresponding extreme Kerr maps associated with each p_i, and we have denoted the mutli-extreme Kerr auxiliary collision configuration map Ψ by $\Theta_0 = (u_0, v_0)$ for uniformity. Furthermore, $0 \leq \chi_i \leq 1$ are cut-off functions described as follows. Let (r, θ, ϕ) be polar coordinates centered at the origin, and take $0 \leq \tilde{\chi}_1, \tilde{\chi}_2 \leq 1$ to be functions of θ alone such that $\tilde{\chi}_1 + \tilde{\chi}_2 = 1$ and $\tilde{\chi}_i = 1$ in B_i. For instance these may be chosen so that $\tilde{\chi}_1 = 0$ for $0 \leq \theta \leq \pi/4$, and $\tilde{\chi}_1 = 1$ for $3\pi/4 \leq \theta \leq \pi$. Then take χ_0 to be a nonnegative function of r alone such that $\chi_0 = 0$ in $B_{\delta/2}$, and $\chi_0 = 1$ outside B_δ, and set $\chi_i = (1 - \chi_0)\tilde{\chi}_i$, $i = 1, 2$. See Figure 2 for a diagram of the regions involved as well as other regions detailed below.

First note that the tension $\tau(\Xi)$ vanishes except in an annulus A and the portion of a sector $S \cap B_{\delta/2}$ where

(5.6) $$S = \{(r,\theta,\phi) \in \mathbb{R}^3 \mid \pi/4 < \theta < 3\pi/4\}, \qquad A = \{(r,\theta,\phi) \in \mathbb{R}^3 \mid \delta/2 < r < \delta\}.$$

In regions where it is nonzero we seek to bound the four quantities labelled by Roman numerals from

(5.7)
$$\begin{aligned}
|\tau(\Xi)| &= \sqrt{|\Delta u - e^{4u}|\nabla v|^2|^2 + e^{4u}|\Delta v + 4\nabla u \cdot \nabla v|^2} \\
&\leq 2\left(|\Delta u| + e^{4u}|\nabla v|^2 + e^{2u}|\Delta v| + 4e^{2u}|\nabla u \cdot \nabla v|\right) \\
&=: \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV},
\end{aligned}$$

where the tension norm is with respect to the hyperbolic space metric. We begin by estimating the tension in $S \cap B_{\delta/2}$, term by term. Observe that in this region $\chi_0 = 0$, hence $\chi_i = \tilde{\chi}_i$ for $i = 1, 2$ and Ξ is a combination of only Θ_1 and Θ_2. Since $\tilde{\chi}_2 = 1 - \tilde{\chi}_1$ we find that

(5.8) $$\mathbf{I} \leq |\Delta\chi_1||u_1 - u_2| + 2|\nabla\chi_1||\nabla(u_1 - u_2)| + \chi_1|\Delta u_1| + \chi_2|\Delta u_2|.$$

The first term of (5.8) may be estimated by C/r^2, where here and in what follows C represents a constant independent of η (or rather n) and of δ. This follows from $|\Delta\chi_1| \leq C/r^2$ and $|u_1 - u_2| \leq C$, with the latter inequality arising from (3.1) together with the leading term inequality $|\ln r_1 - \ln r_2| \leq \ln 5$ outside $B_1 \cup B_2$ which is shown in Corollary A.2; r_1 and r_2 denote the distances to p_1 and p_2 respectively. Similarly, the second term satisfies the same bound since $|\nabla\chi_1| \leq C/r$, and $|\nabla u_1 - \nabla u_2| \leq C(1/r_1 + 1/r_2) \leq C/r$ outside $B_1 \cup B_2$. Moreover, (3.1)-(3.3) imply that the last two terms satisfy $|\Delta u_i| \leq C/r^2$ outside $B_1 \cup B_2$. It follows that $\mathbf{I} \leq C/r^2$. Term $\mathbf{III}$ in (5.7) is handled analogously to $\mathbf{I}$ and thus satisfies the same estimate, using the fact that $e^{2u}|v_1 - v_2|$, $re^{2u}|\nabla v_i|$, and $r^2 e^{2u}|\Delta v_i|$ are uniformly bounded in $S \cap B_{\delta/2}$ as a consequence of Lemma 5.2 below.

Consider now term $\mathbf{IV}$ and use the expression

(5.9) $$\nabla u = (u_1 - u_2)\nabla\chi_1 + \chi_1\nabla u_1 + \chi_2\nabla u_2,$$

as well as a similar expression for v to find

(5.10)
$$\begin{aligned}
\nabla u \cdot \nabla v &= (u_1 - u_2)(v_1 - v_2)|\nabla\chi_1|^2 + \chi_1^2\nabla u_1 \cdot \nabla v_1 + \chi_2^2\nabla u_2 \cdot \nabla v_2 \\
&\quad + (u_1 - u_2)(\chi_1\nabla\chi_1 \cdot \nabla v_1 + \chi_2\nabla\chi_1 \cdot \nabla v_2) \\
&\quad + (v_1 - v_2)(\chi_1\nabla\chi_1 \cdot \nabla u_1 + \chi_2\nabla\chi_1 \cdot \nabla u_2) \\
&\quad + \chi_1\chi_2(\nabla u_1 \cdot \nabla v_2 + \nabla u_2 \cdot \nabla v_1).
\end{aligned}$$

When multiplied by e^{2u} each term on the right-hand side of (5.10) is bounded by C/r^2 in $S \cap B_{\delta/2}$, as can be seen from the uniform boundedness in this region of the following quantities discussed in the previous paragraph: $|u_1 - u_2|$, $e^{2u}|v_1 - v_2|$, $r|\nabla u_i|$, $re^{2u}|\nabla v_i|$ for $i = 1, 2$. Term $\mathbf{II}$ of (5.7) may be handled similarly replacing ∇u with $e^{2u}\nabla v$. Therefore we conclude that

(5.11) $$|\tau(\Xi)| \leq \frac{C}{r^2} \quad \text{in } B_{\delta/2}$$

for some constant C independent of η and δ, since the tension vanishes on $B_{\delta/2} \setminus S$.

Turning now our attention to the annulus A, note that away from the axis in this region $\widetilde{\Theta} = \tilde{\chi}_1 \Theta_1 + \tilde{\chi}_2 \Theta_2 = (\tilde{u}, \tilde{v})$ is bounded. We can perform a computation similar to (5.7), (5.8) utilizing

$$\text{(5.12)} \qquad\qquad \Xi = \chi_0 \Theta_0 + (1 - \chi_0)\widetilde{\Theta}$$

with u_1 replaced by $\tilde{u}$ and u_2 replaced by u_0, to obtain quantities analogous to **I**–**IV** which will be denoted with the same notation. The analogue of (5.8) becomes

$$\text{(5.13)} \qquad \mathbf{I} \leq |\Delta\chi_0||\tilde{u} - u_0| + 2|\nabla\chi_0 \cdot \nabla(\tilde{u} - u_0)| + (1 - \chi_0)|\Delta\tilde{u}| + \chi_0|\Delta u_0|.$$

As a consequence of Lemma A.3 we have that $|\tilde{u} - u_0|$ is uniformly bounded outside $B_{\delta/2}$, and moreover $|\Delta\chi_0| \leq C/\delta^2 \leq C/r^2$ in A. To estimate the gradient term note that (3.1)-(3.3) yield

$$\text{(5.14)} \qquad\qquad \tilde{u} - u_0 = \tilde{\chi}_1 \ln r_1 + \tilde{\chi}_2 \ln r_2 - \ln r + O_1(1),$$

and since $\tilde{\chi}_i$ are independent of r while χ_0 is a function of r alone, we obtain

$$\text{(5.15)} \qquad |\nabla(\tilde{u} - u_0) \cdot \nabla\chi_0| \leq \left(|\tilde{\chi}_1 \nabla \ln r_1 + \tilde{\chi}_2 \nabla \ln r_2 - \nabla \ln r| + O(r^{-1}) \right) |\nabla\chi_0| \leq \frac{C}{r^2}.$$

Furthermore, as indicated above $|\Delta u_i| \leq C/r^2$, $i = 1, 2$ outside of $B_1 \cup B_2$ and hence on A; the same is true also for $i = 0$. Therefore the last two terms of (5.13) admit the inverse square estimate by routine computations. Putting this together then produces $\mathbf{I} \leq C/r^2$ on A. As before, term **II** admits the desired estimate using a version of (5.10), together with the fact that $e^{2u}|\tilde{v} - v_0|$ and $re^{2u}(|\nabla\tilde{v}| + |\nabla v_0|)$ are uniformly bounded in A by Lemma 5.3 below. However, since the estimates for Δv_0 in this same lemma involve a nonzero ς, we cannot directly follow the approach used in $S \cap B_{\delta/2}$ for **III**. Instead, terms **III** and **IV** of (5.7) will be replaced by $e^{2u}|\Delta v + 4\nabla u \cdot \nabla v|$. Observe that a calculation similar to that of (5.13) in which u is replaced by v, combined with the estimates of Lemma 5.3 yields

$$\text{(5.16)} \qquad\qquad e^{2u}\Delta v = e^{2u}\chi_0 \Delta v_0 + O\left(\frac{1}{r^2}\right)$$

in A. Moreover a computation, along with (3.1)-(3.3), and again Lemma 5.3 show that in this region

$$
\begin{aligned}
\nabla u \cdot \nabla v &= \chi_0 \nabla u_0 \cdot \nabla v_0 + (v_0 - \tilde{v})\nabla u_0 \cdot \nabla\chi_0 + (1 - \chi_0)\nabla u_0 \cdot \nabla\tilde{v} \\
&\quad + \nabla[(1 - \chi_0)(\tilde{U} - U_0)] \cdot \nabla[\tilde{v} + \chi_0(v_0 - \tilde{v})] \\
&= \chi_0 \nabla u_0 \cdot \nabla v_0 + O\left(\frac{e^{-2u}}{r^2}\right),
\end{aligned}
$$

$$\text{(5.17)}$$

where $\tilde{U} = \tilde{u} + \ln\rho$ and $U_0 = u_0 + \ln\rho$. Therefore since (u_0, v_0) is harmonic we find that

$$\text{(5.18)} \qquad\qquad e^{2u}|\Delta v + 4\nabla u \cdot \nabla u| \leq \frac{C}{r^2},$$

where C is independent of η and δ. Thus, the tension of Ξ satisfies the same estimate of (5.11) within the annulus A.

At this stage notation indicating the dependence on n will be reintroduced. We may summarize what has been established with the following properties of the model map tension

$$\text{(5.19)} \qquad |\tau(\Xi_n)| \leq \frac{C}{r^2} \quad \text{on } B_\delta \setminus \Gamma, \qquad |\tau(\Xi_n)| = 0 \quad \text{on } (\mathbb{R}^3 \setminus B_\delta) \setminus \Gamma,$$

where C is a constant independent of n and δ. Let

$$(5.20) \qquad \Lambda(\Phi_n, \Xi_n) = \sqrt{1 + d_{\mathbb{H}^2}(\Phi_n, \Xi_n)^2}$$

and observe that by [30, Lemma 1] we have

$$(5.21) \qquad \Delta\Lambda(\Phi_n, \Xi_n) \geq -|\tau(\Xi_n)| \geq \begin{cases} -\frac{C}{r^2} & \text{on } B_\delta \setminus \Gamma \\ 0 & \text{on } (\mathbb{R}^3 \setminus B_\delta) \setminus \Gamma \end{cases}.$$

Now define a radial function

$$(5.22) \qquad w(r) = \begin{cases} C\ln\left(\frac{\delta}{r}\right) + C & \text{on } B_\delta \\ \dfrac{C\delta}{r} & \text{on } \mathbb{R}^3 \setminus B_\delta \end{cases},$$

and note that $w \in C^{1,1}(\mathbb{R}^3 \setminus \{0\})$ with

$$(5.23) \qquad \Delta w = \begin{cases} -\dfrac{C}{r^2} & \text{on } B_\delta \\ 0 & \text{on } \mathbb{R}^3 \setminus B_\delta \end{cases},$$

from which it follows that

$$(5.24) \qquad \Delta\left(\Lambda(\Phi_n, \Xi_n) - w\right) \geq 0 \quad \text{on } \mathbb{R}^3 \setminus (\Gamma \cup \partial B_\delta).$$

Although this inequality is satisfied only outside the axis $\Gamma \cup \partial B_\delta$, [28, Lemma 8] implies it is satisfied weakly on all of $\mathbb{R}^3$ so that the maximum principle applies. Since Φ_n and Ξ_n are asymptotic by Lemma A.4 and Remark A.5, we find that $\Lambda(\Phi_n, \Xi_n) - w \to 1$ at infinity. Moreover, $\Lambda(\Phi_n, \Xi_n) - w < 0$ on the boundary of a small enough ball centered at the origin. Hence, the maximum principle then yields the L^∞-bound

$$(5.25) \qquad d_{\mathbb{H}^2}(\Phi_n, \Psi_n) = d_{\mathbb{H}^2}(\Phi_n, \Xi_n) \leq \sqrt{w(w+2)} = \frac{\sqrt{C\delta(C\delta + 2r)}}{r} \quad \text{on } \mathbb{R}^3 \setminus (B_\delta \cup \Gamma),$$

where C is independent of n and δ.

Case 2. Consider the situation in which there is only a single collision location with multiple punctures colliding, and for convenience denote the z-coordinates of the N_1 punctures p_i colliding at the one location by $z_1 < \cdots < z_{N_1}$. Without loss of generality it may be assumed that $\sum_{i=1}^{N_1} z_i = 0$. Let $z_{0,i} = (z_{i+1} + z_i)/2$, $\eta = 1/4 \cdot \min_{1 \leq i \leq N_1 - 1}(z_{i+1} - z_i)$, and $B_i = B_\eta(p_i)$. Clearly the balls $B_1, \ldots, B_{N_1}$ are mutually disjoint. Now set $\vartheta = \frac{\pi}{2(N_1-1)}$ and for $i = 1, \ldots, N_1 - 1$ define the sectors

$$(5.26) \qquad S_i = \{(\tilde{r}_i, \tilde{\theta}_i, \phi) \in \mathbb{R}^3 \mid \tfrac{3\pi}{4} - i\vartheta < \tilde{\theta}_i < \tfrac{3\pi}{4} - (i-1)\vartheta\},$$

where $(\tilde{r}_i, \tilde{\theta}_i, \phi)$ are polar coordinates centered at the point $\tilde{p}_i \in \Gamma$ with z-coordinate $z_{0,i}$. The sectors S_i are mutually disjoint, and are also disjoint from the balls $\cup_{i=1}^{N_1} B_i$. We will denote the components of the complement of $\cup_{i=1}^{N_1-1} S_i$ by $T_1, \ldots, T_{N_1}$ in order of increasing z. For each $i = 1, \ldots, N_1$ construct cut-off functions $\tilde{\chi}_i \geq 0$ satisfying the following properties: $\tilde{\chi}_i$ is a function of only $\tilde{\theta}_i$ in S_i and is a function of only $\tilde{\theta}_{i-1}$ in S_{i-1}, with $\tilde{\chi}_i = 1$ in T_i, $\tilde{\chi}_i + \tilde{\chi}_{i+1} = 1$

in S_i, and $\sum_{i=1}^{N_1} \tilde{\chi}_i = 1$ globally. Motivated by the arguments from Case 1, the model map will be taken in the form

$$(5.27) \qquad \Xi = (u, v) = \chi_0 \Theta_0 + \chi_1 \Theta_1 + \cdots + \chi_{N_1} \Theta_{N_1},$$

where χ_0 is a nonnegative function of r alone which vanishes in $B_{\delta/2}$ and equals 1 outside B_δ, and $\chi_i = (1 - \chi_0)\tilde{\chi}_i$ so that $\sum \chi_i = 1$. As before n may be taken large enough to guarantee that $\cup_{i=1}^{N_1} B_i$ and the collision location lie within $B_{\delta/4}$, where r respectively B_δ are the Euclidean distance and δ-ball centered at the origin which serves as the center of mass of the punctures. Moreover Θ_i, $i = 1, \ldots, N_1$ denotes the extreme Kerr harmonic map associated with each colliding puncture, and Θ_0 is the multi-extreme Kerr auxiliary collision configuration map Ψ.

The tension $\tau(\Xi)$ vanishes except possibly in the regions $S_i \cap B_{\delta/2}$ and $A = B_\delta \setminus \overline{B}_{\delta/2}$. Note that in each sector S_i only two of the cut-off functions, $\tilde{\chi}_i$ and $\tilde{\chi}_{i+1}$, are ever nonconstant. Therefore, a straightforward generalization of the arguments leading to the tension estimate (5.11) in $B_{\delta/2}$ apply here to yield the same bound with $1/r^2$ replaced by $\sum_{i=1}^{N_1-1} 1/\tilde{r}_i^2$. Similarly, minor modifications to the methods of Case 1 provide the same estimate with $1/r^2$ in the annulus A. Since $\tilde{r}_i$ is comparable with r inside A it follows that

$$(5.28) \qquad |\tau(\Xi_n)| \le C \sum_{i=1}^{N_1-1} \frac{1}{\tilde{r}_i^2} \quad \text{on } B_\delta \setminus \Gamma, \qquad |\tau(\Xi_n)| = 0 \quad \text{on } (\mathbb{R}^3 \setminus B_\delta) \setminus \Gamma,$$

where C is a constant independent of n and δ. Define radial functions $w_i \in C^{1,1}(\mathbb{R}^3 \setminus \{\tilde{p}_i\})$, $i = 1, \ldots N_1 - 1$ by

$$(5.29) \qquad w_i(\tilde{r}_i) = \begin{cases} C \ln\left(\dfrac{2\delta}{\tilde{r}_i}\right) + C & \text{on } B_{2\delta}(\tilde{p}_i) \\ \dfrac{2C\delta}{\tilde{r}_i} & \text{on } \mathbb{R}^3 \setminus B_{2\delta}(\tilde{p}_i) \end{cases},$$

and note that as in (5.21)-(5.24) we have

$$(5.30) \qquad \Delta\left(\Lambda(\Phi_n, \Xi_n) - \sum_{i=1}^{N_1-1} w_i\right) \ge 0 \quad \text{on } \mathbb{R}^3 \setminus \left(\Gamma \cup_{i=1}^{N_1-1} \partial B_{2\delta}(\tilde{p}_i)\right).$$

The weak maximum principle argument may then be applied as before to obtain

$$(5.31) \qquad d_{\mathbb{H}^2}(\Phi_n, \Psi_n) = d_{\mathbb{H}^2}(\Phi_n, \Xi_n) \le \sqrt{w(w+2)} \le \frac{\sqrt{C\delta(C\delta + 2r)}}{r} \quad \text{on } \mathbb{R}^3 \setminus (B_{3\delta} \cup \Gamma),$$

where $w = \sum_{i=1}^{N_1-1} w_i$ for some constant C is independent of n and δ.

Case 3. Consider now the situation in which there are simultaneous collisions at $k > 1$ separate locations, with punctures p_i having z-coordinates $z_1 < \cdots < z_{N_1}$ colliding at the first location, those labelled by $z_{N_1+1} < \cdots < z_{N_2}$ colliding at the second location and so on, up to those labelled by $z_{N_{k-1}+1} < \cdots < z_{N_k}$ colliding at the kth location. For convenience of notation the indices moving from one collision group to another are consecutive, however there may be noncolliding punctures located between the collision groups. Taking η to be $1/4$ of the minimum distance between any two colliding punctures, we can repeat the arguments of Case 2 inside the balls $B_\delta(\bar{p}_j)$, $j = 1, \ldots, k$ where $\bar{p}_j \in \Gamma$ is the point with z-coordinate given by the mean of $z_{N_{j-1}+1}, \ldots, z_{N_j}$, and δ is chosen so that the larger balls $B_{3\delta}(\bar{p}_j)$ are mutually disjoint;

here $N_0 = 0$. When n is sufficiently large, η is small enough to guarantee that each $B_\eta(p_i)$, for $N_{j-1} + 1 \leq i \leq N_j$, is contained within the corresponding ball $B_{\delta/4}(\bar{p}_j)$. The resulting model map Ξ_n transitions within each $B_{\delta/2}(\bar{p}_j)$ from one extreme Kerr solution to another across sector regions, and it further transitions across the associated k annuli out to the multi-extreme Kerr auxiliary collision configuration map Ψ_n on the compliment of the δ-balls. The methods of Case 2 lead to a tension estimate

$$(5.32) \qquad |\tau(\Xi_n)| \leq C \sum_{i=N_{j-1}+1}^{N_j - 1} \frac{1}{\tilde{r}_i^2} \quad \text{on } B_\delta(\bar{p}_j) \setminus \Gamma, \ j = 1, \ldots, k,$$

$$|\tau(\Xi_n)| = 0 \quad \text{on } \left(\mathbb{R}^3 \setminus \cup_{j=1}^{k} B_\delta(\bar{p}_j) \right) \setminus \Gamma,$$

where the $\tilde{r}_i$ denote Euclidean distance to the sector vertices. As above, this leads via the maximum principle to the distance bound

$$(5.33) \qquad d_{\mathbb{H}^2}(\Phi_n, \Psi_n) = d_{\mathbb{H}^2}(\Phi_n, \Xi_n) \leq \frac{\sqrt{C\delta(C\delta + 2\bar{r})}}{\bar{r}} \quad \text{on } \mathbb{R}^3 \setminus \left(\cup_{j=1}^{k} B_{3\delta}(\bar{p}_j) \cup \Gamma \right),$$

for some constant C independent of n and δ where $\bar{r} = \min\{\bar{r}_1, \ldots, \bar{r}_k\}$ with $\bar{r}_j$ representing the distance to $\bar{p}_j$. For convenience we will denote the region on which (5.33) holds by D_δ.

Having achieved the L^∞-bound, we are now able to prove (5.2) and (5.3). According to (5.4), the sequence Ψ_n converges on $\mathbb{R}^3 \setminus \Gamma$ to the collision configuration map Φ_*. Thus, by standard theory of harmonic maps (5.33) implies that Ψ_n subconverges uniformly together with all of its derivatives on compact subsets of D_δ; see [30, Proof of Theorem 1] for a proof in the current setting. Letting now $\delta \to 0$ and passing to a diagonal subsequence (without changing notation), the same is true on $\mathbb{R}^3 \setminus \Gamma$. Let $\widetilde{\Phi}_*$ denote the limit of Φ_n, then we claim that $\widetilde{\Phi}_* = \Phi_*$. Since both of these maps are harmonic, this conclusion would follow immediately if they were known to be asymptotic. However, the estimates above do not show this property due to the blow-up at certain points on the axis. Instead, we can use (5.33) directly to show that $\Phi_n \to \Phi_*$. Indeed, if $\delta_2 >> \delta_1 > 0$ then (5.33) implies that $d_{\mathbb{H}^2}(\Phi_n, \Psi_n) = O(\sqrt{\delta_1})$ on D_{δ_2}. In particular, by keeping $\delta_2 > 0$ fixed and sending δ_1 to zero as $n \to \infty$, we find that Φ_n and Ψ_n must converge to the same limit Φ_* on D_{δ_2}. Since δ_2 is arbitrary, it follows that $\widetilde{\Phi}_* = \Phi_*$. This establishes (5.2).

To complete the proof of the proposition let $\epsilon > 0$ be given, and set $\mathcal{B}_\delta = \cup_{j=1}^{k} B_\delta(p_j^*)$ where p_j^* denote the k distinct collision points having z-coordinates z_j^*. Since Φ_* has finite renormalized energy, δ may be chosen small enough to guarantee that $\mathcal{E}_{\mathcal{B}_\delta}(\Phi_*) < \epsilon/2$. Furthermore, since $\widetilde{\Phi}_* = \Phi_*$ we have that $\mathcal{E}_{\mathbb{R}^3 \setminus \mathcal{B}_\delta}(\Phi_n) \to \mathcal{E}_{\mathbb{R}^3 \setminus \mathcal{B}_\delta}(\Phi_*)$ as $n \to \infty$. To see this in more detail, note that since the difference of energy densities converges to zero in L^1 on any fixed compact subset of $\mathbb{R}^3 \setminus \mathcal{B}_\delta$ with the aid of [12, Theorem 2.1], the same holds if the fixed set is replaced by a slow sequence of exhausting domains. This fact, combined with estimates for the difference of energy densities in the complement of the exhausting domains obtained from the expansions [12, Theorem 2.3], produces the desired conclusion. Thus, there exists a sufficiently large n_ϵ such that

$$(5.34) \qquad \mathcal{E}(\Phi_*) = \mathcal{E}_{\mathcal{B}_\delta}(\Phi_*) + \mathcal{E}_{\mathbb{R}^3 \setminus \mathcal{B}_\delta}(\Phi_*) < \epsilon + \mathcal{E}_{\mathbb{R}^3 \setminus \mathcal{B}_\delta}(\Phi_{n_\epsilon}) \leq \epsilon + \mathcal{E}(\Phi_{n_\epsilon}),$$

which yields (5.3). $\qquad \square$

The proof of Proposition 5.1 relies on two lemmas which we now establish. In what follows the notation will be consistent with that utilized in the proof of this proposition.

Lemma 5.2. *Let $\eta, \delta > 0$ be small parameters with $\delta > 12\eta$. Consider two extreme Kerr harmonic maps (u_i, v_i), $i = 1, 2$ with punctures located at $p_1 = (0, 0, -2\eta)$ and $p_2 = (0, 0, 2\eta)$ in Cartesian coordinates, and set $(u, v) = \sum_{i=1}^{2} \chi_i(u_i, v_i)$ where χ_i are cut-off functions from Case 1 in the proof of Proposition 5.1. Then there exists a constant C independent of η and δ such that*

$$(5.35) \qquad e^{2u}|v_1 - v_2| \le C, \qquad re^{2u}|\nabla v_i| \le C, \qquad r^2 e^{2u}|\Delta v_i| \le C,$$

in sector $S \cap B_\delta$ for $i = 1, 2$.

