WEAK APPROXIMATION AND R-EQUIVALENCE OVER FUNCTION FIELDS OF CURVES

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Abstract. For a rationally connected fibration over a complex curve, and for a closed point of the curve, we prove that every power series section of the fibration near the point is approximated to arbitrary order by polynomial sections provided the "Laurent fiber", i.e., the deleted power series neighborhood of the fiber, as a variety over the Laurent series field is *R-connected* – the analogue of rational connectedness when working over a non-algebraically closed field such as Laurent series. In other words, we prove the Hassett-Tschinkel conjecture when the Laurent fiber is R-connected. For varieties over the fraction field of a complete DVR, we introduce a "continuous variant" of R-connectedness called *pseudo R-connectedness* and we prove pseudo R-connectedness of the Laurent fiber also implies the Hassett-Tschinkel conjecture. Our theorem implies all of the known cases of the Hassett-Tschinkel conjecture, and we also prove some new cases. The key is a new object, a "pseudo ideal sheaf", which generalizes Fulton's notion of effective pseudo divisor.

1. Introduction

sec-int

Disclaimer. This is the current, working draft of this article. This draft is incomplete, but has a complete list of the results. An earlier, complete draft has been posted to the arXiv preprint server.

Let $\pi : \mathcal{X} \to B$ be a surjective morphism from a smooth, projective, complex variety to a smooth, projective, complex curve. Considered differently, π is a 1-parameter family $\{\mathcal{X}_b\}_{b\in B}$ of projective varieties. The morphism π satisfies *weak approximation* if for every finite sequence $(b_1, \dots, b_m)$ of distinct closed points of B, for every sequence $(\widehat{s}_1, \dots, \widehat{s}_m)$ of elements $\widehat{s}_i$ in $\mathcal{X}(\widehat{\mathcal{O}}_{B,b_i})$, i.e., formal power series section of π near b_i, and for every positive integer N, there exists a regular (i.e., polynomial) section s of π which is congruent to $\widehat{s}_i$ modulo $\mathfrak{m}_{B,b_i}^N$ for every $i = 1, \dots, m$.

Hassett observed that if π satisfies weak approximation then every sufficiently general fiber of π is *rationally connected*, i.e., every pair of points in the fiber is in the image of a morphism from $\mathbb{P}^1$ to the fiber. Hassett and Tschinkel conjectured the converse.

Conjecture 1.1 (Hassett-Tschinkel conjecture)**.** [HT06, Conjecture 2] If a general fiber of π is rationally connected, then π satisfies weak approximation. conj-HT

We prove this conjecture assuming an extra condition on the "Laurent fibers" of π. For every closed point b of B, the fraction field $K = \mathrm{Frac}(\widehat{\mathcal{O}}_{B,b})$ of the completed local ring is isomorphic to the Laurent series field $\mathbb{C}((t))$. The *Laurent fiber* is the

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projective scheme $\mathcal{X}_K := \mathcal{X} \otimes_{\mathcal{O}_B} K$. This is an algebraic version of the "deleted tubular neigborhood" of the closed fiber $\mathcal{X}_b := \pi^{-1}(b)$. Even when $\mathcal{X}_b$ is singular, the K-scheme $\mathcal{X}_K$ is smooth. Two K-points s and t of the Laurent fiber are *directly R-equivalent* if there exists a K-morphism from $\mathbb{P}^1_K$ sending 0 to s and sending ∞ to t,

$$f : (\mathbb{P}_K, 0, \infty) \to (\mathcal{X} \otimes_{\mathcal{O}_B} K, s, t).$$

The Laurent fiber is *R-connected* if all K-points are directly R-equivalent. When $\mathcal{X}_b = \pi^{-1}(b)$ is smooth, the Laurent fiber $\mathcal{X}_K$ is R-connected if and only if the closed fiber $\mathcal{X}_b$ is rationally connected. (The usual definition of R-connected is a bit different. But Kollár proved the equivalence of the two definitions when the geometric generic fiber is separably rationally connected, as it is here.)

We prove Conjecture 1.1 when the Laurent fibers of π are R-connected. In fact the proof requires a weaker, "continuous" variant of R-connnectedness which is technical but useful. As above, the underlying hypothesis is that the geometric generic fiber of π is rationally connected. Denote by $\widehat{\mathcal{O}}$ the complete local ring $\widehat{\mathcal{O}}_{B,b}$, and let u be a uniformizer of $\widehat{\mathcal{O}}$. This is a Henselian (even a complete) DVR. Denote by $\mathcal{X}_{\widehat{\mathcal{O}}}$ the base change Spec $(\widehat{\mathcal{O}}) \times_B \mathcal{X}$. The "bidisk", $\mathbf{D}_{\widehat{\mathcal{O}}}$, is the $\widehat{\mathcal{O}}$-scheme

$$\mathbf{D}_{\widehat{\mathcal{O}}} := \text{Spec } (\widehat{\mathcal{O}}\,[\![v]\!]).$$

The "punctured bidisk" is the complement of the closed point,

$$\mathbf{D}^*_{\widehat{\mathcal{O}}} := \mathbf{D}_{\widehat{\mathcal{O}}} \setminus \{\langle u, v\rangle\}.$$

Denote by Δ_v the divisor in $\mathbf{D}_{\widehat{\mathcal{O}}}$

$$\Delta_v := \text{Spec } (\widehat{\mathcal{O}}\,[\![v]\!]\,/\langle v\rangle) \cong \text{Spec } (\widehat{\mathcal{O}}).$$

And for every positive integer N, denote by Δ_{u^N} the divisor in $\mathbf{D}_{\widehat{\mathcal{O}}}$

$$\Delta_{u^N} := \text{Spec } (\widehat{\mathcal{O}}\,[\![v]\!]\,/\langle u^N\rangle) \cong \text{Spec } ((\widehat{\mathcal{O}}/\langle u^N\rangle)\,[\![v]\!]).$$

Finally denote the intersections with $\mathbf{D}^*_{\widehat{\mathcal{O}}}$ by $\Delta^*_v = \Delta_v \cap \mathbf{D}^*_{\widehat{\mathcal{O}}}$, $\Delta^*_{u^N} = \Delta_{u^N} \cap \mathbf{D}^*_{\widehat{\mathcal{O}}}$.

Let $\widehat{s}$ be an $\widehat{\mathcal{O}}$-point of $\mathcal{X}_{\widehat{\mathcal{O}}}$. The restriction of $\widehat{s}$ to Spec (Frac($\widehat{\mathcal{O}}$)) determines an $\widehat{\mathcal{O}}$-morphism from $\Delta^*_v \to \mathcal{X}_{\widehat{\mathcal{O}}}$ via the identification $\Delta^*_v = \text{Spec (Frac}(\widehat{\mathcal{O}}))$. Similarly, the restriction of $\widehat{s}$ to Spec $(\widehat{\mathcal{O}}/\langle u^N\rangle)$ composed with the "projection"

$$\text{Spec } ((\widehat{\mathcal{O}}/\langle u^N\rangle)\,(\!(v)\!)) \to \text{Spec } (\widehat{\mathcal{O}}/\langle u^N\rangle)$$

determines an $\widehat{\mathcal{O}}$-morphism $\Delta^*_{u^N} \to \mathcal{X}_{\widehat{\mathcal{O}}}$. The morphisms $\Delta^*_v \to \mathcal{X}_{\widehat{\mathcal{O}}}$ and $\Delta^*_{u^N} \to \mathcal{X}_{\widehat{\mathcal{O}}}$ defined in this way are the *associated morphisms* of $\widehat{s}$.

defn-psiRprelim

Definition 1.2. Two $\widehat{\mathcal{O}}$-points $\widehat{s}$ and $\widehat{t}$ of $\mathcal{X}_{\widehat{\mathcal{O}}}$ are *directly pseudo R-equivalent* if for every positive integer N there exist $\widehat{\mathcal{O}}$-morphisms

$$q_N, r_N : \mathbf{D}^*_{\widehat{\mathcal{O}}} \to \mathcal{X}_{\widehat{\mathcal{O}}}$$

such that the restriction of q_N, resp. r_N, to Δ^*_v is the morphism associated to $\widehat{s}$, resp. the morphism associated to $\widehat{t}$, and such that the restriction of q_N, resp. r_N, to $\Delta^*_{u^N}$ is the morphism associated to $\widehat{t}$, resp. the morphism associated to $\widehat{s}$. And $\mathcal{X}_{\widehat{\mathcal{O}}}$ is *pseudo R-connected* if every pair of $\widehat{\mathcal{O}}$-points is directly pseudo R-equivalent.