Proof. Let (r_i, θ_i, ϕ) be polar coordinates in $\mathbb{R}^3$ centered at p_i, $i = 1, 2$ and let $a_i > 0$ be parameters, then recall that the extreme Kerr harmonic map [17, Appendix B] with puncture p_i and angular momentum a_i^2 is given by

$$(5.36) \qquad u_i = -\ln \sin \theta_i - \frac{1}{2} \ln \left((r_i + a_i)^2 + a_i^2 + \frac{2a_i^2(r_i + a_i) \sin^2 \theta_i}{(r_i + a_i)^2 + a_i^2 \cos^2 \theta_i} \right),$$

$$(5.37) \qquad v_i = a_i^2 \cos \theta_i (3 - \cos^2 \theta_i) + \frac{a_i^4 \cos \theta_i \sin^4 \theta_i}{(r_i + a_i)^2 + a_i^2 \cos^2 \theta_i}.$$

The extreme Kerr map with angular momentum $-a_i^2$ may be obtained by replacing v_i with $-v_i$. Furthermore, for the purposes of this lemma we may assume without loss of generality that the angular momentum at p_1 is $a/2$ and that the angular momentum at p_2 is $1/2$ for some $a \ne 0$, so that

$$(5.38) \quad v_1 = \frac{a}{2} \cos \theta_1 (3 - \cos^2 \theta_1) - a - 1 + O(\sin^4 \theta_1), \quad v_2 = \frac{1}{2} \cos \theta_2 (3 - \cos^2 \theta_2) + O(\sin^4 \theta_2).$$

Generality is achieved by noting that for any constant c the map $(u, v) \to (u + c, e^{-2c}v)$ is an isometry in the target hyperbolic space. Even though such 'hyperbolic translations' do not preserve asymptotic flatness of the associated spacetime, the estimates in (5.35) will be preserved. Note that the potential constants for v_1 and v_2 agree (and equal -1) on the axis rod between the two punctures.

Using $\rho = r \sin \theta = r_i \sin \theta_i$ and $z = r \cos \theta = r_1 \cos \theta_1 - 2\eta = r_2 \cos \theta_2 + 2\eta$, we find in the region $S \cap B_{\delta/2}$ (where $\theta_1 < \pi/2$ and $\theta_2 > \pi/2$) that

$$(5.39) \qquad \cos \theta_{1,2} = \frac{\pm 1}{\sqrt{1 + \tan^2 \theta_{1,2}}} = \pm \left(1 + \left(\frac{r \sin \theta}{r \cos \theta \pm 2\eta} \right)^2 \right)^{-1/2},$$

and consequently if $x = 2\eta/r$ then

$$(5.40) \qquad \sin^2 \theta_{1,2} = 1 - \cos^2 \theta_{1,2} = \frac{\sin^2 \theta}{1 \pm 2x \cos \theta + x^2}.$$

It follows that

$$(5.41) \qquad v_1 - v_2 = W(x, \theta) + O \left(\frac{\sin^4 \theta}{(1 + 2x \cos \theta + x^2)^2} + \frac{\sin^4 \theta}{(1 - 2x \cos \theta + x^2)^2} \right),$$

where

$$(5.42) \quad W(x,\theta) = \frac{a|\cos\theta + x|}{2\sqrt{1 + 2x\cos\theta + x^2}}\left(3 - \frac{(\cos\theta + x)^2}{1 + 2x\cos\theta + x^2}\right) - a - 1$$
$$+ \frac{|\cos\theta - x|}{2\sqrt{1 - 2x\cos\theta + x^2}}\left(3 - \frac{(\cos\theta - x)^2}{1 - 2x\cos\theta + x^2}\right).$$

Moreover

$$(5.43) \quad u = \chi_1 u_1 + \chi_2 u_2 = -\ln\sin\theta + \frac{\chi_1}{2}\ln(1 + 2x\cos\theta + x^2) + \frac{\chi_2}{2}\ln(1 - 2x\cos\theta + x^2) + O(1),$$

and hence

$$(5.44) \quad e^{2u}|v_1 - v_2| \leq C\frac{(1 - 2x\cos\theta + x^2)^{\chi_1}(1 + 2x\cos\theta + x^2)^{\chi_2}}{\sin^2\theta}W(x,\theta) + O(1)$$

in $S \cap B_\delta$, for some constant C independent of η and δ. By expanding in powers of x and noting several cancellations, a calculation shows that

$$(5.45) \quad |W(x,\theta)| \leq \frac{C}{x^2} \quad \text{for all } x \geq 1 \text{ and } \pi/4 < \theta < 3\pi/4.$$

We also observe that within the region given by $x \leq 1$ and $\pi/4 < \theta < 3\pi/4$, the function $W(x,\theta)$ remains uniformly bounded. These facts, together with $\chi_1 + \chi_2 = 1 - \chi_0 \leq 1$, imply that the first inequality of (5.35) is valid.

Consider now the second inequality of (5.35) and notice that for $i = 1, 2$ a direct computation yields

$$(5.46) \quad |\nabla v_i| \leq C\frac{\sin^3\theta_i}{r_i}.$$

Next use $u_i = -\ln\sin\theta_i + O(1)$ and $\chi_1 + \chi_2 = 1 - \chi_0$ to find

$$(5.47) \quad e^{2u}|\nabla v_i| \leq e^{2|1 - \chi_0 - \chi_i||u_1 - u_2|}e^{2(1 - \chi_0)u_i}|\nabla v_i| \leq C\frac{e^{2|u_1 - u_2|}\sin\theta_i}{r_i}.$$

Moreover, for all $x > 0$ and $\pi/4 < \theta < 3\pi/4$ we obtain

$$(5.48) \quad e^{2(u_1 - u_2)} \leq C\frac{\sin^2\theta_2}{\sin^2\theta_1} \leq C\frac{1 + 2x\cos\theta + x^2}{1 - 2x\cos\theta + x^2} \leq 10C$$

with the help of (5.40), and a similar inequality holds for $e^{2(u_2 - u_1)}$. Since $1/r_i \leq C/r$ for some uniform constant C at all points of S by Corollary A.2, we conclude that the second inequality of (5.35) is valid.

Consider now the third inequality of (5.35) and note that for $i = 1, 2$ we have

$$(5.49) \quad |\Delta v_i| \leq C\frac{\sin^2\theta_i}{r_i^2}.$$

As above this gives rise to

$$(5.50) \quad e^{2u}|\Delta v_i| \leq e^{2|1 - \chi_0 - \chi_i||u_1 - u_2|}e^{2(1 - \chi_0)u_i}|\Delta v_i| \leq C\frac{e^{2|u_1 - u_2|}}{r_i^2},$$

and the desired estimate follows from (5.48). $\qquad\square$

Lemma 5.3. *Let $\eta, \delta > 0$ be small parameters with $\delta > 12\eta$. Using the notation from Case 1 in the proof of Proposition 5.1, consider a multi-extreme Kerr harmonic map (u_0, v_0) having a puncture at the collision location in $B_{\delta/4}$ with all other punctures occurring outside a fixed radius, and set $(\tilde{u}, \tilde{v}) = \sum_{i=1}^{2} \tilde{\chi}_i(u_i, v_i)$ using two extreme Kerr harmonic maps (u_i, v_i), $i = 1, 2$ having punctures located at $p_1 = (0, 0, -2\eta)$ and $p_2 = (0, 0, 2\eta)$ in Cartesian coordinates. Then there exists a constant C independent of η and δ such that*

$$(5.51) \qquad (\sin\theta)^{-1-\varsigma} e^{2u} |\tilde{v} - v_0| \le C, \qquad r e^{2u}\left((\sin\theta)^{-1}|\nabla\tilde{v}| + (\sin\theta)^{-\varsigma}|\nabla v_0|\right) \le C,$$

$$(5.52) \qquad\qquad r^2 e^{2u}\left(|\Delta\tilde{v}| + (\sin\theta)^{1-\varsigma}|\Delta v_0|\right) \le C,$$

in annulus $A = B_\delta \setminus \overline{B}_{\delta/2}$ where $u = \chi_0 u_0 + (1 - \chi_0)\tilde{u}$, and $\varsigma \in (0, 1)$ is arbitrary but fixed.

Proof. Consider the first expression of (5.51) in $A \cap S$, and observe that

$$(5.53) \qquad\qquad e^{2u}|\tilde{v} - v_0| \le e^{2u}|\tilde{\chi}_2(v_1 - v_2)| + e^{2u}|v_1 - v_0|.$$

Applying Lemma 5.2 in this region yields

$$(5.54) \qquad\qquad e^{2u}|\tilde{\chi}_2(v_1 - v_2)| \le e^{2\chi_0 u_0}\left(e^{2(\chi_1 u_1 + \chi_2 u_2)}|v_1 - v_2|\right) \le C,$$

where we have also used that $e^{2\chi_0 u_0}$ remains uniformly bounded due to (3.1)-(3.3) combined with the restriction of angles inherent within $A \cap S$. It should be noted that even though the collision location which serves as the puncture for the map (u_0, v_0) does not necessarily coincide with the origin, its distance to the origin tends to zero as $\eta \to 0$, so that polar coordinate angles measured from the collision location are comparable to those measured from the origin. Moreover, $v_1 - v_0$ may be computed analogously to (5.41), and arguments as in Lemma 5.2 may be utilized to estimate $e^{2(\chi_0 u_0 + \chi_1 u_1)}|v_1 - v_0|$ inside $A \cap S$. Again since $e^{2\chi_2 u_2}$ is controlled here we obtain a bound for $e^{2u}|v_1 - v_0|$, and hence the desired inequality for the first term of (5.51) follows in $A \cap S$. In this process, the expansion (3.1)-(3.3) allows us to effectively replace (u_0, v_0) with a single extreme Kerr. Similar considerations show that the proof of Lemma 5.2 also yields the desired estimates for the remaining quantities of (5.51) and (5.52), in $A \cap S$. In fact, ς plays no role here and may be taken to be zero.

Consider now the region $A \setminus S = A_+ \cup A_-$, where $A_\pm$ are the portions of the annulus with $0 \le \theta \le \pi/4$ and $3\pi/4 \le \theta \le \pi$, respectively. We will treat the A_+ region, and note that similar arguments hold for A_-. First observe that in this region $\tilde{\chi}_1 = 0$, and therefore $\tilde{v} = v_2$. We may assume without loss of generality that the potential constants for v_1 and v_2 are as in the proof of Lemma 5.2. Then (3.3) and (5.38) produce

$$(5.55) \quad v_0 = \frac{a+1}{2}\cos\theta_0(3 - \cos^2\theta_0) - a + O(\sin^{3+\varsigma}\theta_0), \quad v_2 = \frac{1}{2}\cos\theta_2(3 - \cos^2\theta_2) + O(\sin^4\theta_2),$$

where (r_0, θ_0, ϕ) are polar coordinates centered at the collision location and $\varsigma \in (0, 1)$. Using the identity

$$(5.56) \qquad\qquad \frac{1}{2}\cos\theta(3 - \cos^2\theta) = 1 - 6\sin^4(\theta/2) + 4\sin^6(\theta/2),$$

it follows that

$$(5.57) \qquad\qquad \tilde{v} - v_0 = O(\sin^{3+\varsigma}\theta_0 + \sin^4\theta_2).$$

Next utilize (3.1) and similar computations as in (5.43) to find

(5.58) $$u = \chi_0 u_0 + (1 - \chi_0)u_2 = -\ln\sin\theta + O(1),$$

and hence within A_+ we have

(5.59) $$e^{2u}|\tilde{v} - v_0| \leq C\left(\frac{\sin^{3+\varsigma}\theta_0}{\sin^2\theta} + \frac{\sin^4\theta_2}{\sin^2\theta}\right)$$

for some uniform constant C. Furthermore, (5.40) implies

(5.60) $$\frac{\sin^2\theta_2}{\sin^2\theta} = (1 - 2x\cos\theta + x^2)^{-1} = \left((x - \cos\theta)^2 + \sin^2\theta\right)^{-1} \leq (1/\sqrt{2} - 1/3)^{-2}$$

since $x = 2\eta/r < 4\eta/\delta < 1/3$ in A_+, and a similar estimate holds if θ_2 is replaced by θ_0. Hence, the first inequality of (5.51) is satisfied in A_+. Moreover, these observations also show that the derivative inequalities of (5.51) and (5.52) involving $\tilde{v}$ follow from (5.46) and (5.49), together with the fact that r_i, $i = 1, 2$ are comparable with r in A by Lemma A.3. Lastly, the derivative inequalities involving v_0 may be obtained by replacing (5.46) and (5.49) with

(5.61) $$|\nabla v_0| \leq C\frac{\sin^{2+\varsigma}\theta_0}{r_0}, \qquad |\Delta v_0| \leq C\frac{\sin^{1+\varsigma}\theta_0}{r_0^2},$$

which is due to (3.1)-(3.3). We conclude that the desired estimates are valid on all of A. $\qquad\square$

6. SCATTERING OF PUNCTURES

Consider the setup at the beginning of Section 5, in which the flow of harmonic maps $\Phi_t = (u_t, v_t)$ having punctures with z-coordinates $z_i(t)$, $1 \leq i \leq N$ with $z_1(0) < \cdots < z_N(0)$, has a maximal time of existence T (which may be infinite). At the maximal time at least one of the possibilities from (4.1) must transpire. In the former case, namely when $\limsup_{t \to T}(z_{j+1}(t) - z_j(t)) = \infty$ for some $j \in \{1, \ldots, N-1\}$, we say that *scattering* occurs at time T. In a manner similar to that used in the proof of Proposition 5.1, we will first handle the case where only one rod length is unbounded and no collisions occur. We will then subsequently generalize to the other cases.

Suppose that only one axis rod $\Gamma_j = (z_{j-1}, z_j)$ of length ℓ becomes unbounded. There is then a sequence of times $t_n \to T$, and associated maps $\Phi_n = \Phi_{t_n}$ such that the jth-rod lengths satisfy $\ell_n \to \infty$. Without loss of generality, it may be assumed that the origin is always at the midpoint of this rod. In particular, the punctures $p_1, \ldots, p_{j-1}$ lie below the origin, and the punctures $p_j, \ldots, p_N$ lie above the origin. We point out that z_i and p_i here are associated with the harmonic map Φ_n, although the subscript n is suppressed for brevity. Since no collisions occur and only one rod length becomes unbounded, it follows that after translating by $z \mapsto z + \ell_n/2$ the punctures $p_1, \ldots, p_{j-1}$ subconverge to distinct points with z-coordinates $z_1^*, \ldots, z_{j-1}^*$. Similarly, after translating by $-\ell_n/2$ the punctures $p_j, \ldots, p_N$ also subconverge to distinct points with z-coordinates $z_j^*, \ldots, z_N^*$. For simplicity of notation, we will assume that they all converge along the original sequence. The two sets of starred punctures will be referred to as the lower and upper *separation configurations*, respectively. Denote by Φ_*^i, $i = 1, 2$ the harmonic maps associated with these two set of punctures. More precisely, Φ_*^1 admits punctures $z_1^*, \ldots, z_{j-1}^*$ while Φ_*^2 admits punctures $z_j^*, \ldots, z_N^*$, and both maps have the same potential constants as Φ_n

for corresponding rods. The Φ_*^i are referred to as *separation configuration maps.* We point out that different sequences $t_n \to T$ may result in different separation configurations and seaparation configuration maps. Nevertheless, the limiting renormalized energy is the same for all sequences, that is $\lim_{t \to T} \mathcal{E}(\Phi_t)$ exists in light of the monotonicity provided by Theorem 4.8. The next result will allow the flow of harmonic maps to 'pass through' the maximal time of existence when single rod scattering happens, while maintaining the desired monotonicity property with the use of separation configuration maps.

Proposition 6.1. *Let Φ_t be the 1-parameter family of harmonic maps with fixed potential constants defined by the flow of punctures (3.10), in which scattering occurs along a single axis rod and no collisions occur at the limiting time T. If Φ_*^1 and Φ_*^2 are separation configuration maps described above, then*

$$(6.1) \qquad \mathcal{E}(\Phi_*^1) + \mathcal{E}(\Phi_*^2) \le \lim_{t \to T} \mathcal{E}(\Phi_t).$$

Proof. As described in the discussion preceding the statement of the proposition, let $\{t_n\}$ be a sequence of times with $t_n \to T$, which leads to lower and upper separation configurations $\{z_1^*, \ldots, z_{j-1}^*\}$ and $\{z_j^*, \ldots, z_N^*\}$ along with the given separation configuration maps Φ_*^1 and Φ_*^2. For convenience write $\Phi_n = \Phi_{t_n}$. Denote by $\tilde{\Psi}_n^i$, $i = 1, 2$ the harmonic maps associated with the set of punctures below and above the origin. More precisely, $\tilde{\Psi}_n^1$ admits punctures $p_1, \ldots, p_{j-1}$ while $\tilde{\Psi}_n^2$ admits punctures $p_j, \ldots, p_N$, and both maps have the same potential constants as Φ_n for corresponding rods. Furthermore, denote by Ψ_n^i the z-translation of the harmonic maps $\tilde{\Psi}_n^i$, $i = 1, 2$ by $\ell_n/2$ and $-\ell_n/2$, respectively. We call these translated versions the lower and upper *auxiliary separation configuration maps.* According to Theorem 3.1 we have that

$$(6.2) \qquad \Psi_n^i \to \Phi_*^i \quad \text{in } C^3 \text{ away from the axis,}$$

for $i = 1, 2$.

As in the proof of Proposition 5.1, we will construct a model map Ξ_n for Φ_n which coincides with $\tilde{\Psi}_n^i$ outside a fixed set. Let us divide $\mathbb{R}^3$ into four regions: a ball $B = B_\varrho(0)$ where $\varrho > 0$ is independent of n and is chosen small enough so that this set does not contain any punctures, a partial sector

$$(6.3) \qquad S = \{(r, \theta, \phi) \in \mathbb{R}^3 \mid \pi/4 \le \theta \le 3\pi/4\} \setminus B$$

which will serve as a transition region, and their complement $\Omega_1 \sqcup \Omega_2 = \mathbb{R}^3 \setminus (S \cup B)$ where $3\pi/4 < \theta < \pi$ in Ω_1 and $0 < \theta < \pi/4$ in Ω_2. Define a model map by

$$(6.4) \qquad \Xi_n = (u_n, v_n) = \chi_1 \tilde{\Psi}_n^1 + \chi_2 \tilde{\Psi}_n^2,$$

where the cut-off functions χ_i, $i = 1, 2$ are smooth and satisfy the following properties. Namely, they are functions only of θ alone outside of B, $0 \le \chi_i \le 1$, $\chi_1 + \chi_2 = 1$ globally, and $\chi_i = 1$ on Ω_i. These functions may also be chosen such that $|\nabla \chi_i| \le C/(1+r)$ and $|\nabla^2 \chi_i| \le C/(1+r^2)$. Note that $\Xi_n = \tilde{\Psi}_n^i$ in Ω_i for $i = 1, 2$, and hence the tension $\tau(\Xi_n)$ is nonzero only in $S \cup B$. For convenience, we will temporarily omit the subscript n and write $\Xi_n = (u, v)$ as well as $\tilde{\Psi}_n^i = (\tilde{u}_i, \tilde{v}_i)$. Then as in (5.7) we have

$$(6.5) \qquad |\tau(\Xi)| \le 2 \left(|\Delta u| + e^{4u} |\nabla v|^2 + e^{2u} |\Delta v + 4\nabla u \cdot \nabla v| \right) =: 2 \left(\mathbf{I} + \mathbf{II} + \mathbf{III} \right).$$

Estimates for the tension will be made separately in B and S.

Consider the region B. Using $\chi_1 + \chi_2 = 1$ observe that

$$(6.6) \qquad \Delta u = \chi_1 \Delta \tilde{u}_1 + \chi_2 \Delta \tilde{u}_2 + 2\nabla \chi_1 \cdot \nabla(\tilde{u}_1 - \tilde{u}_2) + (\tilde{u}_1 - \tilde{u}_2)\Delta \chi_1.$$

In this domain $|\tilde{u}_1 - \tilde{u}_2|$, $|\nabla(\tilde{u}_1 - \tilde{u}_2)|$, and $|\Delta \tilde{u}_i|$ are all uniformly bounded independent of n. Indeed, since B is far away from the punctures this follows from the expansion of (3.4) which shows that $\tilde{u}_i = -\ln\rho + O(\ell_n^{-1})$ in B, with corresponding fall-off for derivatives. Note here that $\ln\rho$ is harmonic, and that the error estimates are bounded independent of n in light of Theorem 3.1 since the puncture configurations of the maps $\tilde{\Psi}_n^i$ converge modulo translation. Therefore, **I** is uniformly bounded in B. Term **II** may be estimated utilizing (5.10) upon replacing ∇u with $e^{4u}\nabla v$, and noting that the expansions of (3.4) imply $e^{2u}|\tilde{v}_1 - \tilde{v}_2| \le C\rho$ and $e^{2u}|\nabla\tilde{v}_i| \le C$ in B. Thus, we have that **II** is uniformly bounded. In order to control the third term of the tension first notice that (6.6) with u replaced by v, along with the estimates already mentioned, yields

$$(6.7) \qquad e^{2u}\Delta v = e^{2u}\left(\chi_1 \Delta\tilde{v}_1 + \chi_2 \Delta\tilde{v}_2\right) + O(1).$$

Furthermore, a computation shows that

$$(6.8) \qquad \begin{aligned} \nabla u \cdot \nabla v &= \chi_1 \nabla\tilde{u}_1 \cdot \nabla\tilde{v}_1 + \chi_2 \nabla\tilde{u}_2 \cdot \nabla\tilde{v}_2 + (\tilde{v}_1 - \tilde{v}_2)\nabla\tilde{u}_1 \cdot \nabla\chi_1 \\ &\quad + \chi_2 \nabla(\tilde{U}_1 - \tilde{U}_2) \cdot \nabla\tilde{v}_2 + \nabla[\chi_2(\tilde{U}_2 - \tilde{U}_1)] \cdot \nabla[\tilde{v}_2 + \chi_1(\tilde{v}_1 - \tilde{v}_2)] \end{aligned}$$

where $\tilde{U}_i = \tilde{u}_i + \ln\rho$, $i = 1, 2$. Since $|\nabla\tilde{u}_i| \le C/\rho$ and $|\nabla\tilde{U}_i| \le C$ in B by (3.4), together with the estimates mentioned above we then find

$$(6.9) \qquad e^{2u}\nabla u \cdot \nabla v = e^{2u}\left(\chi_1 \nabla\tilde{u}_1 \cdot \nabla\tilde{v}_1 + \chi_2 \nabla\tilde{u}_2 \cdot \nabla\tilde{v}_2\right) + O(1).$$

It now follows that **III** is uniformly bounded, as the maps $(\tilde{u}_i, \tilde{v}_i)$ are harmonic. Hence, $\tau(\Xi) \le C$ in B for some constant independent of n.

Consider now the region S. Let r_i, $i = 1, 2$ be the distances to the punctures p_{j-1}, p_j respectively, which make up the end points of the expanding rod Γ_j. Notice that due to the angle restrictions within S we have the following relation between r and r_i using the law of cosines

$$(6.10) \qquad r_i^2 = r^2 + (\ell_n/2)^2 \pm 2r(\ell_n/2)\cos\theta \ge r^2 + (\ell_n/2)^2 - 2r(\ell_n/2)(1/\sqrt{2}) \ge \frac{1}{2}r^2.$$

According to the expansion (3.4) we then have

$$(6.11) \qquad |\tilde{u}_i + \ln\rho| + \sqrt{2}r|\nabla(\tilde{u}_i + \ln\rho)| \le |\tilde{u}_i + \ln\rho| + r_i|\nabla(\tilde{u}_i + \ln\rho)| \le \frac{C}{r_i} \le \frac{\sqrt{2}C}{r},$$

and therefore $|\tilde{u}_1 - \tilde{u}_2| = O(r^{-1})$, $|\nabla(\tilde{u}_1 - \tilde{u}_2)| = O(r^{-2})$. Moreover, in a similar way the expansion also yields $|\Delta\tilde{u}_i| = O(r^{-3})$, and hence (6.6) implies **I** $\le C/(1 + r^3)$ where C is independent of n. We may break up **III** into two terms. The first, involving a Laplacian, may be estimated in the same manner using the fact that $e^{2u}|\tilde{v}_1 - \tilde{v}_2|$, $re^{2u}|\nabla\tilde{v}_i|$, and $r^2 e^{2u}|\Delta\tilde{v}_i|$ are uniformly decaying on the order of r^{-2}, due to (3.4) and $e^{2u} \le C/\rho^2 \le C/r^2$ in S. For the second term of **III** we employ a computation similar to (5.10), as well as the expansion asymptotics just described to obtain **III** $\le C/(1 + r^4)$. Term **II** is handled analogously to the second term in **III**, by replacing ∇u with $e^{2u}\nabla v$ to find **II** $\le C/(1 + r^4)$. It follows that

$$(6.12) \qquad |\tau(\Xi_n)| \le \frac{C}{1 + r^3} \qquad \text{on } \mathbb{R}^3 \setminus \Gamma$$

for some constant C independent of n.

Consider the quantity

$$(6.13) \qquad \Lambda(\Phi_n, \Xi_n) = \sqrt{1 + d_{\mathbb{H}^2}(\Phi_n, \Xi_n)^2},$$

and recall that [30, Lemma 1] implies

$$(6.14) \qquad \Delta\Lambda(\Phi_n, \Xi_n) \geq -|\tau(\Xi_n)| \geq -\frac{C}{1 + r^3} \quad \text{on } \mathbb{R}^3 \setminus \Gamma.$$

Let $\lambda > 0$ be a sufficiently large constant such that the radial function $w(r) = \lambda(1 + r^2)^{-1/4}$ satisfies

$$(6.15) \qquad \Delta w = -\frac{\lambda}{4}(6 + r^2)(1 + r^2)^{-9/4} \leq -\frac{C}{1 + r^3}.$$

It follows that

$$(6.16) \qquad \Delta\left(\Lambda(\Phi_n, \Xi_n) - w\right) \geq 0 \quad \text{on } \mathbb{R}^3 \setminus \Gamma.$$

As in Section 5, the maximum principle [28, Lemma 8] applies. Since Φ_n and Ξ_n are asymptotic according to Lemma A.4 and Remark A.5, we have that $\Lambda(\Phi_n, \Xi_n) - w \to 1$ at infinity, and hence

$$(6.17) \qquad d_{\mathbb{H}^2}(\Phi_n, \Xi_n) \leq \sqrt{w(w + 2)} \leq \frac{C}{1 + r^{1/4}}$$

on $\mathbb{R}^3 \setminus \Gamma$ where C is independent of n.

Let $\mathcal{B}_n^i \subset \Omega_i$ denote the ball of radius $\ell_n/4$ centered at the puncture p_{j+i-2}, $i = 1, 2$. Since $\Xi_n = \tilde{\Psi}_n^i$ on Ω_i by (6.4), it follows from (6.17) that $d_{\mathbb{H}^2}(\Phi_n, \tilde{\Psi}_n^i)$ is uniformly bounded on Ω_i and tends to zero when restricted to $\mathcal{B}_n^i$. Set $\tilde{\Phi}_n^i$ and $\tilde{\mathcal{B}}_n^i$ to be z-translations by $(-1)^{i-1}\ell_n/2$ of Φ_n and $\mathcal{B}_n^i$, respectively. By (6.2), $\Psi_n^i \to \Phi_*^i$ uniformly on fixed compact subsets of $\tilde{\mathcal{B}}_n^i \setminus \Gamma$. It follows that $\tilde{\Phi}_n^i$ is uniformly bounded on such subsets. Using the fact that $\tilde{\mathcal{B}}_n^i$ exhausts $\mathbb{R}^3$ in the limit, a standard diagonal argument shows that $\tilde{\Phi}_n^i$ subconverges on compact subsets of $\mathbb{R}^3 \setminus \Gamma$ to a harmonic map $\tilde{\Phi}_*^i$; we will henceforth restrict attention to this subsequence but for convenience the notation will remain unchanged. Furthermore, since the hyperbolic distance $d_{\mathbb{H}^2}(\Phi_n, \tilde{\Psi}_n^i)$ tends to zero on these compact sets we must have $\tilde{\Phi}_*^i = \Phi_*^i$. This together with Theorem 4.8 implies that

$$(6.18) \qquad \mathcal{E}_{\tilde{\mathcal{B}}_n^i}(\tilde{\Phi}_n^i) \to \mathcal{E}(\Phi_*^i),$$

for $i = 1, 2$. Moreover

$$(6.19) \qquad \mathcal{E}_{\tilde{\mathcal{B}}_n^1}(\tilde{\Phi}_n^1) + \mathcal{E}_{\tilde{\mathcal{B}}_n^2}(\tilde{\Phi}_n^2) \leq \mathcal{E}_{\Omega_1}(\Phi_n) + \mathcal{E}_{\Omega_2}(\Phi_n) \leq \mathcal{E}(\Phi_n).$$

A simple limit then yields (6.1).   □

For the next proposition we introduce some notation for clarity. Recall that at each time t, there are N punctures $\{p_i\}_{1 \leq i \leq N}$ whose z-coordinates are arranged in increasing order $z_1 < \cdots < z_N$, where we have suppressed the reference to t here for brevity. Suppose now that the lengths of $k > 1$ finite axis rods, say $\Gamma_{N_1}, \ldots, \Gamma_{N_k}$ with $N_1 < \cdots < N_k$, tend to infinity simultaneously along a sequence of times $t_n \to T$ associated with the harmonic maps $\Phi_n = \Phi_{t_n}$. The punctures lying between the midpoints of consecutive expanding rods Γ_{N_i} and $\Gamma_{N_{i+1}}$ are labeled with z-coordinates $z_{N_i}, \ldots, z_{N_{i+1}-1}$ for $i = 0, \ldots, k$; here $N_0 = 0$, $N_{k+1} = N + 2$ while

z_0, Γ_{N_0} are both formally replaced by $-\infty$ and z_{N+1}, $\Gamma_{N_{k+1}}$ are both formally replaced by $+\infty$. We assume that no collision occurs. It follows that the rods Γ_j for $N_i + 1 \leq j \leq N_{i+1} - 1$ and $j \neq 1, N$, have lengths uniformly bounded from above and below away from zero. Thus, there are parameters $\ell_n^i \in \mathbb{R}$, $i = 0, \ldots, k$ such that the translated punctures with z-coordinates $\tilde{z}_j = z_j + \ell_n^i$ subconverge to z_j^* for $N_i \leq j \leq N_{i+1} - 1$. Let Φ_*^i, $i = 0, \ldots, k$ be the harmonic map having punctures with z-coordinates $z_{N_i}^*, \ldots, z_{N_{i+1}-1}^*$ and the same potential constants as Φ_n for corresponding rods. The Φ_*^i are referred to as *separation configuration maps*.

Proposition 6.2. *Let Φ_t be the 1-parameter family of harmonic maps with fixed potential constants defined by the flow of punctures* (3.10)*, in which scattering occurs along $k > 1$ axis rods and no collisions occur at the limiting time T. If Φ_*^i, $i = 0, \ldots, k$ are separation configuration maps described above, then*

$$(6.20) \qquad \sum_{i=0}^{k} \mathcal{E}(\Phi_*^i) \leq \lim_{t \to T} \mathcal{E}(\Phi_t).$$

Proof. As described in the discussion preceding the statement of the proposition, let $\{t_n\}$ be a sequence of times with $t_n \to T$, which leads to separation configuration maps Φ_*^i for $i = 0, \ldots, k$. For convenience write $\Phi_n = \Phi_{t_n}$. Let $\tilde{\Psi}_n^i$, $i = 0, \ldots, k$ be the harmonic map having punctures with z-coordinates $z_{N_i}, \ldots, z_{N_{i+1}-1}$ and the same potential constants as Φ_n for corresponding rods, and let Ψ_n^i be the associated translated maps having punctures with z-coordinates $\tilde{z}_{N_i}, \ldots, \tilde{z}_{N_{i+1}-1}$. We call these translated versions *auxiliary separation configuration maps*. According to Theorem 3.1 we have that

$$(6.21) \qquad \Psi_n^i \to \Phi_*^i \quad \text{in } C^3 \text{ away from the axis,}$$

for $i = 0, \ldots, k$.

Let $p_{0,i}$, $i = 1, \ldots, k$ be the midpoint of the ith expanding rod Γ_{N_i}, and set $B_i = B_\varrho(p_{0,i})$ to be the ball centered at the midpoint with fixed radius $\varrho > 0$ chosen small enough so that no punctures lie inside this set. Consider polar coordinates (r_i, θ_i, ϕ) centered at $p_{0,i}$, set angle $\vartheta = \frac{\pi}{2k}$, and for $i = 1, \ldots, k$ define the disjoint sector portions

$$(6.22) \qquad S_i = \{(r_i, \theta_i, \phi) \mid \tfrac{3\pi}{4} - i\vartheta \leq \theta_i \leq \tfrac{3\pi}{4} - (i-1)\vartheta\} \setminus B_i,$$

We decompose the complement into regions $\sqcup_{j=0}^{k} \Omega_j = \mathbb{R}^3 \setminus (\cup_{i=1}^{k} S_i \cup B_i)$, such that $3\pi/4 < \theta_1 < \pi$ in Ω_0 and $0 < \theta_k < \pi/4$ in Ω_k, while for $j \neq 0, k$ both $\theta_j < 3\pi/4 - j\vartheta$ and $\theta_{j+1} > 3\pi/4 - j\vartheta$ in Ω_j; see Figure 3. Next define smooth cut-off function $\chi_i \geq 0$ for $i = 0, \ldots, k$, such that χ_i is a function of θ_i alone in S_i for $i = 1, \ldots k$, and $\chi_i = 1$ on Ω_i for $i = 0, \ldots, k$. Furthermore, these functions are chosen so that $\sum_{i=0}^{k} \chi_i = 1$ globally, $\chi_i + \chi_{i+1} = 1$ in each $\Omega_i \cup S_{i+1} \cup B_{i+1} \cup \Omega_{i+1}$ for $i = 0, \ldots, k-1$, and

$$(6.23) \qquad |\nabla \chi_{i-1}| + |\nabla \chi_i| \leq \frac{C}{1 + r_i}, \qquad |\nabla^2 \chi_{i-1}| + |\nabla^2 \chi_i| \leq \frac{C}{1 + r_i^2}, \qquad i = 1, \ldots, k.$$

Since in any of the sector portions S_{i+1} only two of the cut-offs χ_i and χ_{i+1} are nonconstant, we can repeat the arguments from the proof of Proposition 6.1 using the model map

$$(6.24) \qquad \Xi_n = (u_n, v_n) = \chi_0 \tilde{\Psi}_n^0 + \cdots + \chi_k \tilde{\Psi}_n^k$$
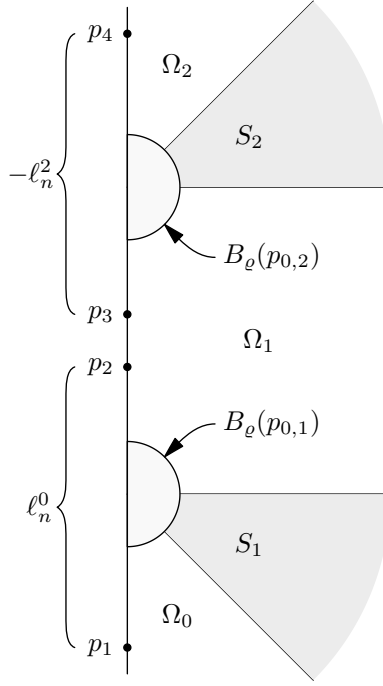
FIGURE 3. Scattering along two rods with four punctures.

to obtain tension estimates

$$(6.25) \qquad |\tau(\Xi_n)| \leq \frac{C}{1 + r_i^3} \qquad \text{on } (S_i \cup B_i) \setminus \Gamma, \quad i = 1, \ldots, k,$$

where C is a constant independent of i and n. Note that $|\tau(\Xi_n)| = 0$ on the compliment $(\mathbb{R}^3 \setminus \Gamma) \setminus (\cup_{i=1}^{k} S_i \cup B_i)$.

Consider the quantity

$$(6.26) \qquad \Lambda(\Phi_n, \Xi_n) = \sqrt{1 + d_{\mathbb{H}^2}(\Phi_n, \Xi_n)^2},$$

and recall that [30, Lemma 1] implies

$$(6.27) \qquad \Delta\Lambda(\Phi_n, \Xi_n) \geq -|\tau(\Xi_n)| \geq \begin{cases} -\frac{C}{1+r_i^3} & \text{on } (S_i \cup B_i) \setminus \Gamma \text{ for } i = 1, \ldots, k \\ 0 & \text{on } (\mathbb{R}^3 \setminus \Gamma) \setminus (\cup_{i=1}^{k} S_i \cup B_i) \end{cases}.$$

As before, we may choose $\lambda > 0$ sufficiently large so that the function $w = \lambda \sum_{i=1}^{k}(1 + r_i^2)^{-1/4}$ satisfies

$$(6.28) \qquad \Delta w = -\sum_{i=1}^{k} \frac{\lambda}{4}(6 + r_i^2)(1 + r_i^2)^{-9/4} \leq -\sum_{i=1}^{k} \frac{C}{1 + r_i^3} \qquad \text{on } \mathbb{R}^3.$$

It follows that

$$(6.29) \qquad \Delta\left(\Lambda(\Phi_n, \Xi_n) - w\right) \geq 0 \qquad \text{on } \mathbb{R}^3 \setminus \Gamma.$$

As a consequence of Lemma A.4 together with the Remark A.5, we find that the maps Φ_n and Ξ_n are asymptotic. Thus, the maximum principle [28, Lemma 8] applies to yield

$$(6.30) \qquad d_{\mathbb{H}^2}(\Phi_n, \Xi_n) \leq \sqrt{w(w+2)} \leq \sum_{i=1}^{k} \frac{C}{1 + r_i^{1/4}}$$

on $\mathbb{R}^3 \setminus \Gamma$ where C is independent of n.

Let $\mathcal{B}_n^i \subset \Omega_i$ denote the ball of radius l_n centered at the midpoint of the i-th *cluster* of punctures having z-coordinates $z_{N_i}, \ldots, z_{N_{i+1}-1}$ for $i = 0, \ldots, k$. Here the radii are chosen such that $l_n \to \infty$ slowly enough to ensure that these balls stay within Ω_i. Note that it is possible to find such radii since the midpoints of the expanding rods, about which the balls B_i are centered, are receding away from each cluster of punctures. Since $\Xi_n = \tilde{\Psi}_n^i$ on Ω_i by (6.24), it follows from (6.30) that $d_{\mathbb{H}^2}(\Phi_n, \tilde{\Psi}_n^i)$ is uniformly bounded on Ω_i and tends to zero when restricted to $\mathcal{B}_n^i$. Set $\tilde{\Phi}_n^i$ and $\tilde{\mathcal{B}}_n^i$ to be z-translations by ℓ_n^i of Φ_n and $\mathcal{B}_n^i$, respectively. By (6.21), $\Psi_n^i \to \Phi_*^i$ uniformly on fixed compact subsets of $\tilde{\mathcal{B}}_n^i \setminus \Gamma$. It follows that $\tilde{\Phi}_n^i$ is uniformly bounded on such subsets. Using the fact that $\tilde{\mathcal{B}}_n^i$ exhausts $\mathbb{R}^3$ in the limit, a standard diagonal argument shows that $\tilde{\Phi}_n^i$ subconverges on compact subsets of $\mathbb{R}^3 \setminus \Gamma$ to a harmonic map $\tilde{\Phi}_*^i$; we will henceforth restrict attention to this subsequence but for convenience the notation will remain unchanged. Furthermore, since the hyperbolic distance $d_{\mathbb{H}^2}(\Phi_n, \tilde{\Psi}_n^i)$ tends to zero on these compact sets we must have $\tilde{\Phi}_*^i = \Phi_*^i$. This together with Theorem 4.8 implies that

$$(6.31) \qquad \mathcal{E}_{\tilde{\mathcal{B}}_n^i}(\tilde{\Phi}_n^i) \to \mathcal{E}(\Phi_*^i),$$

for $i = 0, \ldots, k$. Moreover

$$(6.32) \qquad \sum_{i=0}^{k} \mathcal{E}_{\tilde{\mathcal{B}}_n^i}(\tilde{\Phi}_n^i) \leq \sum_{i=0}^{k} \mathcal{E}_{\Omega_i}(\Phi_n) \leq \mathcal{E}(\Phi_n).$$

A simple limit then yields (6.20). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Finally, we come to the last proposition of this section dealing with the most general event, namely simultaneous scattering along any number of rods while at the same time some collisions occur in any of the clusters. Thus, consider the situation in which the lengths of $k \geq 1$ axis rods, say $\Gamma_{N_1}, \ldots, \Gamma_{N_k}$ with $N_1 < \cdots < N_k$, tend to infinity simultaneously with any number of collisions occurring in the puncture clusters between these rods, along a sequence of times $t_n \to T$ associated with the harmonic maps $\Phi_n = \Phi_{t_n}$. As in Proposition 6.2, there are parameters $\ell_n^i \in \mathbb{R}$, $i = 0, \ldots, k$ such that the translated punctures with z-coordinates $\tilde{z}_j = z_j + \ell_n^i$ subconverge for $N_i \leq j \leq N_{i+1}-1$; the set of these pretranslated punctures will be called the i-th cluster. Let Φ_*^i, $i = 0, \ldots, k$ be the harmonic map having only the punctures from the limit of the translated i-th cluster except that those which are colliding from the i-th cluster are replaced by the limit of translated collision locations. Moreover, the potential constants for Φ_*^i are chosen to be the same as Φ_n for corresponding rods. The Φ_*^i are referred to as *separation/collision configuration maps*.

Proposition 6.3. *Let Φ_t be the 1-parameter family of harmonic maps with fixed potential constants defined by the flow of punctures (3.10), in which scattering occurs along $k \geq 1$ axis rods*

at the limiting time T. If Φ_*^i, $i = 0, \ldots, k$ are separation/collision configuration maps described above, then

$$(6.33) \qquad \sum_{i=0}^{k} \mathcal{E}(\Phi_*^i) \leq \lim_{t \to T} \mathcal{E}(\Phi_t).$$

Proof. This result follows from a combination of the techniques used in the previous section as well as those of the current section. For simplicity of presentation, we will expound on the case when $k = 1$ and there is one collision location, say in the $i = 1$ north cluster. The general case may be obtained in a similar manner, albeit with more complicated notation. In what follows we will adopt the setup preceding the statement of the Proposition.

Let $\{t_n\}$ be a sequence of times with $t_n \to T$, which leads to separation/collision configuration maps Φ_*^i for $i = 0, 1$, and write $\Phi_n = \Phi_{t_n}$. We assume that the length of only one axis rod Γ_{N_1} tends to infinity. All punctures are grouped into two clusters, the one below the midpoint of Γ_{N_1} corresponding to $i = 0$, and another above the midpoint corresponding to $i = 1$. We further assume that collision occurs only in the cluster corresponding to $i = 1$. As in Proposition 6.2, there are parameters $\ell_n^i \in \mathbb{R}$, $i = 0, 1$ such that the translated punctures with z-coordinates $\tilde{z}_j = z_j + \ell_n^i$ subconverge for z_j in the i-th cluster. Let $\tilde{\Psi}_n^i$, $i = 0, 1$ be the harmonic map having only the punctures from the i-th cluster except that those which are colliding from the i-th cluster are replaced by collision locations. Moreover, the potential constants for $\tilde{\Psi}_n^i$ are chosen to be the same as Φ_n for corresponding rods. Then set Ψ_n^i to be the associated harmonic maps translated in the z-direction by ℓ_n^i. We call these translated versions the lower and upper *auxiliary separation/collision configuration maps*. According to Theorem 3.1 we have that

$$(6.34) \qquad \Psi_n^i \to \Phi_*^i \quad \text{in } C^3 \text{ away from the axis},$$

for $i = 0, 1$.

Let there be N_* colliding punctures and set B_j, $j = 1, \ldots, N_*$ to be balls of radius η centered at these points, where 4η is the minimum distance between any two of the colliding punctures. Following Case 2 in the proof of Proposition 5.1 let S_j, $j = 1, \ldots, N_* - 1$ be sectors centered at the midpoints between any two consecutive colliding punctures with opening angles of size $\frac{\pi}{4(N_*-1)}$, ranging from $\pi/4$ to $\pi/2$. Clearly the balls B_j are disjoint, and are also disjoint from all of the sectors S_j. Define B_δ to be a ball centered at the mean $\bar{p} \in \Gamma$ of all the colliding punctures, where δ is chosen (independent of n) sufficiently large so that all of the B_j, $j = 1, \ldots, N_*$ are contained within the radius $\delta/4$, while at the same time δ is chosen small enough so that $B_{3\delta}$ does not contain any non-colliding punctures. Without loss of generality it may be assumed that the midpoint of the expanding axis rod coincides with the origin. We then set S to be the sector having vertex at the origin and opening angle ranging between $\pi/2$ and $3\pi/4$. Define also the ball B centered at the origin of fixed radius ϱ small enough not to intersect any punctures. Next decompose the complement of these domains by $\Omega_0 \sqcup \Omega_1 = \mathbb{R}^3 \setminus (B_\delta \cup S \cup B)$, where Ω_0 lies below S within the range of angles larger than $3\pi/4$ and Ω_1 lies above S within the range of angles less than $\pi/2$. Consider the model map

$$(6.35) \qquad \Xi_n = \chi_0 \tilde{\Psi}_n^0 + \chi_1 \tilde{\Psi}_n^1 + \sum_{j=1}^{N_*} \hat{\chi}_j \Theta_j,$$

where $0 \le \chi_i \le 1$, $i = 0, 1$ and $0 \le \hat{\chi}_j \le 1$, $j = 1, \ldots, N_*$ are smooth cut-off functions such that $\chi_i = 1$ in Ω_i and $\chi_i = 0$ in Ω_l, $l \ne i$, while $\hat{\chi}_i = 1$ in B_i and vanishes outside B_δ, with $\chi_0 + \chi_1 + \hat{\chi}_1 + \ldots \hat{\chi}_{N_*} = 1$ globally. These cut-off functions are constructed using the sectors and balls discussed above as transition regions; details concerning the construction may be found in the proofs of Proposition 5.1 (Case 2) and Proposition 6.2. Furthermore, Θ_j represents the extreme Kerr harmonic map with a single puncture at the j-th colliding puncture, and having the appropriate potential constants that are compatible with Φ_n.

We may now repeat the arguments in the proof of Proposition 5.1 (Case 2) and Proposition 6.2 to obtain the tension estimate

$$(6.36) \qquad |\tau(\Xi_n)| \le C \sum_{j=1}^{N_*-1} \frac{1}{\tilde{r}_j^2} \quad \text{on } B_\delta \setminus \Gamma, \qquad |\tau(\Xi_n)| \le \frac{C}{1 + r^3} \quad \text{on } (S \cup B) \setminus \Gamma,$$

where C is a constant independent of n and δ, and $\tilde{r}_j$ is the distance to the vertex $\tilde{p}_j \in \Gamma$ of sector S_j. Note that the tension vanishes elsewhere on $\mathbb{R}^3 \setminus (B_\delta \cup S \cup B)$. Define now comparison functions $w_0(r) = \lambda(1 + r^2)^{-1/4}$ for $\lambda > 0$ sufficiently large depending on C, and

$$(6.37) \qquad w_j(\tilde{r}_j) = \begin{cases} C \ln\left(\dfrac{2\delta}{\tilde{r}_j}\right) + C & \text{on } B_{2\delta}(\tilde{p}_j) \\ \dfrac{2C\delta}{\tilde{r}_j} & \text{on } \mathbb{R}^3 \setminus B_{2\delta}(\tilde{p}_j) \end{cases},$$

for $j = 1, \ldots N_* - 1$. By setting $w = w_0 + \cdots + w_{N_*-1}$ we find that

$$(6.38) \qquad \Delta\left(\Lambda(\Phi_n, \Xi_n) - w\right) \ge 0 \quad \text{on } \mathbb{R}^3 \setminus \left(\Gamma \cup_{j=1}^{N_1-1} \partial B_{2\delta}(\tilde{p}_j)\right).$$

As before a weak version of the maximum principle then produces

$$(6.39) \quad d_{\mathbb{H}^2}(\Phi_n, \Xi_n) \le \sqrt{w(w+2)} \le C\left(\frac{1}{1 + r^{1/4}} + \frac{\sqrt{\delta(\delta + \bar{r})}}{\bar{r}}\right) \quad \text{on } \mathbb{R}^3 \setminus (B_{3\delta}(\bar{p}) \cup \Gamma) =: D_\delta,$$

where $\bar{r}$ is the distance to $\bar{p}$.

Let $\mathcal{B}_n^i \subset \Omega_i \cup B_\delta$, $i = 0, 1$ denote the ball of radius l_n centered at the midpoint of the i-th cluster of punctures. Here the radii are chosen such that $l_n \to \infty$ slowly enough to ensure that these balls stay within $\Omega_i \cup B_\delta$. Note that B_δ is far removed from Ω_0, but borders part of Ω_1. Since $\Xi_n = \tilde{\Psi}_n^i$ on Ω_i, it follows from (6.39) that $d_{\mathbb{H}^2}(\Phi_n, \tilde{\Psi}_n^i)$ is uniformly bounded on $\Omega_i \cap D_\delta$. Set $\tilde{\Phi}_n^i$ and $\tilde{\mathcal{B}}_n^i$ for $i = 0, 1$ to be z-translations by $\pm \ell_n/2$ of Φ_n and $\mathcal{B}_n^i$ respectively, where ℓ_n is the length of the expanding rod. Since $\Psi_n^i \to \Phi_*^i$ uniformly on fixed compact subsets of $\tilde{\mathcal{B}}_n^i \setminus \Gamma$ by (6.34), it follows that $\tilde{\Phi}_n^i$ is uniformly bounded on such subsets as long as they do not intersect translated image of the balls $B_{3\delta}$. Using the fact that $\tilde{\mathcal{B}}_n^i$ exhausts $\mathbb{R}^3$ in the limit, a standard diagonal argument involving both $n \to \infty$ and $\delta \to 0$ shows that $\tilde{\Phi}_n^i$ subconverges on compact subsets of $\mathbb{R}^3 \setminus \Gamma$ to a harmonic map $\tilde{\Phi}_*^i$; we will henceforth restrict attention to this subsequence but for convenience the notation will remain unchanged. Furthermore, as in the proof of Proposition 5.1 (Case 3) and Proposition 6.2, since the hyperbolic distance $d_{\mathbb{H}^2}(\Phi_n, \tilde{\Psi}_n^i)$ tends to zero on these compact sets we must have $\tilde{\Phi}_*^i = \Phi_*^i$ for $i = 0, 1$.