In fact pseudo R-connectedness depends only on the Laurent fiber $\mathcal{X}_K = \text{Spec } K \times_{\text{Spec } \widehat{\mathcal{O}}} \mathcal{X}_{\widehat{\mathcal{O}}}$. Also R-connectedness implies pseudo R-connectedness. Here is our main theorem.

thm-first

Theorem 1.3. *The morphism π satisfies weak approximation if the Laurent fiber of π over every closed point of B is pseudo R-connected, e.g., if it is R-connected.*

Pseudo ideal sheaves. The key is a new sheaf-theoretic object which we call a "pseudo ideal sheaf". Let S be a scheme or an algebraic space and let Y be a flat, projective S-scheme or a flat, proper, algebraic space over S.

defn-pisprelim

Definition 1.4. For every S-scheme T, denoting $Y_T = T \times_S Y$, a *pseudo ideal sheaf on Y/S* parameterized by T is a pair $(\mathcal{F}, \phi)$ of a T-flat, locally finitely presented, quasi-coherent $\mathcal{O}_{Y_T}$-module $\mathcal{F}$ together with an $\mathcal{O}_{Y_T}$-module homomorphism, $\phi : \mathcal{F} \to \mathcal{O}_{Y_T}$, such that the induced $\mathcal{O}_{Y_T}$-module homomorphism,

$$\phi' : \bigwedge^2 \mathcal{F} \to \mathcal{F}, \quad f_1 \wedge f_2 \mapsto \phi(f_1)f_2 - \phi(f_2)f_1,$$

is the zero homomorphism. A *morphism* from a pseudo ideal sheaf $(\mathcal{F}_1, \phi_1)$ to a pseudo ideal sheaf $(\mathcal{F}_2, \phi_2)$ is an isomorphism of $\mathcal{O}_{Y_T}$-modules, $\psi : \mathcal{F}_1 \to \mathcal{F}_2$, such that $\phi_2 \circ \psi$ equals ϕ_1. The identity morphisms and composition of morphisms are defined in the obvious way. For an S-morphism $g : T' \to T$ and a pseudo ideal sheaf $(\mathcal{F}, \phi)$ parameterized by T, the *pullback by g* is $(g^*\mathcal{F}, g^*\phi)$.

As proved later, this category is an Artin stack $\text{Pseudo}_{Y/S}$. And the Hilbert scheme $\text{Hilb}_{Y/S}$ (or "Hilbert algebraic space", more generally) is an open substack of $\text{Pseudo}_{Y/S}$. Most importantly, when Y is a Cartier divisor in a flat, projective S-scheme X (or a flat, proper algebraic space over S), there is a *restriction morphism*

$$\iota_Y : \text{Hilb}_X \to \text{Pseudo}_Y$$

associating to a flat family of closed subschemes of X with ideal sheaf $\mathcal{I}$, the restriction $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ with its natural map to $\mathcal{O}_Y$. For a geometric point s of S and for an ideal sheaf $\mathcal{I}$ on the fiber X_s which is generated by a regular sequence, the morphism ι_Y is smooth at $[I]$ if $H^1(X, \mathcal{O}_X(-Y) \cdot \mathit{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X/I))$ equals 0. And in this case, the Zariski tangent space of the fiber of ι_Y at $[I]$ is canonically isomorphic to $H^0(X, \mathcal{O}_X(-Y) \cdot \mathit{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X/I))$. Therefore, given a closed subscheme of X and given an infinitesimal deformation of the intersection of the scheme with a "tubular neighborhood" of a Cartier divisor in X, this result allows us to find an infinitesimal deformation of the closed subscheme of X whose intersection with the tubular neighborhood is as prescribed. We will usually apply this when S equals Spec $(\mathbb{C})$, when X equals $\mathcal{X}$, and when Y is an infinitesimal neighborhood of a fiber $\mathcal{X}_b$ of π.

R-connectedness and strong rational connectedness. As mentioned, $\mathcal{X}_K$ is R-connected if $\mathcal{X}_b$ is smooth and rationally connected. More generally, $\mathcal{X}_K$ is R-connected if the smooth locus $(\mathcal{X}_b)_{\text{smooth}}$ is *strongly rationally connected*, i.e., if every pair of closed points of $(\mathcal{X}_b)_{\text{smooth}}$ is connected by a (complete) rational curve in $(\mathcal{X}_b)_{\text{smooth}}$. Thus Theorem 1.3 gives a new proof of a theorem of Hassett and Tschinkel, [HT06]: π satisfies weak approximation if the smooth locus of every fiber $\mathcal{X}_b$ is strongly rationally connected.

Additionally, we prove a structural result about strong rational connectedness using pseudo ideal sheaves. Let X be a normal, projective, complex variety and let D be a closed subset of X containing the singular locus. Assume there exists a very free curve in X which is disjoint from D. Denote by U the dense open subset of X which is the union of all very free curves which are disjoint from D, i.e., the largest open subset of $X \setminus D$ which is strongly rationally connected. Then every connected component of $X \setminus U$ intersects D. In particular, $(X \setminus D) \setminus U$ has no isolated points. This generalizes one of the steps in Chenyang Xu's proof that log Del Pezzo surfaces are strongly rationally connected, [Xu08]. Since Xu's theorem is an important advance in this area, we present our own interpretation of his proof in an appendix to this article.

Special cases of R-connectedness. There are many other cases where the Laurent fibers X_K are known to be R-connected, cf. [CT08, Section 10]:

(i) X_K is a compactification of a connected linear group [CTS77, Corollaire 6],
(ii) X_K is a conic bundle over $\mathbb{P}^1$ with discriminant divisor of degree 4 [CTS87] (such surfaces include Del Pezzo surfaces of degree 4 and Chatelet surfaces),
(iii) X_K is a smooth complete intersection of type $(2,2)$ in $\mathbb{P}^n$ with $n \geq 5$ [CTSSD87, Theorem 3.27(ii)], and
(iv) X_K is a smooth cubic hypersurface in $\mathbb{P}^n$ with $n \geq 5$ [Mad08, Footnote, p. 927].

So if the generic fiber of π satisfies one of (i) – (iv), then Theorem 1.3 implies π satisfies weak approximation. We generalize (i) below. In fact (ii) is one example of a *fibration*, which we consider below. We will also say more about Del Pezzo surfaces. Case (iv) improves a result of Hassett and Tschinkel, [HT], who proved weak approximation for cubic hypersurfaces in $\mathbb{P}^6$.

R-connectedness and rational simple connectedness. Hassett proved π satisfies weak approximation if the geometric generic fiber satisfies a suitably strong version of "rational simple connectedness". Using [GHS03] the Laurent fibers are R-connected if they satisfy a weak version of rational simple connectedness discussed below. This refines Hassett's theorem. In particular, de Jong and the second author proved that every smooth complete intersection of type $(d_1, \dots, d_c)$ in $\mathbb{P}^n$ satisfies the weak version of rational simple connectedness provided that

$$n + 1 \geq d_1^2 + \cdots + d_c^2.$$

We will review the proof of this quickly below. Thus π satisfies weak approximation if the geometric generic fiber of π is a complete intersection as above.

Pseudo R-connectedness and fibrations. A *fibration* over $\mathbb{C}(B)$ is a surjective morphism $\rho : \mathcal{X} \to \mathcal{Y}$ of smooth, projective $\mathbb{C}(B)$-schemes whose geometric generic fiber is connected. If $\mathcal{Y}$ satisfies weak approximation, and if the fiber of ρ over each sufficiently general $\mathbb{C}(B)$-point of $\mathcal{Y}$ satisfies weak approximation, then also $\mathcal{X}$ satisfies weak approximation. The analogous statement for R-connectedness is probably false: the analogue would contradict the Fano conjecture that some conic bundles over $\mathbb{P}^2_{\mathbb{C}}$ are non-unirational. However, pseudo R-connectedness does satisfy the analogous property for a fibration of K-schemes.

Pseudo R-connectedness of homogeneous spaces. Because of the fibration property, conic bundles as in (ii) are pseudo R-connected. This also applies to a compactification of a homogeneous space G/H under a connected, linear algebraic group G. When H is connected, i.e., when the connected component of the identity H_0 equals all of H, then H is (geometrically) rationally connected. Hence R-connectedness of G/H follows from R-connectedness of G. For general H, we do not know if G/H is R-connected. However, it is pseudo R-connected. The Deligne-Mumford stack $B(\pi_0 H)$ is R-connected in a suitable sense. Thus considering G/H as fibered over $B(\pi_0 H)$ with fibers of the form G/H_0 (more correctly, $G/\widetilde{H}_0$ where $\widetilde{H}_0$ is a twist of H_0), it follows that every compactification of G/H is pseudo R-connected. By Theorem 1.3, if the generic fiber of π is a compactification of G/H then π satisfies weak approximation. This gives a new proof of a theorem of Colliot-Thélène and Gille, [CTG04].