Let $\mathcal{B}_\delta \subset \mathcal{B}_n^1$ be a δ-ball centered at the collision location, and denote its z-translation in the amount $-\ell_n/2$ by $\tilde{\mathcal{B}}_\delta$. Then as discussed in the proofs of the previous propositions we have

$$(6.40) \qquad \mathcal{E}_{\tilde{\mathcal{B}}_n^0}(\tilde{\Phi}_n^0) \to \mathcal{E}(\Phi_*^0), \qquad \mathcal{E}_{\tilde{\mathcal{B}}_n^1 \setminus \tilde{\mathcal{B}}_\delta}(\tilde{\Phi}_n^1) \to \mathcal{E}_{\mathbb{R}^3 \setminus \tilde{\mathcal{B}}_\delta}(\Phi_*^1).$$

Note that $\mathcal{E}_{\tilde{\mathcal{B}}_\delta}(\Phi_*^1) \to 0$ as $\delta \to 0$, and hence given $\epsilon > 0$ there exists $\delta > 0$ sufficiently small and n_ϵ sufficiently large to guarantee that

$$(6.41) \qquad \mathcal{E}(\Phi_*^1) = \mathcal{E}_{\tilde{\mathcal{B}}_\delta}(\Phi_*^1) + \mathcal{E}_{\mathbb{R}^3 \setminus \tilde{\mathcal{B}}_\delta}(\Phi_*^1) < \mathcal{E}_{\tilde{\mathcal{B}}_{n_\epsilon}^1 \setminus \tilde{\mathcal{B}}_\delta}(\tilde{\Phi}_{n_\epsilon}^1) + \frac{\epsilon}{2}.$$

Thus, for potentially larger n_ϵ we obtain

$$(6.42) \quad \mathcal{E}(\Phi_*^0) + \mathcal{E}(\Phi_*^1) < \mathcal{E}_{\tilde{\mathcal{B}}_{n_\epsilon}^0}(\tilde{\Phi}_{n_\epsilon}^0) + \mathcal{E}_{\tilde{\mathcal{B}}_{n_\epsilon}^1 \setminus \tilde{\mathcal{B}}_\delta}(\tilde{\Phi}_{n_\epsilon}^1) + \epsilon \leq \mathcal{E}_{\mathcal{B}_{n_\epsilon}^0}(\Phi_{n_\epsilon}) + \mathcal{E}_{\mathcal{B}_{n_\epsilon}^1}(\Phi_{n_\epsilon}) + \epsilon \leq \mathcal{E}(\Phi_{n_\epsilon}) + \epsilon.$$

A simple limit then yields (6.33) for $k = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7. Proof of the Main Theorem

In this section we will complete the proof of Theorem 1.1. Let (M, g, k) be an initial data set satisfying the requisite hypotheses of the theorem. According to (2.16) we have the following relation between the ADM mass m and the renormalized energy of a harmonic map $\Phi = (u, v) :$ $\mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2$ which encodes the angular momenta of the data, namely

$$(7.1) \qquad\qquad m \geq \frac{1}{8\pi} \mathcal{E}(\Phi).$$

For convenience in what follows, we will write $\bar{\mathcal{E}}(\Phi) = \frac{1}{8\pi} \mathcal{E}(\Phi)$. Suppose there are N punctures associated with the map Φ, and that it has total angular momentum $\mathcal{J}$. Assuming that there does not exist an ADM minimizing counterexample to the extreme black hole uniqueness conjecture, it will proven by induction on N that

$$(7.2) \qquad\qquad \bar{\mathcal{E}}(\Phi) \geq \sqrt{|\mathcal{J}|}.$$

This establishes the inequality portion of the theorem. The rigidity statement will be treated after the induction argument.

First note that the case $N = 1$ is true, since Φ must correspond to an extreme Kerr harmonic map with angular momentum $\mathcal{J}$, and therefore $1/8\pi$ times the renormalized energy of the map agrees with $\sqrt{|\mathcal{J}|}$. This latter statement follows from the arguments leading to (2.16) when applied to extreme Kerr initial data. More precisely, in this situation the ADM mass agrees with the appropriate multiple of renormalized energy, and it is known by direct calculation that the mass is given by $\sqrt{|\mathcal{J}|}$ for extreme Kerr.

Suppose now that (7.2) holds for any multi-extreme Kerr harmonic map with n punctures and total angular momentum $\mathcal{J}$, where $1 \leq n \leq N-1$, and let Φ_0 be such a harmonic map with N punctures. We start the flow defined in Section 3 with initial condition Φ_0. By Proposition 4.1 the flow Φ_t exists and is regular up to a maximal time T (possibly infinite), and according to Theorem 4.8 the renormalized energy $\mathcal{E}(\Phi_t)$ is nonincreasing along this flow. There are three possibilities to contend with at time T: either a collision occurs without scattering, a scattering occurs possibly with simultaneous collisions, or no collisions and no scattering occur. Note that in the first two cases T can be either finite or infinite, while in the third case T will be infinite.

Consider the first of these cases in which a collision occurs at time T without any scattering. By Proposition 5.1 we have that

$$\lim_{t \to T} \bar{\mathcal{E}}(\Phi_t) \geq \bar{\mathcal{E}}(\Phi_*),$$ (7.3)

for some collision configuration map Φ_* which has at most $N-1$ punctures and total angular momentum $\mathcal{J}$. By the induction hypothesis we then obtain the desired inequality

$$\bar{\mathcal{E}}(\Phi_0) \geq \lim_{t \to T} \bar{\mathcal{E}}(\Phi_t) \geq \bar{\mathcal{E}}(\Phi_*) \geq \sqrt{|\mathcal{J}|}.$$ (7.4)

Now assume that scattering occurs along $k \geq 1$ axis rods, possibly with collisions as well, at time T. By Proposition 6.3 we have that

$$\lim_{t \to T} \bar{\mathcal{E}}(\Phi_t) \geq \sum_{i=0}^{k} \bar{\mathcal{E}}(\Phi_*^i),$$ (7.5)

for some separation/collision configuration maps Φ_*^i. Let N_i denote the number of punctures associated with Φ_*^i, then $N_i \leq N-1$ and $\sum_{i=0}^{k} N_i \leq N$. By the induction hypothesis we then obtain

$$\bar{\mathcal{E}}(\Phi_0) \geq \lim_{t \to T} \bar{\mathcal{E}}(\Phi_t) \geq \sum_{i=0}^{k} \bar{\mathcal{E}}(\Phi_*^i) \geq \sum_{i=0}^{k} \sqrt{|\mathcal{J}_i^*|} \geq \sqrt{|\mathcal{J}|},$$ (7.6)

where $\mathcal{J}_i^*$ is the total angular momentum associated with Φ_*^i. Since $\sum_{i=0}^{k} \mathcal{J}_i^* = \mathcal{J}$, the last inequality concerning the angular momentum follows from the elementary inequality $\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$ for two nonnegative numbers a, b.

Now consider the last of the three cases, in which no collisions and no scattering occur at time $T = \infty$. Since the function $t \mapsto \mathcal{E}(\Phi_t)$ is continuously differentiable and nonincreasing on $[0, \infty)$ we find that

$$\int_0^\infty \left| \frac{d}{dt} \mathcal{E}(\Phi_t) \right| dt = \mathcal{E}(\Phi_0) - \lim_{t \to \infty} \mathcal{E}(\Phi_t) \leq \mathcal{E}(\Phi_0).$$ (7.7)

It follows that there exists a sequence of times $t_j \to \infty$ such that $\frac{d}{dt} \mathcal{E}(\Phi_{t_j}) \to 0$. Next observe that the N punctures along these times, possibly after recentering by a z-translation and passing to a subsequence, converge to some N distinct fixed limits p_i^*, $i = 1, \ldots, N$. The limiting harmonic map Φ^* must be stationary with respect to the flow, since the associated tangent map parameters satisfy $b_i^* = 0$ for $i = 1, \ldots, N$ by virtue of their continuity (Proposition 4.1) and their relation to the first variation of the renormalized energy (3.11). The axisymmetric stationary vacuum spacetime constructed from Φ^* must be regular, due to the relation (3.14) between b_i^* and logarithmic angle defects of the spacetime. Moreover, the calculation that produced (2.16) shows that the mass m^* of the initial data associated with the constant time slice of this spacetime is given by $\frac{1}{8\pi} \mathcal{E}(\Phi^*)$. Let $\underline{m}^*$ denote the infimum of the ADM mass over all regular axisymmetric stationary vacuum spacetimes with an asymptotically flat end and N degenerate horizons have the same angular momenta as those associated with Φ^*. There is then a nonincreasing sequence of masses, renormalized energies, and harmonic maps $m_j^* = \frac{1}{8\pi} \mathcal{E}(\Phi_j^*) \to \underline{m}^*$. If no collisions and no scattering occur as $j \to \infty$, then as above Φ_j^* subconverges after possible translations to a limit $\underline{\Phi}^*$ with N punctures which achieves the infimum $\underline{m}^* = \frac{1}{8\pi} \mathcal{E}(\underline{\Phi}^*)$. This however, contradicts

the assumption that there does not exist an ADM minimizing counterexample to the extreme black hole uniqueness conjecture. Hence, some combination of collisions and scattering must occur in the limit, and the same arguments used above with the induction hypothesis yield

$$(7.8) \qquad \underline{m}^* = \lim_{j \to \infty} \bar{\mathcal{E}}(\Phi_j^*) \geq \sqrt{|\mathcal{J}|}.$$

The nonexistence of an ADM minimizing counterexample also implies $m^* > \underline{m}^*$, and thus we obtain

$$(7.9) \qquad \bar{\mathcal{E}}(\Phi_0) \geq \lim_{t \to \infty} \bar{\mathcal{E}}(\Phi_t) = \bar{\mathcal{E}}(\Phi^*) = m^* > \underline{m}^* \geq \sqrt{|\mathcal{J}|}.$$

This completes the induction argument, and therefore also the inequality portion of Theorem 1.1.

It remains to establish the rigidity statement for the mass-angular momentum inequality. Consider the case of equality when $m = \sqrt{|\mathcal{J}|}$. This implies equality of renormalized energies $\mathcal{E}(\bar{\Phi}) = \mathcal{E}(\Phi)$, for the map $\bar{\Phi}$ obtained from the initial data (M, g, k) and the associated multi-extreme Kerr harmonic map Φ. Due to the gap bound [17, Theorem 4.1], it follows that $\bar{\Phi} = \Phi$. Moreover, several other restrictions on g and k are obtained from equality in (2.14), as detailed at the end of Section 2 in [17]. We may then construct from Φ an axisymmetric stationary vacuum spacetimes with an asymptotically flat end and N degenerate horizons, from which (M, g, k) arises as a constant time slice. Since the given initial data set is void of conical singularities, this spacetime is regular. Furthermore, in light of the mass-angular momentum inequality, the spacetime minimizes the ADM mass among all such solutions with the same angular momentum configuration. If $N > 1$, then a contradiction is obtained to the assumption concerning the nonexistence of a minimizing counterexample to extreme black hole uniqueness. We conclude that $N = 1$, and [17, Theorem 1.1] yields the desired result that (M, g, k) must arise from a constant time slice of an extreme Kerr black hole.

8. Linearized Harmonic Maps Near Punctures

This section represents an extension of the study of isolated singularities that was presented in [12, Theorem 2.1], to higher order expansions. Let $\Phi = (u, v)$ be an H^1_{loc}-harmonic map from $B_1 \setminus \Gamma$, the Euclidean unit ball minus the z-axis, to $\mathbb{H}^2$. It will not be assumed here that this map is axisymmetric.[3] Suppose that

$$(8.1) \qquad \sup_{B_1 \setminus \Gamma} |u + \ln \sin \theta| < \infty, \qquad |\nabla(u + \ln \rho)|^2 + e^{4u} |\nabla v|^2 \in L^1_{\text{loc}}(B_1 \setminus \{0\}),$$

with

$$(8.2) \qquad v = a \text{ on } B_1 \cap \Gamma_+ \quad \text{and} \quad v = -a \text{ on } B_1 \cap \Gamma_- \quad \text{in the trace sense,}$$

for some positive constant a. We will use the notation $x = (x_1, x_2, x_3)$, $r = |x|$ and $\sin \theta = \frac{\rho}{r}$. By Theorem 2.1 of [12], $(u + \ln \sin \theta, v) \in C^{3,\alpha}(B_1 \setminus \{0\})$ for any $\alpha \in (0, 1)$, and there exists a harmonic map $\bar{\Phi} = (\bar{u}, \bar{v}) \colon \mathbb{S}^2 \setminus \{N, S\} \to \mathbb{H}^2$ satisfying

$$(8.3) \qquad (\bar{u} + \ln \sin \theta, \bar{v}) \in C^{3,\alpha}(\mathbb{S}^2), \quad \bar{v}(N) = a, \quad \bar{v}(S) = -a,$$

---

[3]Note that Φ in the current setting has a different meaning from that in [12].

such that

$$(8.4) \qquad \sup_{\mathbb{S}^2 \setminus \{N,S\}} \left(|(r\partial_r)^l \nabla_{\mathbb{S}^2}^k (u(r) - \bar{u})| + e^{(3+\alpha-k)\bar{u}} |(r\partial_r)^l \nabla_{\mathbb{S}^2}^k (v(r) - \bar{v})| \right) \leq \bar{C} r^\beta$$

for $l + k \leq 3$, where $\bar{C}$ and β are positive constants and $\Phi(r) = \Phi(r, \cdot)$. The map $\bar{\Phi}$ is the tangent map of Φ at the origin, and the constant β depends only on $\bar{\Phi}$.

In the following, we fix a harmonic map $\Phi = (u, v)$ and its tangent map $\bar{\Phi} = (\bar{u}, \bar{v})$ as above. Let $L = (L_1, L_2)$ be the linearized harmonic map operator at Φ, which is given by

$$(8.5) \qquad \begin{aligned} L_1 \varphi &= \Delta\varphi_1 - 8e^{4u}|\nabla v|^2 \varphi_1 - 4e^{4u}\nabla v \cdot \nabla\varphi_2, \\ L_2 \varphi &= \Delta\varphi_2 + 4\nabla u \cdot \nabla\varphi_2 + 4\nabla v \cdot \nabla\varphi_1, \end{aligned}$$

for $\varphi = (\varphi_1, \varphi_2)$ with $\varphi_1 \in C^2(B_1)$ and $\varphi_2 \in C_c^2(B_1 \setminus \Gamma)$. Here, Δ and ∇ are with respect to the flat metric in $\mathbb{R}^3$. The spherical part $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ of L at $\bar{\Phi}$ is given by

$$(8.6) \qquad \begin{aligned} \mathcal{T}_1 \varphi &= \Delta_{g_0}\varphi_1 - 8e^{4\bar{u}}|\nabla_{g_0}\bar{v}|^2 \varphi_1 - 4e^{4\bar{u}}\nabla_{g_0}\bar{v} \cdot \nabla_{g_0}\varphi_2, \\ \mathcal{T}_2 \varphi &= \Delta_{g_0}\varphi_2 + 4\nabla_{g_0}\bar{u} \cdot \nabla_{g_0}\varphi_2 + 4\nabla_{g_0}\bar{v} \cdot \nabla_{g_0}\varphi_1, \end{aligned}$$

where $g_0 = d\theta^2 + \sin^2\theta d\phi^2$ is the round metric on $\mathbb{S}^2$.

Denote by $L^2(\mathbb{S}^2, e^{4\bar{u}})$ the subspace of $L^2(\mathbb{S}^2)$ consisting of all functions f with the bounded norm

$$(8.7) \qquad \|f\|_{L^2(\mathbb{S}^2, e^{4\bar{u}})} = \left(\int_{\mathbb{S}^2} e^{4\bar{u}} f^2 \, dvol_{g_0} \right)^{1/2},$$

and by $H_0^1(\mathbb{S}^2, e^{4\bar{u}})$ the closure of $C_c^\infty(\mathbb{S}^2 \setminus \{N, S\})$ under the norm

$$(8.8) \qquad \|f\|_{H^1(\mathbb{S}^2, e^{4\bar{u}})} = \left(\int_{\mathbb{S}^2} e^{4\bar{u}} (|\nabla_{g_0} f|^2 + f^2) \, dvol_{g_0} \right)^{1/2}.$$

The inner products associated with these norms will be denoted by the braces $\langle \cdot, \cdot \rangle$. Introduce the bilinear form

$$(8.9) \qquad \begin{aligned} \bar{\mathcal{B}}[\varphi, \psi] = \int_{\mathbb{S}^2} \Big(&\nabla_{g_0}\varphi_1 \cdot \nabla_{g_0}\psi_1 + e^{4\bar{u}}\nabla_{g_0}\varphi_2 \cdot \nabla_{g_0}\psi_2 + 8e^{4\bar{u}}|\nabla_{g_0}\bar{v}|^2 \varphi_1\psi_1 \\ &+ 4e^{4\bar{u}}\psi_1 \nabla_{g_0}\bar{v} \cdot \nabla_{g_0}\varphi_2 - 4e^{4\bar{u}}\psi_2 \nabla_{g_0}\bar{v} \cdot \nabla_{g_0}\varphi_1 \Big) \, dvol_{g_0}, \end{aligned}$$

for any $\varphi = (\varphi_1, \varphi_2), \psi = (\psi_1, \psi_2) \in H^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2, e^{4\bar{u}})$. If $\varphi_1 \in C^2(\mathbb{S}^2)$ and $\varphi_2 \in C_c^2(\mathbb{S}^2 \setminus \{N, S\})$, then

$$(8.10) \qquad \bar{\mathcal{B}}[\varphi, \psi] = -\langle \mathcal{T}\varphi, \psi \rangle_{L^2(\mathbb{S}^2, 1 \times e^{4\bar{u}})} = -\langle \mathcal{T}_1\varphi, \psi_1 \rangle_{L^2(\mathbb{S}^2)} - \langle \mathcal{T}_2\varphi, \psi_2 \rangle_{L^2(\mathbb{S}^2, e^{4\bar{u}})}.$$

It was proved in [12] that $\bar{\mathcal{B}}$ is symmetric and nonnegative on $H^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2, e^{4\bar{u}})$, and zero is the least and simple eigenvalue of the problem

$$(8.11) \qquad -\mathcal{T}\varphi = \mu\varphi \quad \text{in } \mathbb{S}^2 \setminus \{N, S\}, \qquad \varphi_2(N) = \varphi_2(S) = 0.$$

Let

$$(8.12) \qquad 0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_i \leq \cdots \to \infty$$

be the collection of eigenvalues, and $\varphi^{(i)}$ be the normalized eigenfunction corresponding to μ_i.

We present a result on the asymptotic behavior of solutions to the linearized equation

$$(8.13) \qquad Lw = f \quad \text{in } B_1 \setminus \Gamma, \quad w_2 = 0 \quad \text{on } \Gamma \cap B_1,$$

as $x \to 0$. The weak solutions will be understood in the integration by parts sense. Their regularity and local derivative estimates can be deduced from [12, Proposition 3.9].

Proposition 8.1. *Assume that $f = (f_1, f_2) \in C^1(B_1 \setminus \Gamma)$ satisfies*

$$(8.14) \qquad |\nabla^k(r^2 f_1 - \bar{f}_1)| + e^{2u} |\nabla^k(r^2 f_2 - \bar{f}_2)| \leq K r^{\gamma - k} \quad in \ B_1 \setminus \Gamma, \qquad k = 0, 1,$$

for some positive constants K, γ, and a C^1-function $\bar{f} = (\bar{f}_1, \bar{f}_2)$ on $\mathbb{S}^2$. Let $w = (w_1, w_2)$ be a weak solution of (8.13) satisfying $(w_1, e^{2u} w_2) \in L^\infty(B_1)$. Then for any $\varphi \in H^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2, e^{4\bar{u}})$ with $\mathcal{T}\varphi = 0$ we have

$$(8.15) \qquad\qquad\qquad \langle \bar{f}, \varphi \rangle_{L^2(\mathbb{S}^2, 1 \times e^{4\bar{u}})} = 0,$$

and there exists a function $\bar{w} = (\bar{w}_1, \bar{w}_2) \in H^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2, e^{4\bar{u}})$ with

$$(8.16) \qquad\qquad\qquad \mathcal{T}\bar{w} = \bar{f} \quad in \ \mathbb{S}^2 \setminus \{N, S\}.$$

Moreover, there is a universal constant $0 < \bar{\beta} \leq \min\{\beta, \gamma\}$ such that for $k = 0, 1$ and any $\alpha \in (0, 1)$ it holds that

$$(8.17) \qquad |\nabla^k(w_1 - \bar{w}_1)|(x) + (\sin\theta)^{k-3-\alpha} |\nabla^k(w_2 - \bar{w}_2)|(x) \leq C|x|^{\bar{\beta}-k} \quad in \ B_1 \setminus \Gamma,$$

where C is a constant depending only on K, α, γ, and the L^∞-norms of w_1, $e^{2u} w_2$, $r^2 f_1$, and $r^2 e^{2u} f_2$ in B_1.

Although w in Proposition 8.1 is assumed to be bounded in B_1, in some applications only its bound in $B_1 \setminus B_\delta$ is known, for some $\delta > 0$. In order to apply Proposition 8.1, we need to estimate the bound of w in B_1 in terms of that in $B_1 \setminus B_\delta$.

Proposition 8.2. *Let f and w be as in Proposition 8.1. Then there exists $0 < \delta < 1/2$ depending only on γ, a, as well as $\bar{C}$ and β in (8.4), such that*

$$(8.18) \qquad\begin{aligned} \||w_1| &+ e^{2u}|w_2|\|_{L^\infty(B_{1/2})} \\ &\leq C \big(\||w_1| + e^{2u}|w_2|\|_{L^\infty(B_{1/2} \setminus B_\delta)} + \||r^2 f_1| + e^{2u}|r^2 f_2|\|_{L^\infty(B_1)} + K \big). \end{aligned}$$

Moreover, (8.17) holds for C depending only on K, α, γ and the L^∞-norms of w_1 and $e^{2u} w_2$ in $B_1 \setminus B_\delta$.

To prove Propositions 8.1 and 8.2, it is convenient to adopt the cylindrical coordinate $t = -\ln r$. Then the Euclidean metric becomes

$$(8.19) \qquad\qquad\qquad (\mathrm{d}x)^2 = \mathrm{d}r^2 + r^2 g_0 = e^{-2t}(\mathrm{d}t^2 + g_0).$$

In what follows we will set

$$(8.20) \qquad\qquad\qquad \hat{g} = \mathrm{d}t^2 + g_0,$$

which is the standard product metric on $\mathbb{R}_+ \times \mathbb{S}^2$. Denote by $\nabla_{\hat{g}}$ and $\Delta_{\hat{g}}$ the gradient operator and the Laplace-Beltrami operator with respect to the metric $\hat{g}$ on $\mathbb{R}_+ \times \mathbb{S}^2$, in particular

$$(8.21) \qquad\qquad\qquad \nabla_{\hat{g}} = (\partial_t, \nabla_{g_0}), \qquad \Delta_{\hat{g}} = \partial_t^2 + \Delta_{g_0}.$$

Recall that $(\bar{u}, \bar{v})$ is the tangent map of Φ and L is the linearized harmonic map operator at Φ. In the new coordinates this takes the form

$$(8.22) \qquad \begin{aligned} e^{-2t} L_1 \varphi &= (\partial_t^2 - \partial_t + \Delta_{g_0}) \varphi_1 - 8 e^{4u} |\nabla_{\hat{g}} v|^2 \varphi_1 - 4 e^{4u} \nabla_{\hat{g}} v \cdot \nabla_{\hat{g}} \varphi_2, \\ e^{-2t} L_2 \varphi &= (\partial_t^2 - \partial_t + \Delta_{g_0}) \varphi_2 + 4 \nabla_{\hat{g}} u \cdot \nabla_{\hat{g}} \varphi_2 + 4 \nabla_{\hat{g}} v \cdot \nabla_{\hat{g}} \varphi_1. \end{aligned}$$

Write

$$(8.23) \qquad e^{-2t} L \varphi = \mathcal{L} \varphi + \mathcal{Q} \varphi,$$

where $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ and $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2)$ are given by

$$(8.24) \qquad \begin{aligned} \mathcal{L}_1 \varphi &= (\partial_t^2 - \partial_t + \Delta_{g_0}) \varphi_1 - 8 e^{4\bar{u}} |\nabla_{g_0} \bar{v}|^2 \varphi_1 - 4 e^{4\bar{u}} \nabla_{g_0} \bar{v} \cdot \nabla_{g_0} \varphi_2, \\ \mathcal{L}_2 \varphi &= (\partial_t^2 - \partial_t + \Delta_{g_0}) \varphi_2 + 4 \nabla_{g_0} \bar{u} \cdot \nabla_{g_0} \varphi_2 + 4 \nabla_{g_0} \bar{v} \cdot \nabla_{g_0} \varphi_1, \end{aligned}$$

and

$$(8.25) \qquad \begin{aligned} \mathcal{Q}_1 \varphi &= (8 e^{4\bar{u}} |\nabla_{g_0} \bar{v}|^2 - 8 e^{4u} |\nabla_{\hat{g}} v|^2) \varphi_1 + (4 e^{4\bar{u}} \nabla_{g_0} \bar{v} \cdot \nabla_{g_0} \varphi_2 - 4 e^{4u} \nabla_{\hat{g}} v \cdot \nabla_{\hat{g}} \varphi_2), \\ \mathcal{Q}_2 \varphi &= 4 \nabla_{\hat{g}} u \cdot \nabla_{\hat{g}} \varphi_2 + 4 \nabla_{\hat{g}} v \cdot \nabla_{\hat{g}} \varphi_1 - 4 \nabla_{g_0} \bar{u} \cdot \nabla_{g_0} \varphi_2 - 4 \nabla_{g_0} \bar{v} \cdot \nabla_{g_0} \varphi_1. \end{aligned}$$

The operator $\mathcal{T}$ introduced earlier is the spherical part of $\mathcal{L}$. We point out that the coefficients of $\mathcal{L}$ do not depend on t. The next result asserts that $\mathcal{Q} w$ is a perturbative term.

Lemma 8.3. *Let $\bar{C}$ and β be as in (8.4). For any $w = (w_1, w_2) \in C^1(B_1 \setminus \Gamma)$, the estimate*

$$(8.26) \qquad |\mathcal{Q}_1 w| + e^{2u} |\mathcal{Q}_2 w| \le C e^{-\beta t} \left(|w_1| + |\nabla_{\hat{g}} w_1| + e^{2u} |\nabla_{\hat{g}} w_2| \right)$$

is valid with a constant C depending only on $\bar{C}$.

Proof. A straightforward calculation yields

$$(8.27) \qquad \begin{aligned} |\mathcal{Q}_1 w| + e^{2u} |\mathcal{Q}_2 w| &\le C \left(|u - \bar{u}| + |\nabla_{\hat{g}}(u - \bar{u})| + e^{2u} |\nabla_{\hat{g}}(v - \bar{v})| \right) \\ &\quad \cdot \left(|w_1| + |\nabla_{\hat{g}} w_1| + e^{2u} |\nabla_{\hat{g}} w_2| \right). \end{aligned}$$

The desired result follows from (8.4), reformulated on the cylinder. $\qquad \square$

We now prove Propositions 8.1 and 8.2 in the new coordinates. Note that the f appearing in these propositions differs from that of the following result, by a scaling factor of e^{-2t}.

Proposition 8.4. *Assume that $f = (f_1, f_2) \in C^1(\mathbb{R}_+ \times (\mathbb{S}^2 \setminus \{N, S\}))$ satisfies*

$$(8.28) \qquad |\nabla_{\hat{g}}^k (f_1 - \bar{f}_1)| + e^{2u} |\nabla_{\hat{g}}^k (f_2 - \bar{f}_2)| \le K e^{-\gamma t} \quad in \ \mathbb{R}_+ \times (\mathbb{S}^2 \setminus \{N, S\}), \qquad k = 0, 1,$$

for some positive constants K, γ, and a C^1-function $\bar{f} = (\bar{f}_1, \bar{f}_2)$ on $\mathbb{S}^2$. Let w satisfying $(w_1, e^{2u} w_2) \in L^\infty(\mathbb{R}_+ \times \mathbb{S}^2)$ be a weak solution of

$$(8.29) \qquad e^{-2t} L w = f \quad in \ \mathbb{R}_+ \times \mathbb{S}^2, \qquad w_2 = 0 \quad on \ \mathbb{R}_+ \times \{N, S\}.$$

Then for any $\varphi \in H^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2, e^{4\bar{u}})$ with $\mathcal{T} \varphi = 0$ we have

$$(8.30) \qquad \langle \bar{f}, \varphi \rangle_{L^2(\mathbb{S}^2, 1 \times e^{4\bar{u}})} = 0,$$

and there exists a function $\bar{w} = (\bar{w}_1, \bar{w}_2) \in H^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2, e^{4\bar{u}})$ with

$$(8.31) \qquad \mathcal{T} \bar{w} = \bar{f} \quad in \ \mathbb{S}^2 \setminus \{N, S\}.$$

Moreover for all $0 < \bar{\beta} \le \min\{\beta, \gamma\}$ with $\bar{\beta}^2 + \bar{\beta} < \mu_2$, $k = 0, 1$, and any $\alpha \in (0, 1)$ it holds that

$$(8.32) \qquad \sup_{\mathbb{S}^2 \setminus \{N,S\}} \left(|\nabla_{\hat{g}}^k (w_1(t) - \bar{w}_1)| + (\sin\theta)^{k-3-\alpha} |\nabla_{\hat{g}}^k (w_2(t) - \bar{w}_2)| \right)$$

$$\le C e^{-\bar{\beta} t} \Big\{ \sup_{\mathbb{R}_+ \times (S^2 \setminus \{N,S\})} \left(|w_1| + e^{2u} |w_2| \right) + \sup_{\mathbb{R}_+ \times (S^2 \setminus \{N,S\})} \left(|f_1| + e^{2u} |f_2| \right) + K \Big\},$$

where C is a constant depending only on K, α, γ, β, and $\bar{\beta}$.

Proof. The proof consists of two steps. For brevity we will make use of the notation

$$(8.33) \quad \|\cdot\|_{L^2} = \|\cdot\|_{L^2(\mathbb{S}^2; 1 \times e^{4\bar{u}})}, \quad \langle\cdot,\cdot\rangle_{L^2} = \langle\cdot,\cdot\rangle_{L^2(\mathbb{S}^2; 1 \times e^{4\bar{u}})}, \quad A = \sup_{t \in [0,\infty)} \left(\|w(t)\|_{L^2} + \|f(t)\|_{L^2} \right).$$

Step 1. By a local estimate for w (see the proof of [12, Proposition 3.9]), for any $t \ge 1$ and $k = 0, 1$ we have

$$(8.34) \quad \sup_{\mathbb{S}^2 \setminus \{N,S\}} \left(|\nabla_{\hat{g}}^k w_1(t)| + (\sin\theta)^{k-3-\alpha} |\nabla_{\hat{g}}^k w_2(t)| \right) \le C \Big(\int_{t-1}^{t+1} \left(\|w(s)\|_{L^2}^2 + \|f(s)\|_{L^2}^2 \right) ds \Big)^{1/2} \le CA.$$

It follows that Lemma 8.3 implies

$$(8.35) \qquad |\mathcal{Q}_1 w| + e^{2u} |\mathcal{Q}_2 w| \le CA e^{-\beta t}, \qquad t \ge 1.$$

By setting $\bar{\gamma} = \min\{\gamma, \beta\}$ we then have that for t in this range

$$(8.36) \qquad \left\| (f - \bar{f}) - \mathcal{Q}w \right\|_{L^2} \le C(A + K) e^{-\bar{\gamma} t}.$$

Now write equations (8.29) as

$$(8.37) \qquad w'' - w' + \mathcal{T}w = \bar{f} + (f - \bar{f}) - \mathcal{Q}w \quad \text{in } \mathbb{R}_+ \times \mathbb{S}^2,$$

where the prime notation indicates t-derivatives. Taking an inner product with the eigenfunction $\varphi^{(1)}$ in $L^2(\mathbb{S}^2, 1 \times e^{4\bar{u}})$ then yields

$$(8.38) \qquad \frac{d^2}{dt^2} \langle w, \varphi^{(1)} \rangle_{L^2} - \frac{d}{dt} \langle w, \varphi^{(1)} \rangle_{L^2} = \langle \bar{f}, \varphi^{(1)} \rangle_{L^2} + \langle f - \bar{f} - \mathcal{Q}w, \varphi^{(1)} \rangle_{L^2},$$

which is a second order ODE for $\langle w, \varphi^{(1)} \rangle_{L^2}$. The first term on the right-hand side is constant and the second term decays exponentially as $t \to \infty$ according to (8.36). Therefore

$$(8.39) \quad \langle w, \varphi^{(1)} \rangle_{L^2} = c_1 + c_2 e^t - t \langle \bar{f}, \varphi^{(1)} \rangle_{L^2} + \int_t^\infty (1 - e^{t-s}) \langle f(s) - \bar{f} - \mathcal{Q}w(s), \varphi^{(1)} \rangle_{L^2} ds,$$

for some constants c_1 and c_2. Since $\langle w, \varphi^{(1)} \rangle_{L^2}$ is bounded, we must have $c_2 = 0$ and $\langle \bar{f}, \varphi^{(1)} \rangle_{L^2} = 0$. Hence, there exists a unique $\bar{w}_* = (\bar{w}_{*1}, \bar{w}_{*2}) \in H^1(\mathbb{S}^2) \times H_0^1(\mathbb{S}^2, e^{4\bar{u}})$ satisfying

$$(8.40) \qquad \mathcal{T}\bar{w}_* = \bar{f} \quad \text{on } \mathbb{S}^2 \quad \text{and} \quad \langle \bar{w}_*, \varphi^{(1)} \rangle_{L^2} = 0.$$

Moreover, note that

$$(8.41) \qquad \left| \langle w(t) - \bar{w}_*, \varphi^{(1)} \rangle_{L^2} - c_1 \right| \le C(A + K) e^{-\bar{\gamma} t}, \qquad t \ge 1.$$

Step 2. We will now establish (8.32), which follows from standard ODE analysis. Set

$$(8.42) \qquad \widehat{w}(t) = w(t) - \bar{w}_* - \langle w(t) - \bar{w}_*, \varphi^{(1)} \rangle_{L^2} \varphi^{(1)},$$

and

$$\widehat{f}(t) = (f(t) - \bar{f}) - \mathcal{Q}w(t) - \langle (f(t) - \bar{f}) - \mathcal{Q}w(t), \varphi^{(1)} \rangle_{L^2} \varphi^{(1)}. \tag{8.43}$$

Then

$$\widehat{w}'' - \widehat{w}' + \mathcal{T}\widehat{w} = \widehat{f} \quad \text{in } \mathbb{R}_+ \times \mathbb{S}^2, \tag{8.44}$$

and

$$\|\widehat{f}(t)\|_{L^2} \le C(A + K)e^{-\bar{\gamma}t}, \qquad t \ge 1. \tag{8.45}$$

Moreover $\langle \widehat{w}(t), \varphi^{(1)} \rangle_{L^2} = 0$ and hence

$$\overline{\mathcal{B}}[\widehat{w}(t), \widehat{w}(t)] \ge \mu_2 \|\widehat{w}(t)\|_{L^2}^2, \tag{8.46}$$

where $\overline{\mathcal{B}}$ is the bilinear form defined by (8.9). Taking an inner product of the equation (8.44) with $-\widehat{w}(t)$ in $L^2(\mathbb{S}^2, 1 \times e^{4\bar{u}})$ produces

$$\langle \widehat{w}(t), -\widehat{w}''(t) + \widehat{w}'(t) \rangle_{L^2} + \overline{\mathcal{B}}[\widehat{w}(t), \widehat{w}(t)] = -\langle \widehat{w}(t), \widehat{f}(t) \rangle_{L^2}. \tag{8.47}$$

For any $t \ge 0$ set

$$y(t) = \|\widehat{w}(t)\|_{L^2} = \left(\int_{\mathbb{S}^2} \left(\widehat{w}_1^2(t) + e^{4\bar{u}} \widehat{w}_2^2(t) \right) dvol_{g_0} \right)^{1/2}, \tag{8.48}$$

and note that y is differentiable at $t > 0$ if $y(t) > 0$. At such t we have

$$y(t)y'(t) = \langle \widehat{w}(t), \widehat{w}'(t) \rangle_{L^2}, \quad y(t)y''(t) + [y'(t)]^2 = \langle \widehat{w}(t), \widehat{w}''(t) \rangle_{L^2} + \langle \widehat{w}'(t), \widehat{w}'(t) \rangle_{L^2}. \tag{8.49}$$

The Cauchy-Schwarz inequality implies that if $y(t) > 0$ then $|y'(t)| \le \|\widehat{w}'(t)\|_{L^2}$ and hence

$$y(t)y''(t) \ge \langle \widehat{w}(t), \widehat{w}''(t) \rangle_{L^2}. \tag{8.50}$$

It follows that for $t \ge 1$ inequalities (8.45) and (8.46) yield

$$-y(t)y''(t) + y(t)y'(t) + \mu_2 y(t)^2 \le y(t)\|\widehat{f}(t)\|_{L^2} \le C(A + K)y(t)e^{-\bar{\gamma}t}, \tag{8.51}$$

and thus when $y(t) > 0$ we obtain

$$-y''(t) + y'(t) + \mu_2 y(t) \le C(A + K)e^{-\bar{\gamma}t}, \qquad t \ge 1. \tag{8.52}$$

Consider $0 < \bar{\beta} \le \bar{\gamma}$ with $\bar{\beta}^2 + \bar{\beta} < \mu_2$. For some positive constant B to be determined, set $z(t) = Be^{-\bar{\beta}(t-1)}$ and observe that

$$-z''(t) + z'(t) + \mu_2 z(t) = B(\mu_2 - \bar{\beta}^2 - \bar{\beta})e^{-\bar{\beta}(t-1)}. \tag{8.53}$$

We choose B large enough such that

$$B(\mu_2 - \bar{\beta}^2 - \bar{\beta}) > C(A + K) \quad \text{and} \quad B > \|\widehat{w}(1)\|_{L^2}. \tag{8.54}$$

Hence, $y(1) < z(1)$ and if $y(t) > 0$ then

$$-(y - z)''(t) + (y - z)'(t) + \mu_2(y - z)(t) < 0, \qquad t \ge 1. \tag{8.55}$$

We now claim that $y(t) \le z(t)$ for any $t \ge 1$. If $y - z$ has a local nonnegative maximum at some $t_* > 1$ then

$$(y - z)(t_*) \ge 0, \quad (y - z)'(t_*) = 0, \quad (y - z)''(t_*) \le 0, \tag{8.56}$$

and in particular, $y(t_*) \geq z(t_*) > 0$, contradicting (8.55) at t_*. Thus, $y - z$ does not have a local nonnegative maximum on $(0, \infty)$. Assume $(y - z)(t_0) > 0$ for some $t_0 > 1$. Then $y(t_0) > 0$ and (8.55) holds at t_0. If $(y - z)(t) < (y - z)(t_0)$ for some $t > t_0$, then $y - z$ has a local positive maximum in $(1, t)$, leading to a contradiction. Therefore, $(y - z)(t) \geq (y - z)(t_0) > 0$ for all $t > t_0$. In particular, $y(t) > z(t) > 0$ for all $t > t_0$ so that (8.55) holds whenever $t > t_0$. Next observe that for any $s > t > t_0$, a similar argument shows that $(y - z)(s) \geq (y - z)(t)$. In other words, $y - z$ is nondecreasing on (t_0, ∞). Hence $(y - z)' \geq 0$ on (t_0, ∞), and by (8.55) we have

$$(8.57) \qquad (y - z)''(t) > (y - z)'(t) + \mu_2(y - z)(t) \geq \mu_2(y - z)(t_0), \qquad t > t_0.$$

It follows that $y(t) \to \infty$, which contradicts the boundedness of w on $\mathbb{R}_+ \times \mathbb{S}^2$. Thus $y(t) \leq z(t)$ for all $t \geq 1$, and hence

$$(8.58) \qquad \|\widehat{w}(t)\|_{L^2} \leq B e^{-\bar{\beta}(t-1)}, \qquad t \geq 1.$$

The proof of [12, Proposition 3.9] applied to equation (8.44) in $(t - 1, t + 1) \times \mathbb{S}^2$, for $t > 2$ and $k = 0, 1$, now implies

$$(8.59) \quad \sup_{\mathbb{S}^2 \setminus \{N, S\}} \left(|\nabla_{\hat{g}}^k \widehat{w}_1(t)| + (\sin\theta)^{k-3-\alpha} |\nabla_{\hat{g}}^k \widehat{w}_2(t)| \right) \leq C \left(\int_{t-1}^{t+1} \left(\|\widehat{w}(s)\|_{L^2}^2 + \|\widehat{f}(s)\|_{L^2}^2 \right) ds \right)^{1/2}$$

$$\leq C(A + K) e^{-\bar{\beta}t}.$$

Note that by (8.42) we have

$$(8.60) \qquad w(t) - \bar{w}_* - c_1 \varphi^{(1)} = \widehat{w}(t) + \left[\langle w(t) - \bar{w}_*, \varphi^{(1)} \rangle - c_1 \right] \varphi^{(1)}.$$

Then for $k = 0, 1$ the estimates (8.41) and (8.59) produce

$$(8.61) \quad \sup_{\mathbb{S}^2 \setminus \{N, S\}} \left(|\nabla_{\hat{g}}^k (w_1(t) - \bar{w}_{*1} - c_1 \varphi_1^{(1)})| + (\sin\theta)^{k-3-\alpha} |\nabla_{\hat{g}}^k (w_2(t) - \bar{w}_{*2} - c_1 \varphi_2^{(1)})| \right)$$

$$\leq C(A + K) e^{-\bar{\beta}t}.$$

This establishes the desired result with $\bar{w} = \bar{w}_* + c_1 \varphi^{(1)}$. $\square$

Proposition 8.5. *Let f and w be as in Proposition 8.4. Then there exists $t_* > 10$ depending only on γ, a, as well as $\bar{C}$ and β in (8.4), such that*

$$(8.62) \qquad \||w_1| + e^{2u}|w_2|\|_{L^\infty(\mathbb{R}_+ \times \mathbb{S}^2)}$$
$$\leq C \left(\||w_1| + e^{2u}|w_2|\|_{L^\infty([0,t_*] \times \mathbb{S}^2)} + \||f_1| + e^{2u}|f_2|\|_{L^\infty(\mathbb{R}_+ \times \mathbb{S}^2)} + K \right).$$

Moreover, (8.32) holds with the L^∞-norms of w_1 and $e^{2u} w_2$ on $[0, \infty) \times \mathbb{S}^2$ replaced by that on $[0, t_*] \times \mathbb{S}^2$.

Proof. For brevity we will write $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(\mathbb{S}^2; 1 \times e^{4\bar{u}})}$. Let $\bar{w}_*$ be as in (8.40) and set

$$(8.63) \qquad \widetilde{w}(t) = w(t) - \bar{w}_*, \qquad \widetilde{f}(t) = (f - \bar{f}) - \mathcal{Q}w.$$

Then

$$(8.64) \qquad \widetilde{w}'' - \widetilde{w}' + \mathcal{T}\widetilde{w} = \widetilde{f} \quad \text{in } \mathbb{R}_+ \times \mathbb{S}^2.$$

We will expand $\widetilde{w}$ and $\widetilde{f}$ according to the eigenfunctions $\{\varphi^{(i)}\}$ as

$$(8.65) \qquad \widetilde{w}(t) = \sum_{i=1}^{\infty} \widetilde{w}^{(i)}(t)\varphi^{(i)}, \quad \widetilde{f}(t) = \sum_{i=1}^{\infty} \widetilde{f}^{(i)}(t)\varphi^{(i)}.$$

The regularity of w obtained from [12, Proposition 3.9] guarantees that the expansion can be differentiated term by term so that

$$(8.66) \qquad \frac{d^2}{dt^2}\widetilde{w}^{(i)} - \frac{d}{dt}\widetilde{w}^{(i)} - \mu_i\widetilde{w}^{(i)} = \widetilde{f}^{(i)} \quad \text{in } \mathbb{R}_+.$$

For $i \geq 1$ define

$$(8.67) \qquad \lambda_i^+ = \frac{1}{2}\big(1 + \sqrt{1 + 4\mu_i}\big), \quad \lambda_i^- = \frac{1}{2}\big(1 - \sqrt{1 + 4\mu_i}\big),$$

and note that $\lambda_i^+ \geq 1$, $\lambda_i^- \leq 0$, as well as $\lambda_1^+ = 1$, $\lambda_1^- = 0$. We will treat $i = 1$ differently from $i \geq 2$. Since $\widetilde{w}^{(i)}$ is bounded on $\mathbb{R}_+$, there exists a constant c_i^- such that

$$(8.68) \qquad \widetilde{w}^{(i)}(t) = c_i^- e^{\lambda_i^- t} + \widetilde{w}_*^{(i)}(t),$$

where for $i = 1$ the remaining function takes the form

$$(8.69) \qquad \widetilde{w}_*^{(1)}(t) = -\int_t^{\infty} e^{t-s}\widetilde{f}^{(1)}(s)\,ds + \int_t^{\infty} \widetilde{f}^{(1)}(s)\,ds,$$

and for $i \geq 2$ it becomes

$$(8.70) \qquad \widetilde{w}_*^{(i)}(t) = \frac{1}{\lambda_i^- - \lambda_i^+}\int_t^{\infty} e^{\lambda_i^+(t-s)}\widetilde{f}^{(i)}(s)\,ds + \frac{1}{\lambda_i^- - \lambda_i^+}\int_2^t e^{\lambda_i^-(t-s)}\widetilde{f}^{(i)}(s)\,ds.$$

Correspondingly, we may write

$$(8.71) \qquad \widetilde{w}(t) = \widetilde{w}_0(t) + \widetilde{w}_*(t), \quad \widetilde{w}_0(t) = \sum_{i=1}^{\infty} c_i^- e^{\lambda_i^- t}\varphi^{(i)}, \quad \widetilde{w}_*(t) = \sum_{i=1}^{\infty} \widetilde{w}_*^{(i)}(t)\varphi^{(i)}.$$

By the explicit expression of $\widetilde{w}_0(t)$ for $t \geq 2$, it follows that

$$(8.72) \qquad \|\widetilde{w}_0(t)\|_{L^2}^2 = \sum_{i=1}^{\infty} (c_i^-)^2 e^{2\lambda_i^- t} \leq \sum_{i=1}^{\infty} (c_i^-)^2 e^{2\cdot 2\lambda_i^-} = \|\widetilde{w}_0(2)\|_{L^2}^2.$$

Moreover, the triangle inequality then gives

$$(8.73) \qquad \begin{aligned} \|\widetilde{w}(t)\|_{L^2} &\leq \|\widetilde{w}_0(t)\|_{L^2} + \|\widetilde{w}_*(t)\|_{L^2} \leq \|\widetilde{w}_0(2)\|_{L^2} + \|\widetilde{w}_*(t)\|_{L^2} \\ &\leq \|\widetilde{w}(2)\|_{L^2} + \|\widetilde{w}_*(2)\|_{L^2} + \|\widetilde{w}_*(t)\|_{L^2}. \end{aligned}$$

For $t \geq 2$ this, together with $w(t) = \widetilde{w}(t) + \bar{w}_*$, implies

$$(8.74) \qquad \begin{aligned} \|w(t)\|_{L^2} &\leq \|w(2)\|_{L^2} + 2\|\bar{w}_*\|_{L^2} + \|\widetilde{w}_*(2)\|_{L^2} + \|\widetilde{w}_*(t)\|_{L^2} \\ &\leq \|w(2)\|_{L^2} + C\|\bar{f}\|_{L^2} + \|\widetilde{w}_*(2)\|_{L^2} + \|\widetilde{w}_*(t)\|_{L^2}. \end{aligned}$$

We next estimate $\widetilde{w}_*$. By (8.70) and the Hölder inequality, we have for $i \geq 2$ and $t \geq 2$ that

(8.75)
$$|\widetilde{w}_*^{(i)}(t)|^2 \leq \frac{C}{(\lambda_i^+)^3} \int_t^\infty e^{\lambda_i^+(t-s)} |\widetilde{f}^{(i)}(s)|^2 \, ds + \frac{C}{(\lambda_i^+)^3} \int_2^t e^{\lambda_i^-(t-s)} |\widetilde{f}^{(i)}(s)|^2 \, ds$$
$$\leq C \int_t^\infty e^{t-s} |\widetilde{f}^{(i)}(s)|^2 \, ds + C \int_2^t e^{\lambda_2^-(t-s)} |\widetilde{f}^{(i)}(s)|^2 \, ds,$$

where in the second inequality $\lambda_i^+ \geq \lambda_2^+ \geq 1$ and $\lambda_i^- \leq \lambda_2^- < 0$ were used. Similarly, for $i = 1$ and $t \geq 2$ we find that (8.69) yields

(8.76)
$$|\widetilde{w}_*^{(1)}(t)|^2 \leq C \int_t^\infty e^{t-s} |\widetilde{f}^{(1)}(s)|^2 \, ds + C \Big(\int_t^\infty |\widetilde{f}^{(1)}(s)| \, ds \Big)^2.$$

Since

(8.77)
$$\|\widetilde{w}_*(t)\|_{L^2}^2 = \sum_{i=1}^\infty |\widetilde{w}_*^{(i)}(t)|^2, \quad \|\widetilde{f}(t)\|_{L^2}^2 = \sum_{i=1}^\infty |\widetilde{f}^{(i)}(t)|^2,$$

for $t \geq 2$ we have

(8.78) $\|\widetilde{w}_*(t)\|_{L^2}^2 \leq C \int_t^\infty e^{t-s} \|\widetilde{f}(s)\|_{L^2}^2 \, ds + C \int_2^t e^{\lambda_2^-(t-s)} \|\widetilde{f}(s)\|_{L^2}^2 \, ds + C \Big(\int_t^\infty |\widetilde{f}^{(1)}(s)| \, ds \Big)^2,$

and hence

(8.79)
$$\|\widetilde{w}_*(t)\|_{L^2}^2 \leq C \sup_{s \in [2,\infty)} \|\widetilde{f}(s)\|_{L^2}^2 + C \Big(\int_t^\infty |\widetilde{f}^{(1)}(s)| \, ds \Big)^2.$$

It is obvious that

(8.80)
$$\|\widetilde{f}(s)\|_{L^2} \leq \|f(s) - \bar{f}\|_{L^2} + \|\mathcal{Q}w(s)\|_{L^2} \leq CKe^{-\gamma s} + \|\mathcal{Q}w(s)\|_{L^2},$$

and

(8.81)
$$|\widetilde{f}^{(1)}(s)| \leq \|\widetilde{f}(s)\|_{L^2} \leq CKe^{-\gamma s} + \|\mathcal{Q}w(s)\|_{L^2},$$

so that

(8.82)
$$\|\widetilde{w}_*(t)\|_{L^2} \leq C \Big(K + \sup_{s \in [2,\infty)} \|\mathcal{Q}w(s)\|_{L^2} + \int_t^\infty \|\mathcal{Q}w(s)\|_{L^2} \, ds \Big).$$

Furthermore, by Lemma 8.3 we have

(8.83)
$$\|\mathcal{Q}w(s)\|_{L^2} \leq Ce^{-\beta s} \big(\|w(s)\|_{L^2} + \|\nabla_{\hat{g}} w(s)\|_{L^2} \big),$$

and a local estimate for w (see Proposition 3.9 of [12]) when $s \geq 2$ produces

(8.84)
$$\|\nabla_{\hat{g}} w(s)\|_{L^2}^2 \leq C \int_{s-1}^{s+1} (\|f(\tau)\|_{L^2}^2 + \|w(\tau)\|_{L^2}^2) \, d\tau$$
$$\leq C \sup_{[s-1,s+1]} \|f(\tau)\|_{L^2}^2 + C \int_{s-1}^{s+1} \|w(\tau)\|_{L^2}^2 \, d\tau,$$

so that

$$\|\mathcal{Q}w(s)\|_{L^2} \leq Ce^{-\beta s} \Big(\sup_{[s-1,s+1]} \|f(\tau)\|_{L^2} + \|w(s)\|_{L^2} + \Big(\int_{s-1}^{s+1} \|w(\tau)\|_{L^2}^2 \, d\tau \Big)^{1/2} \Big).$$

Next note that a simple substitution gives

$$\|\widetilde{w}_*(t)\|_{L^2} \leq C\Big(K + \sup_{s\in[1,\infty)} \|f(s)\|_{L^2} + \sup_{s\in[1,\infty)} e^{-\beta s}\|w(s)\|_{L^2}$$

(8.85)
$$+ \int_t^\infty e^{-\beta s}\|w(s)\|_{L^2}\,ds + \int_t^\infty e^{-\beta s}\Big(\int_{s-1}^{s+1}\|w(\tau)\|_{L^2}^2\,d\tau\Big)^{1/2}ds\Big)$$

$$\leq C\Big(K + \sup_{s\in[1,\infty)} \|f(s)\|_{L^2} + \sup_{s\in[1,\infty)} e^{-\frac{\beta}{2}s}\|w(s)\|_{L^2}\Big);$$

this holds for any $t \geq 2$.