The Hassett-Tschinkel Conjecture. In fact the main importance of Theorem 1.3 is a new perspective on the Hassett-Tschinkel conjecture. We believe this conjecture fails as stated. As we understand it from conversations with experts, there is an *a priori* obstruction to weak approximation defined using notions of $\mathbb{A}^1$-homotopy theory. The two authors do not know how to compute whether or not this obstruction is zero. But we are quite hopeful that this will be possible in the near future, based on the work of Asok, Morel and Bhatt relating rational connectedness to notions of $\mathbb{A}^1$-homotopy theory. Our understanding is that the obstruction is analogous to the Brauer-Manin obstruction to weak approximation for varieties defined over number fields. Just as the "Brauer pairing" is actually an obstruction to R-connectedness of p-adic fibers – and even to pseudo R-connectedness of p-adic fibers – so too any obstruction defined using a cohomology theory with a "homotopy axiom" should factor through R-equivalence. And if the cohomology theory has a "continuous nature", then the obstruction should even factor through pseudo R-equivalence. Thus R-connectedness and pseudo R-connectedness of Laurent fibers is really a stand-in for vanishing of these cohomological obstructions, at the moment beyond our reach. Each positive result listed above supporting the Hassett-Tschinkel conjecture has a hypothesis implying R-connectedness or pseudo R-connectedness of Laurent fibers. Therefore the conjecture which these results more properly support is that the only obstruction to weak approximation for a rationally connected fibration is a cohomological obstruction. This is an analogue for $\mathbb{C}(B)$ of a well-known conjecture of Colliot-Thélène for number fields, [CT03].

New questions and other fields. This perspective leads to an array of enticing questions. Is weak approximation preserved by finite base change? Is it an open condition in families? Is it a closed condition? Is it preserved by "twists"? Is it preserved under quotients by finite groups? Moreover, what can one say about the set of formal sections of π which are *approximable*, i.e., approximated to arbitrary order by rational sections? We will discuss these questions further at the end of the article.

Of course all of these results remain valid when $\mathbb{C}$ is replaced by any algebraically closed field of characteristic 0. We will discuss which results remain valid in positive characteristic, and the issues in extending the other results.

Cubic surfaces with integral special fibers. Finally we prove a new weak approximation result for Del Pezzo surfaces which is elementary and which has nothing to do with Theorem 1.3. If every fiber $\mathcal{X}_b$ of π is an integral scheme whose dualizing sheaf $\omega_{\mathcal{X}_b}$ is antiample with degree $c_1(\omega_{\mathcal{X}_b})^2 \geq 3$, then weak approximation holds. In particular this implies all previously known weak approximation results for cubic surfaces. For Del Pezzo surfaces of degrees 2 and 1, the results of Amanda Knecht and the results of Chenyang Xu give the best current results. In an appendix we present our interpretation of the proof of Xu's theorem.

Overview of this paper. [This still needs to be written]

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sec-G

2. Extension of two results of Grothendieck

This section proves technical results which are useful in the following section. But this section is not strictly necessary for understanding the main ideas. The reader can safely skip this section by assuming all proper morphisms of algebraic spaces in later sections are actually projective morphisms of schemes.

One of Grothendieck's results which is frequently useful in studying coherent sheaves on proper schemes is [Gro63, Corollaire 7.7.8], stated below. The original proof includes a hypothesis which Grothendieck refers to as "*surabondante*" (Grothendieck's emphasis). This hypothesis has subsequently been removed by Lieblich, cf. [Lie06, Proposition 2.1.3]. We would like to briefly explain another way to remove these hypotheses based on the representability of the Quot functor. Martin Olsson proved a very general result about representability of the Quot functor. This leads to the following version of Grothendieck's result.

thm-Hom **Theorem 2.1.** [Gro63, Corollaire 7.7.8], [Lie06, Proposition 2.1.3] *Let Y be an algebraic space. Let $f : \mathcal{X} \to Y$ be a separated, locally finitely presented, algebraic stack over the category (Aff/Y) of affine Y-schemes. Let $\mathcal{F}$ and $\mathcal{G}$ be locally finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules. Assume that $\mathcal{G}$ is Y-flat and has proper support over Y. Then the covariant functor of quasi-coherent $\mathcal{O}_Y$-modules,*

$$\mathcal{T}(\mathcal{M}) := f_*(Hom_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{M})),$$

is representable by a locally finitely presented, quasi-coherent $\mathcal{O}_Y$-module $\mathcal{N}$, i.e., there is a natural equivalence of functors

$$\mathcal{T}(\mathcal{M}) \xrightarrow{\cong} Hom_{\mathcal{O}_Y}(\mathcal{N}, \mathcal{M}).$$

Grothendieck deduced his version of this result as a corollary of another theorem. In fact the theorem also follows from the corollary.

cor-Hom **Corollary 2.2.** [Gro63, Théorème 7.7.6] *Let $f : \mathcal{X} \to Y$ be as in Theorem 2.1. Let $\mathcal{G}$ be a locally finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module which is Y-flat and has proper support over Y. Then there exists a locally finitely presented, quasi-coherent $\mathcal{O}_Y$-module $\mathcal{Q}$ and a natural equivalence of covariant functors of quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{M}$,*

$$f_*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{M}) \xrightarrow{\cong} Hom_{\mathcal{O}_Y}(\mathcal{Q}, \mathcal{M}).$$

Grothendieck in turn deduces [Gro63, Théorème 7.7.6] as a special case of a more general representability result regarding the hyperderived pushforwards of a bounded below complex of locally finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules which are

Y-flat and have proper support over Y. Undoubtedly this general representability result extends to the setting of stacks. However, to deduce [Gro63, Corollaire 7.7.8] from [Gro63, Théorème 7.7.6], Grothendieck requires a hypothesis that $\mathcal{F}$ has a presentation by locally free $\mathcal{O}_{\mathcal{X}}$-modules. This is quite a restrictive hypothesis.

Instead one can try to prove Theorem 2.1 directly using Artin's representability theorems. This is precisely what Lieblich does in [Lie06, Proposition 2.1.3]. Here we point out that Theorem 2.1 also follows from from representability of the Quot functor as proved by Olsson, [Ols05, Theorem 1.5].

lem-Hom1

Lemma 2.3. *Let $f : \mathcal{X} \to Y$ be as in Theorem 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be locally finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules. And let*

$$\phi : \mathcal{F} \to \mathcal{G}$$

be a homomorphism of $\mathcal{O}_{\mathcal{X}}$-modules. Assume $\mathcal{G}$ has proper support over Y, resp. $\mathcal{F}$ and $\mathcal{G}$ have proper support over Y and $\mathcal{G}$ is Y-flat. Then there exists an open subspace U, resp. V, of S with the following property. For every morphism $g : Y' \to Y$ of algebraic spaces, the pullback morphism of sheaves on $Y' \times_Y \mathcal{X}$,

$$(g, Id_{\mathcal{X}})^*\phi : (g, Id_{\mathcal{X}})^*\mathcal{F} \to (g, Id_{\mathcal{X}})^*\mathcal{G}$$

is surjective, resp. an isomorphism, if and only if g factors through U, resp. V.

Proof. Since $\mathrm{Coker}(\phi)$ is a locally finitely presented, quasi-coherent sheaf supported on $\mathrm{Supp}(\mathcal{G})$, $\mathrm{Supp}(\mathrm{Coker}(\phi))$ is closed in $\mathrm{Supp}(\mathcal{G})$. Since $\mathrm{Supp}(\mathcal{G})$ is proper over Y, $f(\mathrm{Supp}(\mathrm{Coker}(\phi)))$ is a closed subset of Y. Define U to be the open complement. For a morphism $g : Y' \to Y$, $(g, \mathrm{Id}_{\mathcal{X}})^*\phi$ is surjective if and only if $\mathrm{Coker}((g, \mathrm{Id}_{\mathcal{X}})^*\phi)$ is zero, i.e., if and only if the support of $\mathrm{Coker}((g, \mathrm{Id}_{\mathcal{X}})^*\phi)$ is empty. Formation of the cokernel is compatible with pullback, i.e.,

$$\mathrm{Coker}((g, \mathrm{Id}_{\mathcal{X}})^*\phi) \cong (g, \mathrm{Id}_{\mathcal{X}})^*\mathrm{Coker}(\phi).$$

Thus the support of $\mathrm{Coker}((g, \mathrm{Id}_{\mathcal{X}})^*\phi)$ is empty if and only if $g(Y')$ is disjoint from $f(\mathrm{Supp}(\mathrm{Coker}(\phi)))$, i.e., if and only if $g(Y')$ is contained in U.