By substituting (8.85) into (8.74) we obtain

(8.86)
$$\|w(t)\|_{L^2} \leq C\big(\|w(2)\|_{L^2} + K + \sup_{s\in[1,\infty)} \|f(s)\|_{L^2} + \sup_{s\in[1,\infty)} e^{-\frac{\beta}{2}s}\|w(s)\|_{L^2}\big).$$

Next, let $t_* > 1$ and break the interval of the last term on the right-hand side into $[1, t_*]$ and $[t_*, \infty)$ to find

(8.87)
$$\sup_{t\in[t_*,\infty)} \|w(t)\|_{L^2} \leq C\big(\|w(2)\|_{L^2} + K + \sup_{s\in[1,\infty)} \|f(s)\|_{L^2} + \sup_{s\in[1,t_*]} e^{-\frac{\beta}{2}s}\|w(s)\|_{L^2}\big)$$
$$+ Ce^{-\frac{\beta}{2}t_*} \sup_{s\in[t_*,\infty)} \|w(s)\|_{L^2}.$$

Choosing $t^* > 2$ large enough such that $Ce^{-\beta t^*/2} < 1/2$ then produces

(8.88)
$$\sup_{t\in[t_*,\infty)} \|w(t)\|_{L^2} \leq C\big(K + \sup_{s\in[1,\infty)} \|f(s)\|_{L^2} + \sup_{s\in[1,t_*]} \|w(s)\|_{L^2}\big).$$

Furthermore by the local estimate [12, Proposition 3.9], for any $t > 3$ and $k = 0, 1$, we have

(8.89)
$$|\nabla_{\hat{g}}^k w_1(t)| + (\sin\theta)^{k-3-\alpha}|\nabla_{\hat{g}}^k w_2(t)| \leq C\big(K + \sup_{s\in[1,\infty)} \|f(s)\|_{L^2} + \sup_{s\in[1,t_*]} \|w(s)\|_{L^2}\big).$$

Following the proof of Proposition 8.4, using (8.89) in place of (8.34), yields the desired (8.32) with the L^∞-norms of w_1 and $e^{2u}w_2$ on $[0,\infty) \times \mathbb{S}^2$ replaced by that on $[0, t_*] \times \mathbb{S}^2$. $\square$

Remark 8.6. Given the propositions above, one can improve (8.4) by identifying higher order limits of (u, v) as $x \to 0$. In particular, we find that $\beta > 0$ depends only on μ_2.

9. Differentiability of Singular Harmonic Maps With Respect to Parameters

We now study the harmonic maps discussed in Section 3 which depend on parameters, and provide a proof of Theorem 3.1 and Corollary 3.2. Let $\mathbf{z}_0 = (z_{0,1}, \ldots, z_{0,N})$ with $z_{0,1} < z_{0,2} < \cdots < z_{0,N}$ denote the collection of z-coordinates for points $p_i^0 = (0, 0, z_{0,i}) \in \mathbb{R}^3$, and set $\delta_0 := \frac{1}{8}\min_{1\leq i\leq N-1}\{z_{0,i+1} - z_{0,i}\}$. Generic points in $\mathbb{R}^3$ will be denoted by $y = (y_1, y_2, y_3)$. Let $\bar{\chi}_1$ be a smooth function on $[0, \infty)$ with support in $[0, \sqrt{2}\delta_0)$, $0 \leq \bar{\chi}_1 \leq 1$, and $\bar{\chi}_1(t) = 1$ for $t \leq \delta_0$, and define

(9.1)
$$\bar{\chi}(y) = \bar{\chi}_1(\rho(y)) \cdot \bar{\chi}_1(|y_3|)$$

where $\rho(y) = \sqrt{y_1^2 + y_2^2}$. Consider the map $\Pi_{\mathbf{z}} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$(9.2) \qquad y \mapsto y + \sum_{i=1}^{N}(z_i - z_{0,i})\bar{\chi}(y_1, y_2, y_3 - z_{0,i})e_3 =: x,$$

where $e_3 = (0, 0, 1)$. Note that it is a diffeomorphism when $\mathbf{z} \in B_{\delta_1}^N(\mathbf{z}_0)$ for some small $\delta_1 > 0$, and that $\Pi_{\mathbf{z}}(p_i^0) = p_i$ for every $1 \le i \le N$. Furthermore, the coefficients of the pull back of the Euclidean metric $g_{\mathbf{z}} = \Pi_{\mathbf{z}}^* \boldsymbol{\delta} = (g_{\mathbf{z}})_{ab} dy^a dy^b$ take the form

$$(9.3) \quad (g_{\mathbf{z}})_{ab}(y) = \boldsymbol{\delta}_{ab} + \sum_{i=1}^{N}\Big((z_i - z_{0,i})(\boldsymbol{\delta}_{a3}\partial_b\bar{\chi} + \boldsymbol{\delta}_{b3}\partial_a\bar{\chi}) + (z_i - z_{0,i})^2\partial_a\bar{\chi}\partial_b\bar{\chi} \Big)(y_1, y_2, y_3 - z_{0,i}).$$

Notice that

$$(9.4) \qquad (g_{\mathbf{z}})_{ab} = \boldsymbol{\delta}_{ab} \quad \text{on } \cup_{i=1}^{N} B_{\delta_0}(p_i^0) \cup (\mathbb{R}^3 \setminus \cup_{i=1}^{N} B_{2\delta_0}(p_i^0)),$$

and

$$(9.5) \qquad \Delta_g \ln \rho = \Delta_{\Pi_{\mathbf{z}}^*\boldsymbol{\delta}}(\Pi_{\mathbf{z}}^* \ln \rho) = \Pi_{\mathbf{z}}^*(\Delta_{\boldsymbol{\delta}} \ln \rho) = 0 \quad \text{in } \mathbb{R}^3.$$

Moreover, we may choose $0 < \varepsilon_0 \le \min\{\delta_0, \delta_1\}$ small enough so that, in the sense of positive definite matrices, $\frac{1}{2}\boldsymbol{\delta} \le g_{\mathbf{z}} \le 2\boldsymbol{\delta}$ for $\mathbf{z} \in B_{\varepsilon_0}(\mathbf{z}_0)$. For convenience, in what follows we will forgo the subscript $\mathbf{z}$ and simply write g in place of $g_{\mathbf{z}}$.

For simplicity, we shall only consider the case when one component z_{i_0} of $\mathbf{z}$ is varying, so that $z_i = z_{0,i}$ for $i \ne i_0$. It will become evident that the general case can be proved similarly, albeit with a tedious but straightforward adaptation. Without loss of generality we may assume that $z_{0,i_0} = 0$, and denote $s := z_{i_0} \in (-\delta_0, \delta_0)$. To streamline the notation set

$$(9.6) \qquad \Phi(s, y) = (u(s, y), v(s, y)) := \Phi_{\mathbf{z}}(\Pi_{\mathbf{z}}(y)),$$

where $\Phi_{\mathbf{z}}$ is the harmonic map defined in Section 3. In the y coordinates, the harmonic map system becomes

$$(9.7) \qquad \Delta_g u - 2e^{4u}|\nabla_g v|^2 = 0, \qquad \Delta_g v + 4\nabla_g u \cdot \nabla_g v = 0.$$

Let σ be a smooth function on $\mathbb{R}^3 \setminus \{p_1^0, \ldots, p_N^0\}$ such that

$$(9.8) \qquad \sigma(y) = \begin{cases} \text{dist}_{\boldsymbol{\delta}}(y, \{p_1^0, \ldots, p_N^0\}) & \text{if dist}_{\boldsymbol{\delta}}(y, \{p_1^0, \ldots, p_N^0\}) \le 2\delta_0, \\ |y| & \text{if } |y| \ge 2\sum_{i=1}^{N}|z_i| + 1, \end{cases}$$

and

$$(9.9) \qquad \frac{1}{2}\text{dist}_{\boldsymbol{\delta}}(y, \{p_1^0, \ldots, p_N^0\}) \le \sigma(y) \le 2\text{dist}_{\boldsymbol{\delta}}(y, \{p_1^0, \ldots, p_N^0\}).$$

Lemma 9.1. *Let $\Phi = \Phi(s) = (u(s), v(s))$ be given by (9.6), for any $s \in [-\varepsilon_0, \varepsilon_0]$. Then*

$$(9.10) \qquad d_{\mathbb{H}^2}(\Phi(s), \Phi(0)) \le \Lambda \quad \text{in } \mathbb{R}^3 \setminus \Gamma,$$

where Λ is a positive constant depending only on ε_0 and $\{p_1^0, \ldots, p_N^0\}$. Moreover for any $\mathbf{r} \ge 2(|z_{0,1}| + |z_{0,N}|)$, $\alpha \in (0, 1)$, and $k = 1, 2, 3$ we have

$$(9.11) \qquad \Big|\nabla_g^k\Big(u(s) + \ln\frac{\rho}{\sigma}\Big)\Big| + \Big(\frac{\rho}{\sigma}\Big)^{k-3-\alpha}|\nabla_g^k v(s)| \le C(\mathbf{r})\sigma^{-k} \quad \text{in } B_{\mathbf{r}} \setminus \Gamma,$$

where $C(\mathbf{r})$ is a constant independent of s.

Proof. Both estimates are consequences of slight modifications of the proofs from previous results, where the flat metric $\boldsymbol{\delta}$ is replaced by g. In particular, the first conclusion follows from Proposition 2.1 of [8] (see also [29]). The second one follows from Proposition 3.8 and Theorem 2.3 of [12], together with standard estimates in regions away from the axis. $\qquad\square$

Remark 9.2. Notice that although the distance between $\Phi(s)$ and $\Phi(0)$ is defined only on $\mathbb{R}^3 \backslash \Gamma$, according to [12, Theorems 2.1 and 2.3] it lies within $C^{3,\alpha}(\mathbb{R}^3 \setminus \{p_1^0, \ldots, p_N^0\})$ for any $\alpha \in (0,1)$.

By Theorem 2.1 of [12] (see also (8.4) above), for each $s \in [-\varepsilon_0, \varepsilon_0]$ and each p_i, $1 \le i \le N$ there exists a unique tangent map $\bar{\Phi}_i = (\bar{u}_i, \bar{v}_i)$ from $\mathbb{S}^2 \backslash \{N, S\}$ to $\mathbb{H}^2$, such that for any $l + k \le 3$ we have

$$
(9.12) \qquad
\begin{aligned}
\sup_{\mathbb{S}^2 \backslash \{N,S\}} & \Big(|(r_i \partial_{r_i})^l \nabla^k_{\mathbb{S}^2}(u(s, r_i, \theta_i) - \bar{u}_i(s, \theta_i))| \\
& + e^{(3+\alpha-k)\bar{u}_i} |(r_i \partial_{r_i})^l \nabla^k_{\mathbb{S}^2}(v(s, r_i, \theta_i) - \bar{v}_i(s, \theta_i))| \Big) \le C_1(s) r_i^\beta
\end{aligned}
$$

in $B_{\delta_0}(p_i^0) \setminus \{p_i^0\}$, where $\beta > 0$ is independent of s, $r_i = |y - p_i^0|$, and $\sin \theta_i = \frac{\rho(y)}{r_i}$. Furthermore, $(\bar{u}_i + \ln \frac{\rho}{r_i}, \bar{v}_i) \in C^{3,\alpha}(\mathbb{S}^2)$ for any $\alpha \in (0,1)$, and $\bar{v}_i = c_i$ at the south pole S while $\bar{v}_i = c_{i+1}$ at the north pole N; see (3.2) for the explicit form. We will now prove that the dependence of $C_1(s)$ on s can be removed.

Lemma 9.3. *Let $\Phi = \Phi(s)$ be given by (9.6), for any $s \in [-\varepsilon_0, \varepsilon_0]$. Then the constant $C_1(s)$ in (9.12) is uniformly bounded for $s \in [-\varepsilon_0, \varepsilon_0]$, and hence may be chosen independent of s.*

Proof. Take an arbitrary sequence $s_n \in [-\varepsilon_0, \varepsilon_0]$, and let $\Phi(s_n)$ be the corresponding solutions of (9.7). By (9.10) and (9.11), the Arzelà-Ascoli theorem may be applied to extract a subsequence, still denoted by $\Phi(s_n)$, such that

$$
(9.13) \qquad \lim_{n,m \to \infty} d_{\mathbb{H}^2}(\Phi(s_n), \Phi(s_m)) = 0 \quad \text{in } \overline{B}_{\delta_0}(p_i^0) \setminus (B_{\delta_0/2}(p_i^0) \cup \Gamma).
$$

By (9.10), we have $d_{\mathbb{H}^2}(\Phi(s_n), \Phi(s_m)) \le C$ in $B_{\delta_0}(p_n^0)$. Note also that the negative curvature of the target space yields

$$
(9.14) \qquad \Delta \sqrt{1 + d_{\mathbb{H}^2}(\Phi(s_n), \Phi(s_m))^2} \ge 0 \quad \text{in } B_{\delta_0}(p_i^0) \setminus \Gamma.
$$

Therefore, the maximum principle [28, Lemma 8] implies

$$
(9.15) \qquad \max_{B_{\delta_0}(p_i^0) \setminus \{p_i^0\}} d_{\mathbb{H}^2}(\Phi(s_n), \Phi(s_m)) \le \max_{\partial B_{\delta_0}} d_{\mathbb{H}^2}(\Phi(s_n), \Phi(s_m)).
$$

These estimates show that the constants C_* and C from Lemmas 5.6 and 5.7 of [12], are independent of n. It then follows from the proof of [12, Theorem 5.8] that $C_1(s_n)$ is uniformly bounded. The desired result may now be obtained from a simple contradiction argument. Suppose that there exists a sequence $s_n \in [-\varepsilon_0, \varepsilon_0]$ such that $C_1(s_n) \to \infty$ as $n \to \infty$. The argument above shows that $C_1(s_n)$ is bounded, leading to a contradiction. $\qquad\square$

Let $\mathbf{r}_0 > 8(|z_{0,1}| + |z_{0,N}|)$ be a large number. According to Theorem 2.3 of [12], for each s and any $\alpha \in (0,1)$ there exist constants ν_0, ν_1, ν_2, C, and spherical harmonics Y_1, Y_2 of degrees

1 and 2 such that within $B_{\mathbf{r}_0}^c$ we have

(9.16)
$$\left| u(s, r, \theta) + \ln \rho - \nu_0 - \nu_1 r^{-1} - Y_1 r^{-2} - Y_2 r^{-3} \right|_{C^3(\mathbb{S}^2)} \leq C r^{-3-\beta_0},$$
$$\left| v(s, r, \theta) - \mathcal{J}_0 \cos \theta (3 - \cos^2 \theta) - \nu_2 r^{-1} \sin^4 \theta \right|_{C^3(\mathbb{S}^2)} \leq C r^{-1-\beta_0} (\sin \theta)^{3+\alpha},$$

where $r = |y|$, $\mathcal{J}_0$ is a multiple of total angular momentum, and $\beta_0 \in (0, 1)$ is independent of s. Moreover, the corresponding asymptotics are valid for the r-derivatives of (u, v). We point out that the constants ν_0, ν_1, ν_2, C and functions $|Y_1|$, $|Y_2|$ have uniform bounds for $s \in [-\varepsilon_0, \varepsilon_0]$, which depend only on Λ and $\mathbf{r}_0$.

Lemma 9.4. *Let $\Phi = \Phi(s)$ be given by* (9.6)*, for any $s \in [-\varepsilon_0, \varepsilon_0]$. Then $\nu_0 = 0$ for all $s \in [-\varepsilon_0, \varepsilon_0]$ and*

(9.17)
$$d_{\mathbb{H}^2}(\Phi(s, y), \Phi(0, y)) \leq C r^{-1} \quad in \ B_{\mathbf{r}_0}^c,$$

where C is a positive constant independent of s.

Proof. For each fixed s, consider the original x coordinates. In this setting, recall the construction of Φ in [8]. In the proof of [8, Proposition 2.1], a model (or reference) map $\tilde{\Phi} = (\tilde{u}, \tilde{v})$ is constructed such that $\tilde{v} = c_i$ on the interval $\Gamma_i \subset \Gamma$, $1 \leq i \leq N$ and $\tilde{\Phi}$ coincides, for all large values of $|x|$, with the extreme Kerr solution having potential constant value c_{N+1} for $x_3 > z_N$ and value c_1 for $x_3 < z_1$ on the x_3-axis. Next, one solves the Dirichlet problem for the harmonic map system in a sequence of exhausting domains $\Omega_n = \{x \mid \rho(x) > 1/n, |x| < n\}$, with boundary values given by the above model map; this generates a sequence of harmonic maps Φ_n in Ω_n. It is then shown that there is a uniform hyperbolic distance bound between Φ_n and $\tilde{\Phi}$ on compact subsets. The desired solution Φ is obtained by sending n to infinity and passing to a subsequence. Furthermore, by Theorem 4.1 of [17] (with motivation from [25]) we have

(9.18)
$$\left(\int_{\mathbb{R}^3} d_{\mathbb{H}^2}(\Phi, \tilde{\Phi})^6 dx \right)^{1/3} < \infty.$$

On the other hand, for large $|x|$ the expansions (9.16) imply

(9.19) $\cosh\left(2 d_{\mathbb{H}^2}(\Phi, \tilde{\Phi})\right) = \cosh\left(2(u - \tilde{u})\right) + 2 e^{2(u+\tilde{u})}(v - \tilde{v})^2 = \cosh[2\nu_0 + O(|x|^{-1})] + O(|x|^{-2})$,

from which it follows that $\nu_0 = 0$. $\qquad\square$

We will begin the study of regularity for $\Phi(s)$, with respect to the s-parameter, by establishing continuity in an L^6-sense.

Proposition 9.5. *Let $\Phi = \Phi(s)$ be given by* (9.6)*, for any $s \in [-\varepsilon_0, \varepsilon_0]$. Then for any $s_1, s_2 \in [-\varepsilon_0, \varepsilon_0]$ we have*

(9.20)
$$\left(\int_{\mathbb{R}^3} d_{\mathbb{H}^2}(\Phi(s_1), \Phi(s_2))^6 dy \right)^{1/6} \leq C |s_1 - s_2|^{1/2},$$

where C is a constant independent of s_1 and s_2.

Proof. Since $\rho(y) = \rho(\Pi_{\mathbf{z}}(y))$, we may renormalize the first term of the harmonic maps with the same expression in x or y-coordinates by $U = u + \ln \rho$, and define the renormalized energy in y-coordinates with

$$(9.21) \qquad \mathcal{E}_s(\Phi(s)) = \int_{\mathbb{R}^3} (|\nabla_{g(s)} U(s)|^2 + e^{4u(s)} |\nabla_{g(s)} v(s)|^2) \, dvol_{g(s)}.$$

Using that $\Phi(s_j)$ is harmonic with respect to $g(s_j)$, $j = 1, 2$ the estimate of [17, Theorem 4.1] (see also [25, Theorem 1.1]) gives

$$(9.22)$$
$$\mathcal{E}_{s_1}(\Phi(s_2)) - \mathcal{E}_{s_1}(\Phi(s_1)) \geq C \left(\int_{\mathbb{R}^3} d_{\mathbb{H}^2}(\Phi(s_1), \Phi(s_2))^6 dy \right)^{1/3},$$

$$\mathcal{E}_{s_2}(\Phi(s_1)) - \mathcal{E}_{s_2}(\Phi(s_2)) \geq C \left(\int_{\mathbb{R}^3} d_{\mathbb{H}^2}(\Phi(s_1), \Phi(s_2))^6 dy \right)^{1/3},$$

where C is a positive constant depending only on ε_0, independent of s_1 and s_2. We point out that these estimates were proved in the original coordinates x, however in light of (9.5) the same arguments may be applied in the current y-coordinate setting. Next observe that

$$(9.23)$$
$$\mathcal{E}_{s_1}(\Phi(s_2)) - \mathcal{E}_{s_1}(\Phi(s_1)) + \mathcal{E}_{s_2}(\Phi(s_1)) - \mathcal{E}_{s_2}(\Phi(s_2))$$
$$= \mathcal{E}_{s_1}(\Phi(s_2)) - \mathcal{E}_{s_2}(\Phi(s_2)) + \mathcal{E}_{s_2}(\Phi(s_1)) - \mathcal{E}_{s_1}(\Phi(s_1)).$$

By a straightforward computation the terms on the right-hand side may be estimated by

$$(9.24)$$
$$(|\nabla_{g(s_1)} U|^2 + e^{4u}|\nabla_{g(s_1)} v|^2)\sqrt{\det g(s_1)} - (|\nabla_{g(s_2)} U|^2 + e^{4u}|\nabla_{g(s_2)} v|^2)\sqrt{\det g(s_2)}$$
$$= (|\nabla_{g(s_1)} U|^2 + e^{4u}|\nabla_{g(s_1)} v|^2)(\sqrt{\det g(s_1)} - \sqrt{\det g(s_2)})$$
$$\quad + [(|\nabla_{g(s_1)} U|^2 + e^{4u}|\nabla_{g(s_1)} v|^2) - (|\nabla_{g(s_2)} U|^2 + e^{4u}|\nabla_{g(s_2)} v|^2)]\sqrt{\det g(s_2)}$$
$$\leq C(|\nabla_{g(s_1)} U|^2 + e^{4u}|\nabla_{g(s_1)} v|^2)|s_1 - s_2| \cdot \chi_{\{\text{supp}(g(s_1)-g(s_2))\}}$$
$$\quad + C|(g^{ij}(s_1) - g^{ij}(s_2))\partial_i U \partial_j U| + e^{4u}|(g^{ij}(s_1) - g^{ij}(s_2))\partial_i v \partial_j v|,$$

where $\chi_{\{\text{supp}(g(s_1)-g(s_2))\}}$ is the indicator function of the support set of $g(s_1) - g(s_2)$. Since $g(s_1) - g(s_2)$ is supported in $B_{2\delta_0} \setminus B_{\delta_0}$, we may apply (9.11) to obtain the desired result. $\square$

We are now able to establish the Hölder continuity of Φ with respect to s.

Proposition 9.6. *Let $\Phi = \Phi(s)$ be given by (9.6), for any $s \in [-\varepsilon_0, \varepsilon_0]$. Then for any $s_1, s_2 \in [-\varepsilon_0, \varepsilon_0]$ we have*

$$(9.25) \qquad d_{\mathbb{H}^2}(\Phi(s_1, y), \Phi(s_2, y)) \leq C(1 + |y|)^{-1}|s_1 - s_2|^{1/2} \quad in \ \mathbb{R}^3 \setminus \Gamma,$$

and for $k = 0, 1, 2, 3$ as well as any $\alpha \in (0, 1)$ it holds that

$$(9.26)$$
$$|\nabla_g^k(u(s_1) - u(s_2))|(y) + \left(\frac{\rho}{\sigma} \right)^{k-3-\alpha} |\nabla_g^k(v(s_1) - v(s_2))|(y)$$
$$\leq C\sigma^{-k}(1 + |y|)^{-1}|s_1 - s_2|^{1/2} \quad in \ \mathbb{R}^3 \setminus \Gamma,$$

where C is a constant independent of s_1 and s_2, and $g = g(s)$ for any $s \in [-\varepsilon_0, \varepsilon_0]$.

Proof. Recall that $p_{i_0}^0$ is the origin and $z_{0,1} < \cdots < z_{0,N}$. It follows from Proposition 9.5 above and (the proof of) Proposition 3.9 of [12] that for any $\mathbf{r} > 2(|z_{0,1}| + |z_{0,N}|)$ and $\alpha \in (0,1)$ there exists a constant C, depending only on $\mathbf{r}$, δ_0, and the boundary data $c_1, \ldots, c_{N+1}$ such that for all $s_1, s_2 \in [-\varepsilon_0, \varepsilon_0]$ and $k = 0,1,2,3$ we have

$$(9.27) \qquad |\nabla_g^k(u(s_1) - u(s_2))| + \left(\frac{\rho}{\sigma}\right)^{k-3-\alpha} |\nabla_g^k(v(s_1) - v(s_2))|$$
$$\leq C|s_1 - s_2|^{1/2} \quad \text{in } B_{\mathbf{r}} \setminus \left(\Gamma \cup_{i=1}^N B_{\delta_0}(p_i^0)\right).$$

With the aid of (9.19), this in particular implies

$$(9.28) \qquad \sup_{B_{\mathbf{r}} \setminus \cup_{i=1}^N B_{\delta_0}(p_i^0)} d_{\mathbb{H}^2}(\Phi(s_1, y), \Phi(s_2, y)) \leq C|s_1 - s_2|^{1/2}.$$

Furthermore, since the metrics $g(s_1) = g(s_2) = \boldsymbol{\delta}$ agree in $B_{\delta_0} \cup B_{2\delta_0}^c$, it follows that

$$(9.29) \qquad \Delta\sqrt{1 + d_{\mathbb{H}^2}(\Phi(s_1, y), \Phi(s_2, y))^2} \geq 0 \quad \text{in } B_{\delta_0} \cup B_{2\delta_0}^c.$$

The first conclusion then follows from a maximum principle argument as in the proof of Lemma 9.3, together with Lemma 9.4. This inequality (9.25), in turn may be used to obtain the second conclusion from the proof of [12, Proposition 3.9]. $\qquad\square$

In the remainder of this section, we will study higher order regularity with respect to s. Let $g = g(s)$ be the metric as in (9.3). In view of (9.7) and $\Phi = (u, v)$, define

$$(9.30) \qquad \mathcal{F}_1(s, \Phi) = \Delta_g u - 2e^{4u}|\nabla_g v|^2, \qquad \mathcal{F}_2(s, \Phi) = \Delta_g v + 4\nabla_g u \cdot \nabla_g v,$$

and write $\mathcal{F}(s, \Phi) = (\mathcal{F}_1(s, \Phi), \mathcal{F}_2(s, \Phi))$. Let $L_s = (L_{s,1}, L_{s,2})$ be the linearized operator of $\mathcal{F}$ at Φ. A straightforward computation yields

$$(9.31) \qquad \begin{aligned} L_{s,1}\varphi &= \Delta_g\varphi_1 - 8e^{4u}|\nabla_g v|^2\varphi_1 - 4e^{4u}\nabla_g v \cdot \nabla_g\varphi_2, \\ L_{s,2}\varphi &= \Delta_g\varphi_2 + 4\nabla_g u \cdot \nabla_g\varphi_2 + 4\nabla_g v \cdot \nabla_g\varphi_1, \end{aligned}$$

for $\varphi = (\varphi_1, \varphi_2) \in C_c^1(\mathbb{R}^3) \times C_c^1(\mathbb{R}^3 \setminus \Gamma)$. Define

$$(9.32) \qquad \begin{aligned} \mathcal{B}_s[\varphi, \psi] &= \int_{\mathbb{R}^3} \left(\nabla_g\varphi_1 \cdot \nabla_g\psi_1 + 8e^{4u}|\nabla_g v|^2\varphi_1\psi_1 + e^{4u}\nabla_g\varphi_2 \cdot \nabla_g\psi_2\right) dvol_g \\ &\quad + 4\int_{\mathbb{R}^3} e^{4u}\left((\nabla_g v \cdot \nabla_g\varphi_2)\psi_1 - (\nabla_g v \cdot \nabla_g\varphi_1)\psi_2\right) dvol_g. \end{aligned}$$

This is the quadratic form associated with L_s.

Proposition 9.7. *Let* $\Phi = \Phi(s)$ *be given by* (9.6), *for any* $s \in [-\varepsilon_0, \varepsilon_0]$. *Then for all* $\varphi = (\varphi_1, \varphi_2)$ *with* $\varphi_1 \in C_c^1(\mathbb{R}^3)$ *and* $\varphi_2 \in C_c^1(\mathbb{R}^3 \setminus \Gamma)$ *we have*

$$(9.33) \qquad \mathcal{B}_s[\varphi, \varphi] \geq C\left(\int_{\mathbb{R}^3}(|\varphi_1| + e^{2u}|\varphi_2|)^6 dy\right)^{1/3},$$

where C *is a positive constant independent of* s.

Proof. By Taylor's theorem, for any $t > 0$ there exists $t_* \in (0, t)$ such that

$$(9.34) \quad \mathcal{E}_s(\Phi + t\varphi) - \mathcal{E}_s(\Phi) = t\frac{d}{dt}\mathcal{E}_s(\Phi + t\varphi)\Big|_{t=0} + \frac{t^2}{2}\frac{d^2}{dt^2}\mathcal{E}_s(\Phi + t\varphi)\Big|_{t=t_*} = \frac{t^2}{2}\frac{d^2}{dt^2}\mathcal{E}_s(\Phi + t\varphi)\Big|_{t=t^*}.$$

Moreover, as in the proof of Proposition 9.5 the estimate of [17, Theorem 4.1] applies to the current setting and yields

$$(9.35) \qquad \mathcal{E}_s(\Phi + t\varphi) - \mathcal{E}_s(\Phi) \geq C \left(\int_{\mathbb{R}^3} d_{\mathbb{H}^2}(\Phi + t\varphi, \Phi)^6 dy \right)^{1/3},$$

where C is a positive constant independent of s. Therefore

$$(9.36) \qquad \mathcal{B}_s[\varphi, \varphi] = \lim_{t \to 0} \frac{1}{2} \frac{d^2}{dt^2} \mathcal{E}_s(\Phi + t\varphi) \Big|_{t=t_*} \geq \liminf_{t \to 0} C \left(\int_{\mathbb{R}^3} \left(\frac{d_{\mathbb{H}^2}(\Phi + t\varphi, \Phi)}{t} \right)^6 dy \right)^{1/3}.$$

Next note that the hyperbolic distance expression (9.19) implies

$$(9.37) \qquad d_{\mathbb{H}^2}(\Phi + t\varphi, \Phi) = \frac{1}{2} \ln(\zeta + \sqrt{\zeta^2 - 1}), \qquad \zeta = \cosh(2t\varphi_1) + 2e^{2(2u + t\varphi_1)} t^2 \varphi_2^2.$$

Since $\zeta = 1 + 2t^2(\varphi_1^2 + e^{4u}\varphi_2^2) + O(t^3)$ we obtain

$$(9.38) \qquad \zeta + \sqrt{\zeta^2 - 1} = 1 + 2t\sqrt{\varphi_1^2 + e^{4u}\varphi_2^2} + O(t^2),$$

and hence

$$(9.39) \qquad \lim_{t \to 0} \frac{d_{\mathbb{H}^2}(\Phi + t\varphi, \Phi)}{t} = \sqrt{\varphi_1^2 + e^{4u}\varphi_2^2}.$$

The stated result then follows easily. $\qquad\qquad\square$

For each $s \in [-\varepsilon_0, \varepsilon_0]$, denote by $\mathcal{H}$ the closure of $C_c^1(\mathbb{R}^3) \times C_c^1(\mathbb{R}^3 \setminus \Gamma)$ under the norm

$$(9.40) \qquad \|\varphi\|_{\mathcal{H}} = \sqrt{\mathcal{B}_s[\varphi, \varphi]}.$$

By Proposition 9.7 we have the following Sobolev type inequality

$$(9.41) \qquad \left(\int_{\mathbb{R}^3} (|\varphi_1| + e^{2u}|\varphi_2|)^6 \, dy \right)^{1/6} \leq C \|\varphi\|_{\mathcal{H}} \quad \text{for any } \varphi \in \mathcal{H},$$

where C is a positive constant independent of s. We point out that the space $\mathcal{H}$ is independent of s and the norms $\|\varphi\|_{\mathcal{H}}$ corresponding to different $s \in [-\varepsilon_0, \varepsilon_0]$ are comparable.

Consider the linear equation with boundary condition

$$(9.42) \qquad L_s w = f \quad \text{in } \mathbb{R}^3 \setminus \Gamma, \qquad w_2 = 0 \quad \text{on } \Gamma \setminus \{p_1^0, \ldots, p_N^0\}.$$

Given $f = (f_1, f_2)$ with $f_1, f_2 \in L_{loc}^1(\mathbb{R}^3)$, we say that $w \in \mathcal{H}$ is a weak solution of (9.42) if

$$(9.43) \qquad \mathcal{B}_s[w, \varphi] = - \int_{\mathbb{R}^3} (f_1 \varphi_1 + e^{4u} f_2 \varphi_2) dvol_g,$$

for all $\varphi = (\varphi_1, \varphi_2) \in C_c^1(\mathbb{R}^3) \times C_c^1(\mathbb{R}^3 \setminus \Gamma)$.

Lemma 9.8. *Suppose that $w \in \mathcal{H}$ is a weak solution of (9.42) with f as above and satisfying*

$$(9.44) \qquad |f_1(y)| + e^{2u(y)}|f_2(y)| \leq C_1 \sigma(y)^{-2}(1 + |y|)^{-1} \quad \text{for } y \in \mathbb{R}^3 \setminus \Gamma.$$

Then

$$(9.45) \qquad \|w\|_{\mathcal{H}} \leq CC_1,$$

where C is a constant independent of $s \in [-\varepsilon_0, \varepsilon_0]$ and w.

Proof. By the density of compact support functions in $\mathcal{H}$, together with Hölder's inequality, and the definition of weak solutions we have

$$
\begin{aligned}
\mathcal{B}_s[w, w] &= -\int_{\mathbb{R}^3} \left(f_1 w_1 + e^{4u} f_2 w_2 \right) dvol_g \\
&\leq CC_1 \int_{\mathbb{R}^3} \sigma(y)^{-2} (1 + |y|)^{-1} (|w_1| + e^{2u}|w_2|) \, dy \\
&\leq CC_1 \left(\int_{\mathbb{R}^3} \sigma^{-\frac{12}{5}} (1 + |y|)^{-\frac{6}{5}} \, dy \right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} (|w_1| + e^{2u}|w_2|)^6 \, dy \right)^{\frac{1}{6}} \\
&\leq CC_1 \left(\int_{\mathbb{R}^3} (|w_1| + e^{2u}|w_2|)^6 \, dy \right)^{\frac{1}{6}}.
\end{aligned}
$$
(9.46)

The lemma then follows from (9.41) immediately. □

It is now possible to improve Proposition 9.6 to obtain Lipschitz regularity in s.

Proposition 9.9. *Let $\Phi = \Phi(s)$ be given by (9.6), for any $s \in [-\varepsilon_0, \varepsilon_0]$. Then for any $s_1, s_2 \in [-\varepsilon_0, \varepsilon_0]$ we have*

$$
d_{\mathbb{H}^2}(\Phi(s_1, y), \Phi(s_2, y)) \leq C(1 + |y|)^{-1} |s_1 - s_2| \quad \text{in } \mathbb{R}^3 \setminus \Gamma,
$$
(9.47)

and for $k = 0, 1, 2, 3$ as well as any $\alpha \in (0, 1)$ it holds that

$$
\begin{aligned}
|\nabla_g^k (u(s_1) - u(s_2))|(y) &+ \left(\frac{\rho}{\sigma} \right)^{k-3-\alpha} |\nabla_g^k (v(s_1) - v(s_2))|(y) \\
&\leq C\sigma^{-k} (1 + |y|)^{-1} |s_1 - s_2| \quad \text{in } \mathbb{R}^3 \setminus \{p_1^0, \ldots, p_N^0\},
\end{aligned}
$$
(9.48)

where C is a constant independent of s_1 and s_2, and $g = g(s)$ for any $s \in [-\varepsilon_0, \varepsilon_0]$.

Proof. Take any s, τ such that $s, s + \tau \in [-\varepsilon_0, \varepsilon_0]$. We will first derive an equation satisfied by $\Phi(s + \tau) - \Phi(s)$. Note that $\mathcal{F}(s + \tau, \Phi(s + \tau)) = \mathcal{F}(s, \Phi(s)) = 0$, and thus

$$
\begin{aligned}
0 &= \mathcal{F}(s + \tau, \Phi(s + \tau)) - \mathcal{F}(s, \Phi(s)) \\
&= \mathcal{F}(s + \tau, \Phi(s + \tau)) - \mathcal{F}(s, \Phi(s + \tau)) + \mathcal{F}(s, \Phi(s + \tau)) - \mathcal{F}(s, \Phi(s)) \\
&\quad - L_s \big(\Phi(s + \tau) - \Phi(s) \big) + L_s \big(\Phi(s + \tau) - \Phi(s) \big).
\end{aligned}
$$
(9.49)

This may be rewritten as

$$
L_s \big(\Phi(s + \tau) - \Phi(s) \big) = P(s, \tau; y) + Q(s, \tau; y),
$$
(9.50)

where

$$
\begin{aligned}
P(s, \tau; y) &= -\big[\mathcal{F}(s + \tau, \Phi(s + \tau)) - \mathcal{F}(s, \Phi(s + \tau)) \big], \\
Q(s, \tau; y) &= -\big[\mathcal{F}(s, \Phi(s + \tau)) - \mathcal{F}(s, \Phi(s)) - L_s \big(\Phi(s + \tau) - \Phi(s) \big) \big].
\end{aligned}
$$
(9.51)

We point out that P involves $g(s + \tau) - g(s)$ and hence is supported in $B_{2\delta_0} \setminus B_{\delta_0}$, while Q is quadratic in $\Phi(s + \tau) - \Phi(s)$. Next set $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$, and write $g = g(s)$ for

brevity. Then a straightforward computations yields

$$
\begin{aligned}
Q_1(s, \tau; y) = {} & 2e^{4u(s)}\big(e^{4(u(s+\tau)-u(s))} - 1 - 4(u(s+\tau) - u(s))\big)|\nabla_g v(s)|^2 \\
& + 2e^{4u(s)}\big(e^{4(u(s+\tau)-u(s))} - 1\big)\big(|\nabla_g v(s+\tau)|^2 - |\nabla_g v(s)|^2\big) \\
& + 2e^{4u(s)}|\nabla_g v(s+\tau) - \nabla_g v(s)|^2,
\end{aligned}
\tag{9.52}
$$

and

$$
Q_2(s, \tau; y) = -4(\nabla_g u(s+\tau) - \nabla_g u(s)) \cdot (\nabla_g v(s+\tau) - \nabla_g v(s)).
\tag{9.53}
$$

For any $s_1, s_2 \in [-\varepsilon_0, \varepsilon_0]$ write $s_1 = s$ and $s_2 = s + \tau$, then according to (9.50) we find that $\Phi(s_2) - \Phi(s_1)$ is a solution of (9.42) at $s = s_1$ with

$$
f = P(s_1, s_2 - s_1; \cdot) + Q(s_1, s_2 - s_1; \cdot).
\tag{9.54}
$$

By the explicit expressions of Q_1 and Q_2, Proposition 9.6, as well as (9.11) and (9.16) we have

$$
|Q_1(s_1, s_2 - s_1; y)| + e^{2u(s_1)}|Q_2(s_1, s_2 - s_1; y)| \le C|s_1 - s_2|\sigma^{-2}(1 + |y|)^{-2}
\tag{9.55}
$$

on $\mathbb{R}^3 \setminus \Gamma$. A similar estimate holds for P, since P_1, P_2 involve $g(s_1) - g(s_2)$ and are therefore supported in $B_{2\delta_0} \setminus B_{\delta_0}$. Denoting $f = (f_1, f_2)$, it follows that

$$
|f_1(y)| + e^{2u(y)}|f_2(y)| \le C|s_1 - s_2|\sigma^{-2}(1 + |y|)^{-2} \quad \text{in } \mathbb{R}^3 \setminus \Gamma.
\tag{9.56}
$$

By Lemma 9.8, noting that $\Phi(s_1) - \Phi(s_2) \in \mathcal{H}$ in light of (9.16) and (9.26), we get

$$
\|\Phi(s_1) - \Phi(s_2)\|_{\mathcal{H}} \le C|s_1 - s_2|.
\tag{9.57}
$$

Inequality (9.41) then produces

$$
\left(\int_{\mathbb{R}^3} (|u(s_1) - u(s_2)| + e^{2u(s_1)}|v(s_1) - v(s_2)|)^6 \, dy\right)^{1/6} \le C|s_1 - s_2|,
\tag{9.58}
$$

and hence the distance formula (9.19) yields

$$
\left(\int_{\mathbb{R}^3} d_{\mathbb{H}^2}(\Phi(s_1), \Phi(s_2))^6 dy\right)^{1/6} \le C|s_1 - s_2|.
\tag{9.59}
$$

This is an improvement of Proposition 9.5 from the power $1/2$ to 1 for $|s_1 - s_2|$. We can now repeat the proof of Proposition 9.6 to obtain the desired result. □

We will next prove differentiability of $\Phi(s)$ with respect to s.

Proposition 9.10. *Let $\Phi = \Phi(s)$ be given by (9.6), for any $s \in [-\varepsilon_0, \varepsilon_0]$. Then $\nabla_g^k \Phi(s)$ is differentiable in s except on the axis for $k = 0, 1, 2, 3$, and for any $\alpha \in (0, 1)$ it holds that*

$$
|\nabla_g^k \partial_s u(s)|(y) + \left(\frac{\rho}{\sigma}\right)^{k-3-\alpha}|\nabla_g^k \partial_s v(s)|(y) \le C\sigma^{-k}(1 + |y|)^{-1} \quad \text{in } \mathbb{R}^3 \setminus \Gamma,
\tag{9.60}
$$

where C is a constant independent of s.

Proof. We will adopt the notation from the proof of Proposition 9.9. Consider the difference quotient

$$
D_\tau \Phi(s) = \frac{1}{\tau}[\Phi(s+\tau) - \Phi(s)],
\tag{9.61}
$$

for $\tau \neq 0$. By Proposition 9.9, for $k = 0, 1, 2, 3$ and any $s, s + \tau \in [-\varepsilon_0, \varepsilon_0]$ we have

$$(9.62) \qquad |\nabla_g^k D_\tau u(s)|(y) + \left(\frac{\rho}{\sigma}\right)^{k-3-\alpha} |\nabla_g^k D_\tau v(s)|(y) \leq C \sigma^{-k}(1 + |y|)^{-1} \quad \text{in } \mathbb{R}^3 \setminus \Gamma.$$

We will prove that the limits of $\nabla^k D_\tau u(s)$ and $\nabla^k D_\tau v(s)$ exist as $\tau \to 0$, therefore showing that $\nabla_g^k u(s)$ and $\nabla_g^k v(s)$ are differentiable in s and satisfy (9.60).

Dividing (9.50) by τ produces

$$(9.63) \qquad L_s D_\tau \Phi(s) = \frac{1}{\tau} P(s, \tau; y) + \frac{1}{\tau} Q(s, \tau; y).$$

For any s, τ, and $\tau' \neq 0$ with $s, s + \tau, s + \tau' \in [-\varepsilon_0, \varepsilon_0]$ define $w(s, \tau, \tau') = D_\tau \Phi(s) - D_{\tau'} \Phi(s)$, and observe that a simple subtraction yields

$$(9.64) \qquad L_s \big[w(s, \tau, \tau') \big] = f(s, \tau, \tau'),$$

where

$$(9.65) \qquad f(s, \tau, \tau') = \frac{1}{\tau} P(s, \tau; \cdot) - \frac{1}{\tau'} P(s, \tau'; \cdot) + \frac{1}{\tau} Q(s, \tau; \cdot) - \frac{1}{\tau'} Q(s, \tau'; \cdot).$$

The two differences on the right-hand side will be estimated below. In particular, by writing $f = (f_1, f_2)$ we claim that

$$(9.66) \qquad |f_1(s, \tau, \tau')|(y) + e^{2u(y)} |f_2(s, \tau, \tau')|(y) \leq C |\tau - \tau'| \sigma^{-2} (1 + |y|)^{-2} \quad \text{in } \mathbb{R}^3 \setminus \Gamma.$$

With this, Lemma 9.8 applies to give

$$(9.67) \qquad \|w(s, \tau, \tau')\|_{\mathcal{H}} \leq C |\tau - \tau'|.$$

Moreover, after setting $w = (w_1, w_2)$ we find that (9.41) implies

$$(9.68) \qquad \left(\int_{\mathbb{R}^3} (|w_1(s, \tau, \tau')| + e^{2u(s)} |w_2(s, \tau, \tau')|)^6 \, dy \right)^{1/6} \leq C |\tau - \tau'|,$$

and hence by the definition of w it follows that

$$(9.69) \qquad \left(\int_{\mathbb{R}^3} d_{\mathbb{H}^2}(D_\tau \Phi(s), D_{\tau'} \Phi(s))^6 \, dy \right)^{1/6} \leq C |\tau - \tau'|.$$

We now proceed as in the proof of Proposition 9.6. Take any $\mathbf{r}$ sufficiently large, δ sufficiently small, and any $\alpha \in (0, 1)$. By (the proof of) Proposition 3.9 of [12], for $k = 0, 1, 2, 3$ we have

$$(9.70) \qquad \begin{aligned} |\nabla_g^k [D_\tau u(s) - D_{\tau'} u(s)]| &+ \left(\frac{\rho}{\sigma}\right)^{k-3-\alpha} |\nabla_g^k [D_\tau v(s) - D_{\tau'} v(s)]| \\ &\leq C |\tau - \tau'| \quad \text{in } B_{\mathbf{r}} \setminus \left(\Gamma \cup_{i=1}^N B_\delta(p_i^0) \right), \end{aligned}$$

where C is a constant depending on $\mathbf{r}$ and δ, independent of s. Hence, for $k = 0, 1, 2, 3$ the limits of $D_\tau \nabla^k u(s)$ and $D_\tau \nabla^k v(s)$ exist as $\tau \to 0$, and thus $\nabla^k u(s)$ and $\nabla^k v(s)$ are differentiable with respect to s in $B_{\mathbf{r}} \setminus \left(\Gamma \cup_{i=1}^N B_\delta(p_i^0) \right)$ for any $\mathbf{r}, \delta > 0$.

We now establish (9.66). Consider first the term

$$(9.71) \qquad Q_2(s, \tau; \cdot) = -4\tau^2 \nabla_g D_\tau u(s) \cdot \nabla_g D_\tau v(s),$$

and observe that

$$
\begin{aligned}
\frac{1}{4\tau}Q_2(s,\tau;\cdot) - \frac{1}{4\tau'}Q_2(s,\tau';\cdot) &= -\tau\nabla_g D_\tau u(s)\cdot\nabla_g D_\tau v(s) + \tau'\nabla_g D_{\tau'}u(s)\cdot\nabla_g D_{\tau'}v(s)\\
&= -\nabla_g\left(u(s+\tau) - u(s+\tau')\right)\cdot\nabla_g D_\tau v(s)\\
&\quad + (\tau-\tau')\nabla_g D_{\tau'}u(s)\cdot\nabla_g D_\tau v(s)\\
&\quad + \nabla_g\left(v(s+\tau') - v(s+\tau)\right)\cdot\nabla_g D_{\tau'}u(s).
\end{aligned}
$$
(9.72)

Then Proposition 9.9 and (9.62) imply

$$
(9.73)\qquad e^{2u(y)}\left|\frac{1}{\tau}Q_2(s,\tau;y) - \frac{1}{\tau'}Q_2(s,\tau';y)\right| \le C|\tau-\tau'|\sigma^{-2}(1+|y|)^{-2}\quad\text{in }\mathbb{R}^3\setminus\Gamma.
$$

Estimates for Q_1, P_1, and P_2 may be achieved similarly, yielding the desired conclusion. $\qquad\square$

For a later purpose, we shall improve (9.70) so that for any large $\mathbf{r}$, $\alpha\in(0,1)$, and $k=0,1,2,3$ the following estimate is valid

$$
\begin{aligned}
(9.74)\qquad &|\nabla_g^k[D_\tau u(s) - D_{\tau'}u(s)]| + \left(\frac{\rho}{\sigma}\right)^{k-3-\alpha}|\nabla_g^k[D_\tau v(s) - D_{\tau'}v(s)]|\\
&\le C\sigma^{-k}|\tau-\tau'|\quad\text{in }B_{\mathbf{r}}\setminus\Gamma,
\end{aligned}
$$

where C is a constant depending only on $\mathbf{r}$, independent of s. To see this, first rewrite (9.70) as

$$
(9.75)\qquad |\nabla_g^k w_1(s,\tau,\tau')| + \left(\frac{\rho}{\sigma}\right)^{k-3-\alpha}|\nabla_g^k w_2(s,\tau,\tau')| \le C|\tau-\tau'|\quad\text{in }B_{\mathbf{r}}\setminus\left(\Gamma\cup_{i=1}^N B_\delta(p_i^0)\right).
$$

By applying Proposition 8.2 to the equation (9.64) near each puncture p_i^0, we find that there exists a $\delta>0$ sufficiently small such that

$$
\begin{aligned}
(9.76)\qquad \sup_{B_{\mathbf{r}}\setminus\Gamma}&\left(|w_1(s,\tau,\tau')| + \left(\frac{\rho}{\sigma}\right)^{-2}|w_2(s,\tau,\tau')|\right)\\
&\le \sup_{B_{\mathbf{r}}\setminus\left(\Gamma\cup_{i=1}^N B_\delta(p_i^0)\right)}\left(|w_1(s,\tau,\tau')| + \left(\frac{\rho}{\sigma}\right)^{-2}|w_2(s,\tau,\tau')|\right)\\
&\quad + \sup_{B_{\mathbf{r}}\setminus\Gamma}\left(|\sigma^2 f_1(s,\tau,\tau')| + \left(\frac{\rho}{\sigma}\right)^{-2}|\sigma^2 f_2(s,\tau,\tau')|\right)\\
&\le C|\tau-\tau'|.
\end{aligned}
$$

This inequality, together with the proof of Proposition 3.9 of [12], then yields (9.74). We next study the asymptotic behavior of $\partial_s\Phi(s)$ as $y\to\{p_1^0,\dots,p_N^0\}$.

Proposition 9.11. *Let* $\Phi=\Phi(s)$ *be given by* (9.6) *and* $\bar\Phi_i(s)=(\bar u_i(s),\bar v_i(s))$ *be the map satisfying* (9.12)*, for any* $s\in[-\varepsilon_0,\varepsilon_0]$ *and any puncture* p_i^0*. Then* $\nabla_{\mathbb{S}^2}^k\bar\Phi_i(s)$ *is differentiable in* s *except at the north and south pole for* $k=0,1,2,3$*, and for any* $\alpha\in(0,1)$ *we have*

$$
(9.77)\qquad |\nabla_{\mathbb{S}^2}^k\partial_s\bar u_i(s)| + (\sin\theta_i)^{k-3-\alpha}|\nabla_{\mathbb{S}^2}^k\partial_s\bar v_i(s)| \le C\quad\text{on }\mathbb{S}^2\setminus\{N,S\},
$$

and for $l+k\le 3$ *it holds that*

$$
\begin{aligned}
(9.78)\qquad \sup_{\mathbb{S}^2\setminus\{N,S\}}&\left(|(r_i\partial_{r_i})^l\nabla_{\mathbb{S}^2}^k[\partial_s u(s,r_i,\theta_i) - \partial_s\bar u_i(s,\theta_i)]|\right.\\
&\left.+ e^{(3+\alpha-k)u(s)}|(r_i\partial_{r_i})^l\nabla_{\mathbb{S}^2}^k[\partial_s v(s,r_i,\theta_i) - \partial_s\bar v_i(s,\theta_i)]|\right) \le Cr_i^{\bar\beta}
\end{aligned}
$$

in $B_{\delta_0}(p_i^0) \setminus \{p_i^0\}$, where C and $\bar{\beta}$ are positive constants independent of s, $r_i = |y - p_i^0|$, and $\sin \theta_i = \frac{\rho(y)}{r_i}$.

Proof. The proof consists of three steps. Throughout, we will fix a puncture p_i^0.