Next assume that $\mathcal{F}$, $\mathcal{G}$ each have proper support over Y and $\mathcal{G}$ is Y-flat. Since $\mathcal{G}$ is Y-flat, the kernel of ϕ is locally finitely presented by [Gro67, Lemme 11.3.9.1] (the property of being locally finitely presented can be checked locally in the fppf topology on $\mathcal{X}$, thus reduces to the case of a morphism of schemes). The support of the kernel is contained in the support of the kernel of $\mathcal{F}$. Thus $\mathrm{Ker}(\phi)$ is a locally finitely presented, quasi-coherent sheaf on the support of $\mathcal{F}$, which is proper over Y. So the support of $\mathrm{Ker}(\phi)$ is also proper over Y. Thus its image under f is a closed subset of Y. Define V to be the open complement of this closed subset in U.

For every morphism $g : Y' \to U$, since $\mathcal{G}$ is Y-flat, the following sequence is exact

$$0 \to (g, \mathrm{Id}_{\mathcal{X}})^*\mathrm{Ker}(\phi) \to (g, \mathrm{Id}_{\mathcal{X}})^*\mathcal{F} \to (g, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \to 0.$$

Thus $(g, \mathrm{Id}_{\mathcal{X}})^*\phi$ is an isomorphism if and only if $(g, \mathrm{Id}_{\mathcal{X}})^*\mathrm{Ker}(\phi)$ is zero, i.e., if and only if $g(Y')$ is contained in V. ☐

lem-Hom2

Lemma 2.4. *Let $f : \mathcal{X} \to Y$, $\mathcal{F}$ and $\mathcal{G}$ be as in Theorem 2.1. There exists a locally finitely presented, separated morphism of algebraic spaces $h : Z \to Y$ and a morphism of quasi-coherent sheaves on $Z \times_Y \mathcal{X}$*

$$\phi : (h, Id_{\mathcal{X}})^*\mathcal{F} \to (h, Id_{\mathcal{X}})^*\mathcal{G}$$

which represents the contravariant functor associating to every morphism $g : Y' \to Y$ the set of morphisms of quasi-coherent sheaves on $Y' \times_Y \mathcal{X}$

$$\psi : (g, Id_{\mathcal{X}})^*\mathcal{F} \to (g, Id_{\mathcal{X}})^*\mathcal{G}.$$

Proof. By [Ols05, Theorem 1.5], there exists a locally finitely presented, separated morphism $i : W \to Y$ of algebraic spaces and a quotient

$$\theta : (i, \mathrm{Id}_{\mathcal{X}})^*(\mathcal{F} \oplus \mathcal{G}) \to \mathcal{H}$$

representing the Quot functor of flat families of locally finitely presented, quasi-coherent quotients of the pullback of $\mathcal{F} \oplus \mathcal{G}$ having proper support over the base. Denote by

$$\theta_{\mathcal{G}} : (i, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \to \mathcal{H}$$

the composition of the summand

$$e_{\mathcal{G}} : (i, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \hookrightarrow (i, \mathrm{Id}_{\mathcal{X}})^*(\mathcal{F} \oplus \mathcal{G})$$

with θ.

By Lemma 2.3, there is an open subspace Z of W such that for every morphism $j : Y' \to W$, the pullback

$$(j, \mathrm{Id}_{\mathcal{X}})^*\theta_{\mathcal{G}} : (i \circ j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \to (j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{H}$$

is an isomorphism if and only if $j(Y')$ factors through Z. Denote by $h : Z \to Y$ the restriction of i to Z. If $j(Y')$ factors through Z, then $(j, \mathrm{Id}_{\mathcal{X}})^*\theta$ equals the composition

$$(i \circ j, \mathrm{Id}_{\mathcal{X}})^*(\mathcal{F} \oplus \mathcal{G}) \xrightarrow{(\psi, \mathrm{Id})} (i \circ j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \xrightarrow{j^*\theta_{\mathcal{G}}} (j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{H}$$

for a unique morphism of quasi-coherent sheaves

$$\psi : (i \circ j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{F} \to (i \circ j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G}.$$

In particular, applied to $\mathrm{Id}_Z : Z \to Z$, this produces the homomorphism ϕ. And it is straightforward to see that the natural transformation associating to every morphism $j : Y' \to Z$ the homomorphism ψ is an equivalence of functors, i.e., $(h : Z \to Y, \phi)$ is universal. □

Proof of Theorem 2.1. Let $h : Z \to Y$ and ϕ be as in Lemma 2.4. The zero homomorphism $0 : \mathcal{F} \to \mathcal{G}$ defines a Y-morphism $z : Y \to Z$. Since Z is separated over Y, the Y-morphism z is a closed immersion. Since Z is locally finitely presented over Y, the ideal sheaf $\mathcal{I}$ of this closed immersion is a locally finitely presented, quasi-coherent $\mathcal{O}_Z$-module. Thus $z^*\mathcal{I}$ is a locally finitely presented, quasi-coherent $\mathcal{O}_Y$-module. Denote this $\mathcal{O}_Y$-module by $\mathcal{N}$.

Consider the closed subspace Z_1 of Z with ideal sheaf $\mathcal{I}^2$. The restriction of h to Z_1 is a finite morphism, thus equivalent to the locally finitely presented $\mathcal{O}_Y$-algebra $h_*\mathcal{O}_{Z_1}$. Of course this fits into a short exact sequence

$$0 \to z^*\mathcal{I} \to h_*\mathcal{O}_{Z_1} \to \mathcal{O}_Y \to 0$$

where the injection is an ideal sheaf and the surjection is a homomorphism of $\mathcal{O}_Y$-algebras. And the morphism z defines a splitting of this surjection of $\mathcal{O}_Y$-algebras. Thus, as an $\mathcal{O}_Y$-algebra, there is a canonical isomorphism

$$h_*\mathcal{O}_{Z_1} \cong \mathcal{O}_Y \oplus z^*\mathcal{I}.$$

The restriction of ϕ to Z_1 together with adjunction of h^* and h_* defines a homomorphism of $\mathcal{O}_Y$-modules
$$\mathcal{F} \to \mathcal{G} \otimes_{\mathcal{O}_Y} h_*\mathcal{O}_{Z_1}.$$
Using the canonical isomorphism, this homomorphism is of the form
$$\mathcal{F} \xrightarrow{(0,\chi)} \mathcal{G} \oplus (\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{N}).$$
The homomorphism χ defines a natural transformation of covariant functors of quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{M}$
$$Hom_{\mathcal{O}_Y}(\mathcal{N}, \mathcal{M}) \to f_*(Hom_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{M}).$$
By the same argument used to prove the equivalence of parts (a) and (d) of [Gro63, Théorème 7.7.5], this is an equivalence of functors and the induced Y-morphism
$$\underline{\mathrm{Spec}}_Y(\mathrm{Sym}^\bullet(\mathcal{N})) \to Z$$
is an isomorphism. □

Proof of Corollary 2.2. This follows from Theorem 2.1 by taking $\mathcal{F}$ to be $\mathcal{O}_\mathcal{X}$. □

3. Pseudo ideal sheaves

sec-pis

Given a fibration over a curve, given a fiber, and given a section, the N-jet of the section at the fiber is simply the intersection of the section and the N^{th} infinitesimal neighborhood of the fiber, considered as a closed subscheme of the infinitesimal neighborhood. The rule associating to a section its N-jet gives a morphism from the parameter scheme of sections to the Hilbert scheme of the N^{th} infinitesimal neighborhood of the fiber. In order to prove Theorem 1.3, it is necessary to extend this morphism to the locus parameterizing "combs" whose intersection with the fiber may not be transverse. This is done using "pseudo ideal sheaves", a generalization of Fulton's notion of effective pseudo divisor.

Let $f : X \to Y$ be a flat, locally finitely presented, proper morphism of algebraic spaces. For every morphism of algebraic spaces, $g : Y' \to Y$, denote by $f_{Y'} : X_{Y'} \to Y'$ the basechange of f. Using results of Martin Olsson, the following definitions and results are still valid whenever X is a flat, locally finitely presented, proper algebraic stack over Y. But since our application is to algebraic spaces, we leave the case of stacks to the interested reader.

defn-pis

Definition 3.1. For every morphism $g : Y' \to Y$ of algebraic spaces, a flat family of *pseudo ideal sheaves of* X/Y *over* Y' is a pair $(\mathcal{F}, u)$ consisting of

(i) a Y'-flat, locally finitely presented, quasi-coherent $\mathcal{O}_{X_{Y'}}$-module $\mathcal{F}$, and
(ii) an $\mathcal{O}_{X_{Y'}}$-homomorphism $u : \mathcal{F} \to \mathcal{O}_{X_{Y'}}$

such that the following induced morphism is zero,
$$u' : \bigwedge^2 \mathcal{F} \to \mathcal{F}, \quad f_1 \wedge f_2 \mapsto u(f_1)f_2 - u(f_2)f_1.$$

For every pair $g_1 : Y'_1 \to Y$, $g_2 : Y'_2 \to Y$ of morphisms of algebraic spaces, for every pair $(\mathcal{F}_1, u_1)$, resp. $(\mathcal{F}_2, u_2)$, of flat families of pseudo ideal sheaves of X/Y over Y'_1, resp. over Y'_2, and for every Y-morphism $h : Y'_1 \to Y'_2$, a *pullback map* from $(\mathcal{F}_1, u_1)$ to $(\mathcal{F}_2, u_2)$ over h is an isomorphism of $\mathcal{O}_{X_{Y'_1}}$-modules
$$\eta : \mathcal{F}_1 \to h^*\mathcal{F}_2$$

such that $h^*u_2 \circ \eta$ equals u_1.

The *category of pseudo ideal sheaves of* X/Y, $\text{Pseudo}_{X/S}$, is the category whose objects are data $(g : Y' \to Y, (\mathcal{F}, u))$ of an affine Y-scheme Y' together with a flat family of pseudo ideal sheaves of X/Y over Y', and whose Hom sets

$$\text{Hom}((g_1 : Y_1' \to Y, (\mathcal{F}_1, u_1)), (g_2 : Y_2' \to Y, (\mathcal{F}_2, u_2)))$$

are the sets of pairs (h, η) of a Y-morphism $h : Y_1' \to Y_2'$ together with a pullback map η from $(\mathcal{F}_1, u_1)$ to $(\mathcal{F}_2, u_2)$ over h. Identity morphisms and composition of morphisms are defined in the obvious manner. There is an obvious functor from $\text{Pseudo}_{X/Y}$ to the category (Aff/Y) of affine Y-schemes sending every object $(g : Y' \to Y, (\mathcal{F}, u))$ to $(g : Y' \to Y)$ and sending every morphism (h, η) to h.

prop-algebraic

Proposition 3.2. *The category $\text{Pseudo}_{X/Y}$ is a limit-preserving algebraic stack over the category (Aff/Y) of affine Y-schemes. Moreover, the diagonal is quasi-compact and separated.*

Proof. Denote by $\text{Coh}_{X/Y}$ the category of coherent sheaves on X/Y, cf. [LMB00, (2.4.4)]. There is a functor $G : \text{Pseudo}_{X/Y} \to \text{Coh}_{X/Y}$ sending every object $(g : Y' \to Y, (\mathcal{F}, u))$ to $(g : Y' \to Y, \mathcal{F})$ and sending every morphism (h, η) to (h, η). This is a 1-morphism of categories over (Aff/Y). By [LMB00, Théorème 4.6.2.1] and [Sta06, Proposition 4.1], $\text{Coh}_{X/Y}$ is a limit-preserving algebraic stack over (Aff/Y) with quasi-compact, separated diagonal. Thus to prove the proposition, it suffices to prove that G is representable by locally finitely presented, separated algebraic spaces.

Let Y' be a Y-algebraic space and let $\mathcal{F}$ be a locally finitely presented, quasi-coherent $\mathcal{O}_{X_{Y'}}$-module. Since $f_{Y'} : X_{Y'} \to Y'$ is flat, locally finitely presented and proper, by Lemma 2.4, there exists a locally finitely presented, separated morphism $h : Z \to Y'$ of algebraic spaces and a universal homomorphism

$$u : (h, \text{Id}_{X_{Y'}})^*\mathcal{F} \to \mathcal{O}_{X_Z}.$$

In fact, as follows from the proof of Theorem 2.1, there is a locally finitely presented, quasi-coherent $\mathcal{O}_{Y'}$-module $\mathcal{N}$ and a homomorphism of $\mathcal{O}_{X_{Y'}}$-modules

$$\chi : \mathcal{F} \to \mathcal{O}_{X_{Y'}} \otimes_{\mathcal{O}_{Y'}} \mathcal{N}$$

such that $Z = \underline{\text{Spec}}_{Y'}(\text{Sym}^\bullet(\mathcal{N}))$ and such that u is the homomorphism induced by χ.

Since $\mathcal{F}$ is a locally finitely presented, quasi-coherent $\mathcal{O}_{X_{Y'}}$-module which is Y'-flat and has proper support over Y', the same is true of $(h, \text{Id}_{X_{Y'}})^*\mathcal{F}$ relative to Z. Thus, by Theorem 2.1, there exists a locally finitely presented, quasi-coherent $\mathcal{O}_Z$-module $\mathcal{N}$ and a universal homomorphism

$$\chi : \bigwedge^2 (h, \text{Id}_{X_{Y'}})^*\mathcal{F} \to (h, \text{Id}_{X_{Y'}})^*\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{N}.$$

Thus there exists a unique homomorphism of $\mathcal{O}_Z$-modules,

$$\zeta : \mathcal{N} \to \mathcal{O}_Z$$

such that the induced homomorphism of $\mathcal{O}_{X_Z}$-modules

$$\bigwedge^2 (h, \text{Id}_{X_{Y'}})^*\mathcal{F} \xrightarrow{\chi} (h, \text{Id}_{X_{Y'}})^*\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{N} \xrightarrow{\text{Id}\otimes\zeta} (h, \text{Id}_{X_{Y'}})^*\mathcal{F}$$

equals u'. Let P denote the closed subspace of Z whose ideal sheaf equals $\zeta(\mathcal{N})$. Chasing diagrams, the restriction morphism $h|_P : P \to Y'$ together with the pull-back of u to P is a pair representing $Y' \times_{\mathrm{Coh}_{X/Y}} \mathrm{Pseudo}_{X/Y} \to Y'$. Since $Z \to Y'$ is a locally finitely presented, separated morphism of algebraic spaces (also schematic), and since P is a closed subspace of Y' whose ideal sheaf is locally finitely generated (being the image of the locally finitely presented sheaf $\mathcal{N}$), also $P \to Y$ is a locally finitely presented, separated morphism of algebraic spaces (also schematic). □

Denote by $(g : \mathrm{Hilb}_{X/Y} \to Y, C)$ a universal pair of a morphism of algebraic spaces g and a closed subspace C of $\mathrm{Hilb}_{X/Y} \times_Y X$ which is flat, locally finitely presented, and proper over $\mathrm{Hilb}_{X/Y}$, i.e., an object representing the Hilbert functor, cf. [Art69, Corollary 6.2]. Denote by

$$0 \to \mathcal{I} \xrightarrow{u} \mathcal{O}_{\mathrm{Hilb}_{X/Y} \times_Y X} \to \mathcal{O}_C \to 0.$$

the natural exact sequence, where $\mathcal{I}$ is the ideal sheaf of C in $\mathrm{Hilb}_{X/Y} \times_Y X$.

prop-Hilbopen

Proposition 3.3. *The pair $(\mathcal{I}, u)$ is a family of pseudo ideal sheaves of X/Y over $Hilb_{X/Y}$. The induced 1-morphism*

$$\iota : Hilb_{X/Y} \to Pseudo_{X/Y}$$

is representable by open immersions.

Proof. Since the kernel of a surjection of flat modules is flat, $\mathcal{I}$ is flat over $\mathrm{Hilb}_{X/Y}$. By [Gro67, Lemme 11.3.9.1], $\mathcal{I}$ is a locally finitely presented, quasi-coherent sheaf. Since the homomorphism u is injective, to prove that u' is zero it suffices to prove the composition $u \circ u'$ is zero. This follows immediately from the definition of u'. Thus the pair $(\mathcal{I}, u)$ is a family of pseudo ideal sheaves of X/Y over $\mathrm{Hilb}_{X/Y}$.

To prove that ι is representable by open immersions, it suffices to prove that it is representable by quasi-compact, étale monomorphisms of schemes. To this end, let Y' be an affine Y-scheme and let $(\mathcal{F}, v)$ be a flat family of pseudo ideal sheaves of X/Y over Y'. Denote the cokernel of v by

$$w : \mathcal{O}_{X_{Y'}} \to \mathcal{G}.$$

By [OS03, Theorem 3.2], there is a morphism of algebraic spaces $\sigma : \Sigma \to Y'$ such that $(\sigma, \mathrm{Id}_{X_{Y'}})^*\mathcal{G}$ is flat over Σ and such that Σ is universal among Y'-spaces with this property. Moreover, Σ is a surjective, finitely-presented, quasi-affine monomorphism (in particular schematic). As an aside, please note that the remark preceding [OS03, Theorem 3.2] is incomplete – [OS03, Proposition 3.1] should be properly attributed to Laumon and Moret-Bailly, [LMB00, Théorème A.2].