Step 1. We will show that $\nabla_{\mathbb{S}^2}^k \bar{\Phi}_i(s)$ is differentiable in s for $k = 0, 1, 2, 3$, and that (9.77) holds. To this end recall that for $k = 0, 1, 2, 3$, any $s_1, s_2 \in [-\varepsilon_0, \varepsilon_0]$, and any $\alpha \in (0, 1)$, Proposition 9.9 implies

$$
(9.79) \quad \begin{aligned}
|\nabla_g^k [u(s_1) - u(s_2)]|(y) + \left(\frac{\rho}{\sigma}\right)^{k-3-\alpha} |\nabla_g^k [v(s_1) - v(s_2)]|(y) \\
\leq C\sigma^{-k}(1 + |y|)^{-1} |s_1 - s_2| \quad \text{in } \mathbb{R}^3 \setminus \Gamma.
\end{aligned}
$$

By letting $y \to p_i^0$ and using (9.12) we obtain

$$
(9.80) \quad \begin{aligned}
|\nabla_{\mathbb{S}^2}^k [\bar{u}_i(s_1) - \bar{u}_i(s_2)]| + (\sin \theta)^{k-3-\alpha} |\nabla_{\mathbb{S}^2}^k [\bar{v}_i(s_1) - \bar{u}_i(s_2)]| \\
\leq C|s_1 - s_2| \quad \text{on } \mathbb{S}^2 \setminus \{N, S\}.
\end{aligned}
$$

In other words, $\nabla_{\mathbb{S}^2}^k \bar{u}_i(s)$ is Lipschitz in s for $k = 0, 1, 2, 3$. This is the counterpart of Proposition 9.9 for $\bar{\Phi}_i = (\bar{u}_i, \bar{v}_i)$. Note that $\bar{\Phi}_i$ is a harmonic map from $\mathbb{S}^2 \setminus \{N, S\}$ to $\mathbb{H}^2$. By proceeding as in the proof of Proposition 9.10, we obtain the desired result. In fact, the proof in the present case is easier since $\mathbb{S}^2$ is compact.

Step 2. We will show that there exists a map $\bar{\Phi}_i^{(1)}(s) = (\bar{u}_i^{(1)}(s), \bar{v}_i^{(1)}(s))$ with both component functions in $C^{3,\alpha}(\mathbb{S}^2)$ for any $\alpha \in (0, 1)$, while the second satisfies $\bar{v}_i^{(1)}(s) = 0$ at N and S, such that for $l + k \leq 3$ it holds that

$$
(9.81) \quad \begin{aligned}
\sup_{\mathbb{S}^2 \setminus \{N, S\}} \left(|(r_i \partial_{r_i})^l \nabla_{\mathbb{S}^2}^k [\partial_s u(s, r_i, \theta_i) - \bar{u}_i^{(1)}(s, \theta_i)]| \right. \\
\left. + e^{(3+\alpha-k)u(s)} |(r_i \partial_{r_i})^l \nabla_{\mathbb{S}^2}^k [\partial_s v(s, r_i, \theta_i) - \bar{v}_i^{(1)}(s, \theta_i)]| \right) \leq C r_i^{\bar{\beta}}
\end{aligned}
$$

in $B_{\delta_0}(p_i^0) \setminus \{p_i^0\}$, where C and $\bar{\beta}$ are independent of s. To prove (9.81) differentiate (9.7) with respect to s, and observe that in view of (9.31) we have

$$
(9.82) \quad L_s \partial_s \Phi = \mathcal{P}_1,
$$

where $\mathcal{P}_1 = (\mathcal{P}_{1,1}, \mathcal{P}_{1,2})$ with

$$
(9.83) \quad \begin{aligned}
\mathcal{P}_{1,1} &= -\partial_s g^{ab} \partial_{ab} u - \partial_s (\partial_a g^{ab} + g^{ab} \partial_a \ln \sqrt{\det g}) \partial_b u + 2e^{4u} \partial_s g^{ab} \partial_a v \partial_b v, \\
\mathcal{P}_{1,2} &= -\partial_s g^{ab} \partial_{ab} v - \partial_s (\partial_a g^{ab} + g^{ab} \partial_a \ln \sqrt{\det g}) \partial_b v - 4 \partial_s g^{ab} \partial_a u \partial_b v.
\end{aligned}
$$

Note that $\mathcal{P}_1$ is supported in $B_{2\delta_0}(p_i^0) \setminus B_{\delta_0}(p_i^0)$ so that

$$
(9.84) \quad L_s \partial_s \Phi = 0 \quad \text{in } B_{\delta_0}(p_i^0) \setminus \Gamma, \qquad \partial_s v = 0 \quad \text{on } \Gamma \cap B_{\delta_0}(p_i^0).
$$

Utilizing (9.60), Proposition 8.1 as well as the proof of Proposition 3.8 of [12] for higher order derivatives, we obtain (9.81).

Step 3. We will now prove that $\partial_s \bar{\Phi}_i = \bar{\Phi}_i^{(1)}$. Since $D_{\tau'} \Phi(s) \to \partial_s \Phi(s)$ as $\tau' \to 0$ by Proposition 9.10, it follows that letting $\tau' \to 0$ and taking $k = 0$ in (9.74) produces

$$
(9.85) \quad |D_\tau u(s) - \partial_s u(s)| + \left(\frac{\rho}{\sigma}\right)^{-3-\alpha} |D_\tau v(s) - \partial_s v(s)| \leq C|\tau| \quad \text{in } B_{\mathbf{r}} \setminus \Gamma.
$$

Sending $y \to p_i^0$ then yields

(9.86) $\qquad |D_\tau \bar{u}(s) - \bar{u}_i^{(1)}(s)| + (\sin \theta_i)^{-3-\alpha} |D_\tau \bar{v}(s) - \bar{v}_i^{(1)}(s)| \le C|\tau| \quad \text{on } \mathbb{S}^2 \setminus \{N, S\}.$

Finally, by $\tau \to 0$ we obtain $\partial_s \bar{\Phi}_i = \bar{\Phi}_i^{(1)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Lastly, we will establish the smooth dependence of $\Phi(s)$ on s.

Theorem 9.12. *Let $\Phi = \Phi(s)$ be given by (9.6) and $\bar{\Phi}_i(s) = (\bar{u}_i(s), \bar{v}_i(s))$ be the map satisfying (9.12), for any $s \in [-\varepsilon_0, \varepsilon_0]$ and any puncture p_i^0. Then for $k = 0, 1, 2, 3$, the maps $\nabla_g^k \Phi(s)$ and $\nabla_{\mathbb{S}^2}^k \bar{\Phi}_i(s)$ are smooth in s, except on the axis. Moreover for these k, any $j \ge 1$, and any $\alpha \in (0, 1)$ we have*

(9.87) $\qquad |\nabla_g^k \partial_s^j u(s)|(y) + \left(\dfrac{\rho}{\sigma}\right)^{k-3-\alpha} |\nabla_g^k \partial_s^j v(s)|(y) \le C\sigma^{-k}(1+|y|)^{-1} \quad in \ \mathbb{R}^3 \setminus \Gamma,$

(9.88) $\qquad |\nabla_{\mathbb{S}^2}^k \partial_s^j \bar{u}_i(s)| + (\sin \theta_i)^{k-3-\alpha} |\nabla_{\mathbb{S}^2}^k \partial_s^j \bar{v}_i(s)| \le C \quad on \ \mathbb{S}^2 \setminus \{N, S\},$

and for $l + k \le 3$ and $0 < r_i \le \delta_0$ it holds that

(9.89)
$$\sup_{\mathbb{S}^2 \setminus \{N,S\}} \left(|(r_i \partial_{r_i})^l \nabla_{\mathbb{S}^2}^k [\partial_s^j u(s, r_i, \theta_i) - \partial_s^j \bar{u}_i(s, \theta_i)]| \right.$$
$$\left. + e^{(3+\alpha-k)u(s)} |(r_i \partial_{r_i})^l \nabla_{\mathbb{S}^2}^k [\partial_s^j v(s, r_i, \theta_i) - \partial_s^j \bar{v}_i(s, \theta_i)]| \right) \le C r_i^{\bar{\beta}},$$

where C and $\bar{\beta}$ are positive constants independent of s, and $r_i = |y - p_i^0|$ with $\sin \theta_i = \dfrac{\rho(y)}{r_i}$.

Proof. By Propositions 9.10 and 9.11 we have that $\nabla_g^k \Phi(s)$ and $\nabla_{\mathbb{S}^2}^k \bar{\Phi}_i(s)$ are differentiable in s, and that (9.87)-(9.89) hold for $j = 1$. We now prove that $\nabla_g^k \Phi(s)$ and $\nabla_{\mathbb{S}^2}^k \bar{\Phi}_i(s)$ are twice differentiable in s, and that (9.87)-(9.89) hold for $j = 2$. The proof consists of several steps. Recall that $\partial_s \Phi$ satisfies (9.82).

Step 1. We claim that for $k = 0, 1, 2, 3$ any $s_1, s_2 \in [-\varepsilon_0, \varepsilon_0]$, and any $\alpha \in (0, 1)$ it holds that

(9.90)
$$|\nabla_g^k [\partial_s u(s_1) - \partial_s u(s_2)]|(y) + \left(\dfrac{\rho}{\sigma}\right)^{k-3-\alpha} |\nabla_g^k [\partial_s v(s_1) - \partial_s v(s_2)]|(y)$$
$$\le C\sigma^{-k}(1+|y|)^{-1}|s_1 - s_2| \quad \text{in } \mathbb{R}^3 \setminus \Gamma,$$

where C is a positive constant independent of s_1 and s_2. The proof is similar to that of Proposition 9.9, with Φ there replaced by $\partial_s \Phi$.

Step 2. We claim that $\nabla_g^k \partial_s \Phi(s)$ is differentiable in s for $k = 0, 1, 2, 3$, and that (9.87) holds for $j = 2$. The proof is similar to that of Proposition 9.10, again with Φ replaced by $\partial_s \Phi$.

Step 3. We claim that $\nabla_{\mathbb{S}^2}^k \partial_s \bar{\Phi}_i(s)$ is differentiable in s for $k = 0, 1, 2, 3$, and that (9.88) holds for $j = 2$. The proof is similar to Step 1 in the proof of Proposition 9.11.

Step 4. We will show that there exists a map $\bar{\Phi}_i^{(2)}(s) = (\bar{u}_i^{(2)}(s), \bar{v}_i^{(2)}(s))$ with both component functions in $C^{3,\alpha}(\mathbb{S}^2)$ for any $\alpha \in (0, 1)$, while the second satisfies $\bar{v}_i^{(2)}(s) = 0$ at N and S, such that for $l + k \le 3$ and $0 < r_i \le \delta_0$ it holds that

(9.91)
$$\sup_{\mathbb{S}^2 \setminus \{N,S\}} \left(|(r_i \partial_{r_i})^l \nabla_{\mathbb{S}^2}^k [\partial_s^2 u(s, r_i, \theta_i) - \bar{u}_i^{(2)}(s, \theta_i)]| \right.$$
$$\left. + e^{(3+\alpha-k)u(s)} |(r_i \partial_{r_i})^l \nabla_{\mathbb{S}^2}^k [\partial_s^2 v(s, r_i, \theta_i) - \bar{v}_i^{(2)}(s, \theta_i)]| \right) \le C r_i^{\bar{\beta}},$$

where C and $\bar{\beta}$ are independent of s. To this end differentiate (9.82) with respect to s, or differentiate (9.7) with respect to s twice to obtain

$$(9.92) \qquad\qquad L_s \partial_s^2 \Phi = \mathcal{P}_2,$$

where $\mathcal{P}_2 = (\mathcal{P}_{2,1}, \mathcal{P}_{2,2})$ with

$$
\begin{aligned}
(9.93) \qquad \mathcal{P}_{2,1} &= \partial_s \mathcal{P}_{1,1} - \partial_s g^{ab} \partial_{ab} \partial_s u - \partial_s (\partial_a g^{ab} + g^{ab} \partial_a \ln \sqrt{\det g}) \partial_b \partial_s u \\
&\quad + 8 \partial_s (e^{4u} |\nabla_g v|^2) \partial_s u + 4 \partial_s (e^{4u} \nabla_g v \cdot \nabla_g) \partial_s v, \\
\mathcal{P}_{2,2} &= \partial_s \mathcal{P}_{1,2} - \partial_s g^{ab} \partial_{ab} \partial_s v - \partial_s (\partial_a g^{ab} + g^{ab} \partial_a \ln \sqrt{\det g}) \partial_b \partial_s v \\
&\quad - 4 \partial_s (\nabla_g u \cdot \nabla_g) \partial_s v - 4 \partial_s (\nabla_g v \cdot \nabla_g) \partial_s u.
\end{aligned}
$$

We may then proceed similarly to Step 2 in the proof of Proposition 9.11 to obtain the desired conclusion.

Step 5. We claim that $\partial_s^2 \bar{\Phi}_i = \bar{\Phi}_i^{(2)}$. The proof is similar to Step 3 in the proof of Proposition 9.11. This completes the proof for $j = 2$.

The general case may be treated by induction. $\qquad\qquad\square$

Proof of Theorem 3.1. Assuming all components of $\mathbf{z}$ are allowed to vary, the same conclusions as above remain valid. In particular the map $B_\varepsilon^N(\mathbf{z}_0) \to C_{loc}^{3,\varsigma}(\mathbb{R}^3 \setminus \cup_{i=1}^N \{p_i^0\}, \mathbb{R}^2)$ given by $\mathbf{z} \mapsto (U_{\mathbf{z}} \circ \Pi_{\mathbf{z}}, v_{\mathbf{z}} \circ \Pi_{\mathbf{z}})$, and the map $B_\varepsilon^N(\mathbf{z}_0) \to C_{loc}^{3,\varsigma}(\mathbb{S}^2, \mathbb{R}^2)$ given by $\mathbf{z} \mapsto (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i})$ for each $i = 1, \dots, N$, are smooth in s for some small $\varepsilon > 0$. Since $\Pi_{\mathbf{z}}$ is a diffeomorphism, the first conclusion of Theorem 3.1 holds.

We will now consider estimates for the derivatives with respect to $\mathbf{z}$. Observe that for each $j = 1, \dots, N$, the chain rule along with $\Pi_{\mathbf{z}}(y) = x$ produces

$$
\begin{aligned}
(9.94) \qquad \partial_{z_j}(\Phi_{\mathbf{z}} \circ \Pi_{\mathbf{z}}(y)) &= \partial_{z_j} \Phi_{\mathbf{z}}(x) + \nabla \Phi_{\mathbf{z}}(x) \cdot \partial_{z_j} \Pi_{\mathbf{z}}(y) \\
&= \partial_{z_j} \Phi_{\mathbf{z}}(x) + \partial_{x_3} \Phi_{\mathbf{z}}(x) \bar{\chi}(y_1, y_2, y_3 - z_{0,j}).
\end{aligned}
$$

Let $\zeta_j(x) := \bar{\chi}(\Pi_{\mathbf{z}}^{-1}(x) - z_{0,j} e_3)$ and note that this function is supported in $B_{2\delta_0}(p_j^0)$, and set $\tilde{\sigma}(x) = \sigma(\Pi_{\mathbf{z}}^{-1}(x))$. It follows from Theorem 9.12 that for $k = 0, 1, 2, 3$ and any $\alpha \in (0,1)$ we have

$$(9.95) \qquad |\nabla^k(\partial_{z_j} u_{\mathbf{z}} + \partial_{x_3} u_{\mathbf{z}} \zeta_j)|(x) + \left(\frac{\rho}{\tilde{\sigma}}\right)^{k-3-\alpha} |\nabla^k(\partial_{z_j} v_{\mathbf{z}} + \partial_{x_3} v_{\mathbf{z}} \zeta_j)|(x) \le C \tilde{\sigma}^{-k} (1 + |x|)^{-1}$$

in $\mathbb{R}^3 \setminus \Gamma$, as well as

$$(9.96) \qquad |\nabla_{\mathbb{S}^2}^k \partial_{z_j} \bar{u}_{\mathbf{z},i}| + (\sin \theta_i)^{k-3-\alpha} |\nabla_{\mathbb{S}^2}^k \partial_{z_j} \bar{v}_{\mathbf{z},i}| \le C \quad \text{on } \mathbb{S}^2 \setminus \{N, S\},$$

and for $l + k \le 3$ and $0 < r_i \le \delta_0$ it holds that

$$
\begin{aligned}
(9.97) \qquad \sup_{\mathbb{S}^2 \setminus \{N,S\}} \big(&|(r_i \partial_{r_i})^l \nabla_{\mathbb{S}^2}^k [(\partial_{z_j} u_{\mathbf{z}} + \partial_{x_3} u_{\mathbf{z}} \zeta_j) - \partial_{z_j} \bar{u}_{\mathbf{z},i}]| \\
&+ e^{(3+\alpha-k)u_{\mathbf{z}}} |(r_i \partial_{r_i})^l \nabla_{\mathbb{S}^2}^k [(\partial_{z_j} v_{\mathbf{z}} + \partial_{x_3} v_{\mathbf{z}} \zeta_j) - \partial_{z_j} \bar{v}_{\mathbf{z},i}]| \big) \le C r_i^{\bar{\beta}},
\end{aligned}
$$

for some constants C and $\bar{\beta} > 0$. $\qquad\qquad\square$

Proof of Corollary 3.2. By a direct computation

$$(9.98) \qquad (\dot{U}_t, \dot{v}_t) = \sum_{j=1}^{N} \partial_{z_j} (U_{\mathbf{z}}, v_{\mathbf{z}}) \dot{z}_j.$$

In each ball $B_{\varepsilon/2}(p_i)$ we have the following expansion by (3.6) of Theorem 3.1, namely

$$(9.99) \qquad \partial_{z_j} (U_{\mathbf{z}}, v_{\mathbf{z}}) = -\boldsymbol{\delta}_{ij} (\partial_{x_3} U_{\mathbf{z}}, \partial_{x_3} v_{\mathbf{z}}) + \partial_{z_j} (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i}) + O(r_i^{\bar{\beta}}).$$

Next observe that

$$(9.100) \qquad \partial_{z_j} (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i}) \dot{z}_j = \partial_t (\bar{U}_{\mathbf{z}(t),i}, \bar{v}_{\mathbf{z}(t),i}) = \partial_{b_i} (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i}) \dot{b}_i,$$

and by [12, Theorems 2.1 and 2.2] (see (3.1)-(3.3)) it holds that

$$(9.101) \qquad \begin{aligned} (\partial_{x_3} U_{\mathbf{z}}, \partial_{x_3} v_{\mathbf{z}}) &= \left(\tfrac{x_3 - z_i}{r_i^2}, 0 \right) + \partial_{\theta_i} (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i}) \partial_{x_3} \theta_i + O(r_i^{\beta_i - 1}) \\ &= \left(\tfrac{x_3 - z_i}{r_i^2}, 0 \right) - \tfrac{\rho}{r_i^2} \partial_{\theta_i} (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i}) + O(r_i^{\beta_i - 1}). \end{aligned}$$

Therefore

$$(9.102) \qquad \left(\dot{U}_t, \dot{v}_t \right) = \left[- \left(\tfrac{x_3 - z_i}{r_i^2}, 0 \right) + \tfrac{\rho}{r_i^2} \partial_{\theta_i} (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i}) \right] \dot{z}_i + \partial_{b_i} (\bar{U}_{\mathbf{z},i}, \bar{v}_{\mathbf{z},i}) \dot{b}_i + O(r_i^{\beta_i - 1}).$$

Regarding $\dot{v}_t$, the remainder $O(r_i^{\beta_i - 1})$ can be refined to $O(r_i^{\beta_i - 1}(\sin \theta_i)^3)$ as we have the weighted estimates. Hence (3.7) is proved. The remaining two estimates of this corollary may be established similarly, and we omit the details. $\qquad \square$

APPENDIX A. MISCELLANEOUS COMPUTATIONS

Lemma A.1. *Let r_1, r_2 be the respective Euclidean distances to the (Cartesian) points $(0, 0, \mp 2\eta) \in \mathbb{R}^3$ where $\eta > 0$, and set $D = \{(\rho, z, \phi) \in \mathbb{R}^3 \mid z > 0, \ r_2 > \eta\}$. Then within D we have*

$$(A.1) \qquad \frac{r_1}{r_2} \leq 5.$$

Proof. On D, consider the function

$$(A.2) \qquad f(\rho, z) = \frac{r_1^2}{r_2^2} = \frac{\rho^2 + (z - 2\eta)^2 + 8\eta z}{r_2^2} = 1 + \frac{8\eta z}{r_2^2}.$$

Clearly f is a decreasing function of ρ, and hence $f(\rho, z) \leq f(\bar{\rho}, z)$ for $(\bar{\rho}, z) \in \tilde{\partial} D$ where $\tilde{\partial} D$ is the left-hand portion of the projected boundary of D within the ρz-half plane. This left-hand portion of the projected boundary consists of three parts within the half plane: I $= \{\rho = 0, \ 0 \leq z \leq \eta\}$, II $= \{r_2 = \eta, \ \eta \leq z \leq 3\eta\}$, and III $= \{\rho = 0, \ 3\eta \leq z < \infty\}$. As a function of z, we find that f is increasing on I $\cup$ II and decreasing on III. Therefore, its maximum value on the closure of D is achieved at the z-axis when $z = 3\eta$, and (A.1) follows. $\qquad \square$

Corollary A.2. *Let r_1, r_2 be the respective Euclidean distances to two distinct points $p_1, p_2 \in \mathbb{R}^3$, and set $4\eta = |p_1 - p_2|$ to be the distance between them. Then*

$$(A.3) \qquad \frac{1}{5} \leq \frac{r_1}{r_2} \leq 5$$

outside of the two balls of radius η centered at p_1 and p_2.

Proof. Without loss of generality it may be assumed that the points p_1 and p_2 are located on the z-axis, and that the origin is located at the midpoint between them. Lemma A.1 then states that $r_1/r_2 \leq 5$ in the upper half-space $z > 0$ outside $B_\eta(p_2)$, and hence $r_2/r_1 \geq 1/5$ there. Also clearly $r_2 < r_1$ in the upper half-space, so that $r_1/r_2 \geq 1$ there. By symmetry we obtain $r_1/r_2 \geq 1/5$ in the lower half-space outside $B_\eta(p_1)$, and also $r_1/r_2 \leq 1$ there. Combining these inequalities yields the desired result.   □

Lemma A.3. *Let $\delta, \eta > 0$ and $p_0, p_1, p_2 \in \mathbb{R}^3$ be such that $B_\eta(p_i) \subset B_{\delta/4}(p_0)$, $i = 1, 2$ with $|p_1 - p_2| = 4\eta$ and $p_0 = (p_1 + p_2)/2$. Set r_i to be the Euclidean distance to p_i, $i = 0, 1, 2$. Then for any $\lambda \in [0, 1]$ we have*

$$(A.4) \qquad |\lambda \ln r_1 + (1 - \lambda) \ln r_2 - \ln r_0| \leq \ln 2, \qquad |\nabla \ln r_i| \leq \frac{2}{r_0},$$

outside $B_{\delta/2}(p_0)$.

Proof. Let $i = 1, 2$. By the triangle inequality

$$(A.5) \qquad r_0 \leq r_i + 2\eta < r_i + \frac{\delta}{4},$$

and thus dividing by r_i produces

$$(A.6) \qquad \frac{r_0}{r_i} \leq 1 + \frac{\delta/4}{r_i} \leq 2$$

outside $B_{\delta/2}(p_0)$, since in this region $r_i \geq \delta/4$. Similarly

$$(A.7) \qquad r_i \leq r_0 + 2\eta < r_0 + \frac{\delta}{4},$$

and hence

$$(A.8) \qquad \frac{r_i}{r_0} \leq 1 + \frac{\delta/4}{r_0} \leq \frac{3}{2}$$

outside $B_{\delta/2}(p_0)$. Therefore in this region

$$(A.9) \qquad |\ln r_i - \ln r_0| \leq \ln 2,$$

and (A.4) follows by the triangle inequality after rewriting $\ln r_0$ as a convex combination

$$(A.10) \qquad |\lambda \ln r_1 + (1 - \lambda) \ln r_2 - \ln r_0| \leq \lambda |\ln r_1 - \ln r_0| + (1 - \lambda)|\ln r_2 - \ln r_0| \leq \ln 2.$$

Lastly, for the gradient bound observe that (A.6) implies $1/r_i \leq 2/r_0$ outside $B_{\delta/2}(p_0)$. The desired estimate now follows.   □

Lemma A.4. *Let $\Phi_i = (u_i, v_i)$, $i = 1, 2$ be multi-extreme Kerr harmonic maps with the same potential constants on $\Gamma_a^+ = [a, \infty) \subset z$-axis, for $a > 0$ sufficiently large so that there are no punctures on this axis interval. Then $d_{\mathbb{H}^2}(\Phi_1, \Phi_2) \to 0$ as $r \to \infty$ with $0 \leq \theta \leq \pi/2$.*

Proof. Recall the hyperbolic distance formula [27, Lemma 3] given by

$$(A.11) \qquad \cosh\left(2d_{\mathbb{H}^2}(\Phi_1, \Phi_2)\right) = \cosh\left(2(u_1 - u_2)\right) + 2e^{2(u_1 + u_2)}(v_1 - v_2)^2.$$

Observe that the expansions of [12, Theorem 2.3], on the compliment of ball $B_a(0)$ with $0 \leq \theta \leq \pi/2$, imply that

$$(A.12) \qquad u_1 - u_2 = O(r^{-1}), \qquad v_1 - v_2 = O(\sin^3 \theta),$$

as $r \to \infty$. Here we have utilized the fact that the constant term in the expansion of u_i vanishes, see Lemma 9.4. Therefore

$$(A.13) \qquad \cosh\left(2d_{\mathbb{H}^2}(\Phi_1, \Phi_2)\right) = 1 + O(r^{-1})$$

in this region, and the desired result follows. $\qquad\square$

Remark A.5. We note that an analogous result holds for $\Gamma_a^- = (-\infty, -a]$ and $\pi/2 \leq \theta \leq \pi$, using the same asymptotics (A.12). Similarly, the result is valid for $0 < \theta_1 \leq \theta \leq \theta_2 < \pi$ without hypotheses concerning potential constants, where we only need to use the first inequality in (A.12) and boundedness of v_1, v_2.

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