By [Gro67, Lemma 11.3.9.1], $\mathrm{Ker}((\sigma, \mathrm{Id}_{X_{Y'}})^*w)$ is a locally finitely presented, quasi-coherent sheaf. Moreover, because it is the kernel of a surjection of sheaves which are flat over Σ, it is also flat over Σ. By Lemma 2.3, there is an open subscheme W of Σ such that a morphism $S \to \Sigma$ factors through W if and only if the pullback of

$$(\sigma, \mathrm{Id}_{X_{Y'}})^*\mathcal{F} \to \mathrm{Ker}((\sigma, \mathrm{Id}_{X_{Y'}})^*w)$$

is an isomorphism. Chasing universal properties, it is clear that $W \to Y'$ represents

$$Y' \times_{\mathrm{Pseudo}_{X/Y}} \mathrm{Hilb}_{X/Y}.$$

Thus ι is representable by finitely-presented, quasi-affine monomorphisms of schemes.

It only remains to prove that ι is étale. Because ι is a finitely-presented it remains to prove that ι is formally étale. Thus, let $Y' = \operatorname{Spec} A'$ where A' is a local Artin $\mathcal{O}_Y$-algebra with maximal ideal $\mathfrak{m}$ and residue field κ. And let

$$0 \to J \to A' \to A \to 0$$

be an infinitesimal extension, i.e., $\mathfrak{m}J$ is zero. Let $(\mathcal{F}, u)$ be a pseudo ideal sheaf of X/Y over Y', and assume the basechange to Spec A is an ideal sheaf with A-flat cokernel. Since ι is a monomorphism, formal étaleness for ι precisely says that $Y' \to \mathrm{Pseudo}_{X/Y}$ factors through ι, i.e., u is injective and $\operatorname{Coker}(u)$ is A'-flat.

To prove this use the local flatness criterion, e.g., as formulated in [Gro67, Proposition 11.3.7]. This criterion is an equivalence between the conditions of

(i) injectivity of u is injective and A'-flatness of $\operatorname{Coker}(u)$

(ii) and injectivity of

$$u \otimes_{A'} \kappa : \mathcal{F} \otimes_{A'} \kappa \to \mathcal{O}_{X_{Y'}} \otimes_{A'} \kappa$$

By hypothesis, (i) holds after basechange to A. Thus (ii) holds after basechange to A. But since $A'/\mathfrak{m}$ equals $A/\mathfrak{m}$, (ii) for the original family over A' is precisely the same as (ii) for the basechange family over A. Thus also (i) holds over A'. $\square$

The significance of pseudo ideal sheaves has to do with restriction to Cartier divisors. Let D be an effective Cartier divisor in X, considered as a closed subscheme of X, and assume D is flat over Y. Denote by $\mathcal{I}_D$ the pullback

$$\mathcal{I}_D := \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_D$$

on $\mathrm{Hilb}_{X/Y} \times_X D$. And denote by

$$u_D : \mathcal{I}_D \to \mathcal{O}_{\mathrm{Hilb}_{X/Y} \times_X D}$$

the restriction of u.

prop-flat

Proposition 3.4. *The locally finitely presented, quasi-coherent sheaf $\mathcal{I}_D$ is flat over $Hilb_{X/Y}$. Thus the pair $(\mathcal{I}_D, u_D)$ is a flat family of pseudo ideal sheaves of D/Y over $Hilb_{X/Y}$.*

Proof. Associated to the Cartier divisor D there is an injective homomorphism of invertible sheaves

$$t' : \mathcal{O}_X(-D) \xrightarrow{\mathcal{O}}_X .$$

This induces a morphism of locally finitely presented, quasi-coherent sheaves

$$t : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to \mathcal{I}.$$

The cokernel of t is $\mathcal{I}_D$. By the local flatness criterion, [Gro67, Proposition 11.3.7], to prove that t is injective and $\mathcal{I}_D$ is flat over $\mathrm{Hilb}_{X/Y}$, it suffices to prove that the "fiber" of t over every point of $\mathrm{Hilb}_{X/Y}$ is injective. Thus, let κ be a field, let $y : \operatorname{Spec} \kappa Y$ be a morphism, and let $\mathcal{I}_y \xrightarrow{\mathcal{O}}_{X_y}$ be an ideal sheaf. Since D is Y-flat, the homomorphism of locally free sheaves

$$t'_y : \mathcal{O}_X(-D) \otimes_{\mathcal{O}_Y} \kappa \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} \kappa$$

is injective, thus a flat resolution of $\mathcal{O}_D \otimes_{\mathcal{O}_Y} \kappa$. In particular, $Tor_2^{\mathcal{O}_{X_y}}(\mathcal{O}_{X_y}/\mathcal{I}_y, \mathcal{O}_{D_y})$ equals zero because there is a flat resolution of $\mathcal{O}_{D_y}$ with amplitude $[-1, 0]$. By the long exact sequence of Tor associated to the short exact sequence

$$0 \to \mathcal{I}_y \to \mathcal{O}_{X_y} \to \mathcal{O}_{X_y}/\mathcal{I}_y \to 0,$$

there is an isomorphism

$$Tor_1^{\mathcal{O}_{X_y}}(\mathcal{I}_y, \mathcal{O}_{D_y}) \cong Tor_2^{\mathcal{O}_{X_y}}(\mathcal{O}_{X_y}/\mathcal{I}_y, \mathcal{O}_{D_y}) = 0.$$

But this Tor sheaf is precisely the kernel of

$$t_y : \mathcal{I}_y \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to \mathcal{I}_y.$$

Thus t_y is injective, and so $\mathcal{I}_D$ is flat over $\mathrm{Hilb}_{X/Y}$. $\square$

notat-iotaD

Notation 3.5. Denote by

$$\iota_D : \mathrm{Hilb}_{X/Y} \to \mathrm{Pseudo}_{D/Y}$$

the 1-morphism associated to the flat family $(\mathcal{I}_D, u_D)$ of pseudo ideal sheaves of D/Y over $\mathrm{Hilb}_{X/Y}$. This is the *divisor restriction map.*

Since $\mathrm{Hilb}_{X/Y}$ and $\mathrm{Pseudo}_{D/Y}$ are both locally finitely presented over Y, ι_D is locally finitely presented. Since $\mathrm{Hilb}_{X/Y}$ is an algebraic space, ι_D is representable (by morphisms of algebraic spaces). Since the diagonal morphism of $\mathrm{Pseudo}_{D/Y}$ over Y is separated, and since $\mathrm{Hilb}_{X/Y}$ is separated over Y, ι_D is separated.

ssec-inf

3.1. **Infinitesimal study of the divisor restriction map.** Let A' be a local Artin $\mathcal{O}_Y$-algebra with maximal ideal $\mathfrak{m}$ and residue field κ. And let

$$0 \to J \to A' \to A \to 0$$

be an infinitesimal extension, i.e., $\mathfrak{m}J$ is zero. Denote by $X_{A'}$, resp. X_A, X_κ, the fiber product of $X \to Y$ with Spec $A' \to Y$, resp. Spec $A \to Y$, Spec $\kappa \to Y$.

Let $(\mathcal{F}_{A'}, u_{A'})$ be a pseudo ideal sheaf of D/Y over Spec A'. Denote by $(\mathcal{F}_A, u_A)$, resp. $(\mathcal{F}_\kappa, u_\kappa)$, the restriction of $(\mathcal{F}_{A'}, u_{A'})$ to A, resp. to κ. Let $\mathcal{I}_A$ be the ideal sheaf of a flat family C_A of closed subschemes of X/Y over Spec A. Denote by $\mathcal{I}_\kappa$, resp. C_κ, the restriction of $\mathcal{I}_A$ to κ, resp. of C_A to κ. And assume that ι_D sends $\mathcal{I}_A$ to $(\mathcal{F}_A, u_A)$.

prop-inf

Proposition 3.6. *Let n be a nonnegative integer. Assume that C_κ is a regular immersion of codimension n in X_κ, cf.* [Gro67, Définition 16.9.2] *(since X_κ is an algebraic space, in that definition one must replace the Zariski covering by affine schemes by an étale covering by affine schemes).*

(i) *The morphism ι_D is* locally unobstructed *at C_κ in the following sense. For every étale morphism Spec $R_{A'} \to X_{A'}$ such that the pullback I_κ of $\mathcal{I}_\kappa$ in R_κ is generated by a regular sequence, there exists an ideal $I_{A'}$ in $R_{A'}$ whose restriction I_A to R_A equals the pullback of $\mathcal{I}_A$ and whose "local pseudo ideal sheaf" $(I_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D, v)$ equals the pullback of $(\mathcal{F}_{A'}, u_{A'})$. Moreover, the set of such ideals $I_{A'}$ is naturally a torsor for the R_κ-submodule*

$$J \otimes_\kappa \mathcal{O}_X(-D) \cdot Hom_{R_\kappa}(I_\kappa, R_\kappa/I_\kappa)$$

of

$$J \otimes_\kappa Hom_{R_\kappa}(I_\kappa, R_\kappa/I_\kappa)$$

(here $\mathcal{O}_X(-D)\cdot$ denote multiplication by the inverse image ideal of $\mathcal{O}_X(-D)$).

(ii) *There exists an element ω in*

$$J \otimes_\kappa H^1(C_\kappa, \mathcal{O}_X(-D) \cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$$

which equals 0 if and only if there exists a flat family $C_{A'}$ of closed subschemes of X/Y over Spec A' whose restriction to Spec A equals C_A and whose image under ι_D equals $(\mathcal{F}_{A'}, u_{A'})$. When it equals 0, the set of such families $C_{A'}$ is naturally a torsor for the κ-vector space

$$J \otimes_\kappa H^0(C_\kappa, \mathcal{O}_X(-D) \cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa})).$$

In particular, if $h^1(C_\kappa, \mathcal{O}_X(-D) \cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$ equals 0, then ι_D is smooth at $[C_\kappa]$.

Proof. **(i)** Let $v_{(A,1)}, \dots, v_{(A,n)}$ be a regular sequence in R_A generating I_A. Denote by v_A the R_A-module homomorphism

$$v_A : R_A^{\oplus n} \to R_A, \quad v_A(\mathbf{e}_i) = v_{(A,i)}.$$

Denote the associated R_A-module homomorphism by

$$v'_A : R_A^{\oplus \binom{n}{2}} \to R_A^{\oplus n}, \quad v'_A(\mathbf{e}_i \wedge \mathbf{e}_j) = v_{(A,i)}\mathbf{e}_j - v_{(A,j)}\mathbf{e}_i.$$

Since $(v_{(A,1)}, \dots, v_{(A,n)})$ is a regular sequence, the following sequence is exact

$$R_A^{\oplus \binom{n}{2}} \xrightarrow{v'_A} R_A^{\oplus n} \to I_A \to 0.$$

Thus there is an exact sequence

$$(R_A \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}} \xrightarrow{v'_A \otimes \mathrm{Id}} (R_A \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \to (I_A \otimes_{\mathcal{O}_X} \mathcal{O}_D) \to 0.$$

Denote by $F_{A'}$ the $R_{A'}$-module whose associated quasi-coherent sheaf is the pullback of $\mathcal{F}_{A'}$. Since $\mathcal{F}_A$ equals $\mathcal{I}_A \otimes_{\mathcal{O}_X} \mathcal{O}_D$, there is an isomorphism

$$F_{A'}/JF_{A'} \cong I_A \otimes_{\mathcal{O}_X} \mathcal{O}_D$$

compatible with the maps $u_{A'}$ and u_A. Since $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n}$ is a projective $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-module, there exists an $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-module homomorphism

$$a : (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \to F_{A'}$$

whose restriction to A is the surjection above. So, by Nakayama's lemma, this map is also surjective. Since both the source and target of the surjection are A'-flat, also the kernel is A'-flat. Thus there is also a lifting of the set of generators of the kernel, i.e., there is an exact sequence of $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-modules

$$(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}} \xrightarrow{b} (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \xrightarrow{a} F_{A'} \to 0$$

whose restriction to A is the short exact sequence above.

The composition of the surjection with $u_{A'}$ defines an $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-module homomorphism

$$w_{A'} : (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \to (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$$

whose restriction to A equals $v_A \otimes \mathrm{Id}$. There is an associated map

$$w'_{A'} : (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}} \to (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n}, \ \ w'_{A'}(\mathbf{e}_i \wedge \mathbf{e}_j) = w_{A'}(\mathbf{e}_i)\mathbf{e}_j - w_{A'}(\mathbf{e}_j)\mathbf{e}_i.$$

Since $(\mathcal{F}_{A'}, u_{A'})$ is a pseudo ideal sheaf, the induced map $u'_{A'}$ equals 0. Therefore the image of $w'_{A'}$ is contained in the kernel of a. Since $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus\binom{n}{2}}$ is a free $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-module, there is a lifting

$$c : (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus\binom{n}{2}} \to (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus\binom{n}{2}}$$

such that $w'_{A'} = b \circ c$. In particular, the restriction of c to A is an isomorphism. Thus, by Nakayama's lemma, c is surjective. A surjection of free modules of the same finite rank is automatically an isomorphism. Thus there is a presentation

$$(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus\binom{n}{2}} \xrightarrow{w'_{A'}} (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \xrightarrow{a} F_{A'} \to 0.$$

Because both $R_{A'}$ and $R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ are A'-flat, there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J \otimes_\kappa R_\kappa & \longrightarrow & R_{A'} & \longrightarrow & R_A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J \otimes_\kappa R_\kappa \otimes_{\mathcal{O}_X} \mathcal{O}_D & \longrightarrow & R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D & \longrightarrow & R_A \otimes_{\mathcal{O}_X} \mathcal{O}_D & \longrightarrow & 0 \end{array}$$

where the vertical maps are each surjective. By the snake lemma, the induced map

$$R_{A'}/(J\mathcal{O}_X(-D) \cdot R_{A'}) \to R_A \times_{(R_A \otimes_{\mathcal{O}_X} \mathcal{O}_D)} (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$$

is an isomorphism. Thus, for every integer $i = 1, \dots, n$, there exists an element $v_{(A',i)}$ in $R_{A'}$ whose image in R_A equals $v_{(A,i)}$ and whose image in $R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ equals $w_{A'}(\mathbf{e}_i)$. Moreover, the set of all such elements is naturally a torsor for $J \otimes_\kappa (\mathcal{O}_X(-D) \cdot R_\kappa)$. In other words, there is an $R_{A'}$-module homomorphism

$$v_{A'} : R_{A'}^{\oplus n} \to R_{A'}$$

whose restriction to A equals v_A and such that $v_{A'} \otimes \mathrm{Id}$ equals $w_{A'}$.

Since $(v_{(A,1)}, \dots, v_{(A,n)})$ is a regular sequence in R_A, also $(v_{(A',1)}, \dots, v_{(A',n)})$ is a regular sequence in $R_{A'}$. One way to see this is to tensor the Koszul complex $K^\bullet(R_{A'}, v_{A'})$ of $v_{A'}$ with the short exact sequence of A'-modules

$$0 \to J \to A' \to A \to 0.$$

Since the terms in the Koszul complex are free $R_{A'}$-modules, and since $R_{A'}$ is A'-flat, the associated sequence of complexes is exact

$$0 \to J \otimes_A K^\bullet(R_A, v_A) \to K^\bullet(R_{A'}, v_{A'}) \to K^\bullet(R_A, v_A) \to 0.$$

Thus there is a long exact sequence of Koszul cohomology

$$\dots \to J \otimes_A H^n(K^\bullet(R_A, v_A)) \to H^n(K^\bullet(R_{A'}, v_{A'})) \to H^n(K^\bullet(R_A, v_A)) \to J \otimes_A H^{n+1}(K^\bullet(R_A, v_A)) \to \dots$$

Since v_A is regular, $H^{n-1}(K^\bullet(R_A, v_A))$ is zero, which then implies $H^{n-1}(K^\bullet(R_{A'}, v_{A'}))$ by the long exact sequence above. Thus also $v_{A'}$ is regular. Moreover, this gives a short exact sequence

$$0 \to J \otimes_\kappa (R_A/I_A) \to R'_A/\mathrm{Image}(v_{A'}) \to R_A/I_A \to 0$$

from which it follows that $R_{A'}/\mathrm{Image}(v_{A'})$ is A'-flat. Denote by $I_{A'}$ the image of $v_{A'}$.

Both the pseudo ideal $I_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ and $F_{A'}$ equal the cokernel of $w'_{A'}$. Thus there is a unique isomorphism between them compatible with $w'_{A'}$. Moreover, since the compositions

$$(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \xrightarrow{a} F'_A \xrightarrow{u_{A'}} (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$$

and

$$(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \xrightarrow{v_{A'} \otimes \mathrm{Id}} I'_A \otimes_{\mathcal{O}_X} \mathcal{O}_D \to (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$$

both equal $w_{A'}$, the isomorphism above is compatible with the maps to $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$. Thus it is an isomorphism of pseudo ideal sheaves. Therefore there exists $I_{A'}$ satisfying all the conditions in the proposition.

For every lift $I_{A'}$ of I_A which is A-flat, the surjection $v_A : R_A^{\oplus n} \to I_A$ lifts to a surjection $R_{A'}^{\oplus n} \to I_{A'}$. Composing this surjection with the injection $I_{A'} \hookrightarrow R_{A'}$, it follows that every lift $I_{A'}$ arises from a lift $v_{A'}$ of v_A. As mentioned previously, the set of lifts $v_{A'}$ whose restriction to A equals v_A and with $v_{A'} \otimes \mathrm{Id}$ equal to $w_{A'}$ is naturally a torsor for

$$J \otimes_\kappa \mathrm{Hom}_{R_{A'}}(R_{A'}^{\oplus n}, \mathcal{O}_X(-D) \cdot R_\kappa).$$

But the translate of a lift by a homomorphism with image in I_κ gives the same ideal $I_{A'}$ (just different surjections from $R_{A'}^{\oplus n}$ to the ideal). Thus, the set of lifts $I_{A'}$ of I_A whose pseudo ideal sheaf is the pullback of $(\mathcal{F}_{A'}, u_{A'})$ is naturally a torsor for

$$J \otimes_\kappa \mathrm{Hom}_{R_{A'}}(R_{A'}^{\oplus n}, \mathcal{O}_X(-D) \cdot (R_\kappa/I_\kappa)) = J \otimes_\kappa \mathrm{Hom}_{R_\kappa}(I_\kappa, R_\kappa/I_\kappa).$$

(ii) Let Γ be an indexing set and let $(\mathrm{Spec}\, R_{A'}^\gamma \to X_{A'})_{\gamma \in \Gamma}$ be an étale covering such that for every γ, either $I_\kappa^\gamma \subset R_\kappa^\gamma$ is generated by a regular sequence of length n or else equals R_κ^γ. By the hypothesis that C_κ is a regular immersion of codimension n, there exists such a covering.

Suppose first that I_κ^γ equals R_κ^γ. Then also $(F_\kappa^\gamma, u_\kappa^\gamma)$ is isomorphic to $R_\kappa^\gamma \otimes_{\mathcal{O}_X} \mathcal{O}_D \xrightarrow{=} R_\kappa^\gamma \otimes_{\mathcal{O}_X} \mathcal{O}_D$. It is straightforward to see that the only deformations over A' of $R_\kappa^\gamma \xrightarrow{=} R_\kappa^\gamma$, resp. $R_\kappa^\gamma \otimes_{\mathcal{O}_X} \mathcal{O}_D \xrightarrow{=} R_\kappa^\gamma \otimes_{\mathcal{O}_X} \mathcal{O}_D$, as pseudo ideals are $R_{A'}^\gamma \xrightarrow{=} R_{A'}^\gamma$, resp. $R_{A'}^\gamma \otimes_{\mathcal{O}_X} \mathcal{O}_D \xrightarrow{=} R_{A'}^\gamma \otimes_{\mathcal{O}_X} \mathcal{O}_D$. Thus there is a lifting $I_{A'}^\gamma$ of I_A^γ, and it is unique.

On the other hand, if I_κ^γ is generated by a regular sequence of length n, by (i) there exist liftings $I_{A'}^\gamma$ and the set of all liftings is a torsor for

$$J \otimes_\kappa \mathcal{O}_X(-D) \cdot \mathrm{Hom}_{R_\kappa^\gamma}(I_\kappa^\gamma, R_\kappa^\gamma/I_\kappa^\gamma).$$

For a collection of liftings $(I_{A'}^\gamma)_{\gamma \in \Gamma}$, for every $\gamma_1, \gamma_2 \in \Gamma$, the basechanges of $I_{A'}^{\gamma_1}$ and $I_{A'}^{\gamma_2}$ to

$$R_{A'}^{\gamma_1,\gamma_2} := R_{A'}^{\gamma_1} \otimes_{\mathcal{O}_X} R_{A'}^{\gamma_2}$$

differ by an element $\omega^{\gamma_1,\gamma_2}$ in

$$J \otimes_\kappa \mathcal{O}_X(-D) \cdot \mathrm{Hom}_{R_\kappa^{\gamma_1,\gamma_2}}(I_\kappa^{\gamma_1,\gamma_2}, R_\kappa^{\gamma_1,\gamma_2}/I_\kappa^\gamma).$$

It is straightforward to see that $(\omega^{\gamma_1,\gamma_2})_{\gamma_1,\gamma_2 \in \Gamma}$ is a 1-cocycle for

$$\mathcal{O}_X(-D) \cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa})$$

with respect to the given étale covering. Moreover, changing the collection of lifts $(I_{A'}^\gamma)$ by translating by elements in

$$(J \otimes_\kappa \mathcal{O}_X(-D) \cdot \mathrm{Hom}_{R_\kappa^\gamma}(I_\kappa^\gamma, R_\kappa^\gamma/I_\kappa^\gamma))_{\gamma \in \Gamma}$$

precisely changes $(\omega^{\gamma_1,\gamma_2})_{\gamma_1,\gamma_2\in\Gamma}$ by a 1-coboundary. Therefore, the cohomology class

$$\omega \in J\otimes_\kappa H^1(C_\kappa, \mathcal{O}_X(-D)\cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$$

is well-defined and equals 0 if and only if there is a lifting $\mathcal{I}_{A'}$ as in the proposition. And in this case, the set of liftings is a torsor for the set of compatible families $(t^\gamma)_{\gamma\in\Gamma}$ of elements t^γ in

$$J\otimes_\kappa \mathcal{O}_X(-D)\cdot \mathrm{Hom}_{R_\kappa^\gamma}(I_\kappa^\gamma, R_\kappa^\gamma/I_\kappa^\gamma),$$

i.e., it is a torsor for the set of elements t in

$$J\otimes_\kappa H^0(C_\kappa, \mathcal{O}_X(-D)\cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa})).$$

□

4. Pseudo R-equivalence

sec-2q

Overview. The definition of R-equivalence in the introduction is valid over any field. But in applications to number theory and geometry, R-equivalence is most often applied when the field is the fraction field of a Henselian DVR. (Although it is interesting to see what can be proved over other fields.) For varieties over a DVR there is another notion, "pseudo R-equivalence", which follows from R-equivalence and which captures the "continuous nature" of the known cohomological obstructions to weak approximation.

Let $\widehat{\mathcal{O}}$ be a DVR. In later sections $\widehat{\mathcal{O}}$ will equal the complete local ring of a smooth, closed point of a curve, i.e., $k[\![u]\!]$ for an algebraically closed field k. Let $\mathcal{Y}$ be a flat, quasi-projective $\widehat{\mathcal{O}}$-scheme whose generic fiber is smooth. Let $\widehat{s}$ and $\widehat{t}$ be $\widehat{\mathcal{O}}$-points of $\mathcal{Y}$ and consider the corresponding $\mathrm{Frac}(\widehat{\mathcal{O}})$-points of the generic fiber. Since $\mathcal{Y}$ is separated, $\widehat{t}$ specializes to $\widehat{s}$ only if $\widehat{t}$ equals $\widehat{s}$. However, if $\widehat{s}$ and $\widehat{t}$ are R-equivalent then $\widehat{s}$ is the specialization of a family of sections of $\mathcal{Y}$ which agree with $\widehat{t}$ to arbitrary order. We say that $\widehat{t}$ "pseudo specializes" to $\widehat{s}$. The same holds with $\widehat{s}$ and $\widehat{t}$ permuted. Thus we say that $\widehat{s}$ and $\widehat{t}$ are "pseudo R-equivalent". Pseudo specialization is invariant under modification of the closed fiber of $\mathcal{Y}$. If $\widehat{\mathcal{O}}$ is Henselian and if $\widehat{t}$ pseudo specializes to $\widehat{s}$, then both $\widehat{t}$ and $\widehat{s}$ determine the same "Brauer pairing",

$$\mathrm{Br}(\mathcal{Y}_{\mathrm{Frac}(\widehat{\mathcal{O}})}) \to \mathrm{Br}(\mathrm{Frac}(\widehat{\mathcal{O}})).$$

Finally, pseudo R-equivalence satisfies the fibration property from the introduction which is not currently known for (usual) R-equivalence. For all these reasons we develop these "pseudo" notions in this section. Some of the arguments are technical. The reader who wants to skip this section may safely replace "pseudo R-equivalence" by "R-equivalence" wherever it occurs in the later sections.

Definitions. As above, let $\widehat{\mathcal{O}}$ be a DVR, and denote by u a generator of the maximal ideal. Let $\mathcal{Y}$ be a flat, quasi-projective $\widehat{\mathcal{O}}$-scheme whose generic fiber is smooth. The *bidisk* $\mathbf{D}_{\widehat{\mathcal{O}}}$ is the power series ring,

$$\mathbf{D}_{\widehat{\mathcal{O}}} := \mathrm{Spec}\ (\widehat{\mathcal{O}}[\![v]\!]).$$

The *punctured bidisk* is the scheme

$$\mathbf{D}^*_{\widehat{\mathcal{O}}} := \mathbf{D}_{\widehat{\mathcal{O}}} \setminus \{\langle u,v\rangle\} = \mathrm{Spec}\ (\widehat{\mathcal{O}}[\![v]\!]) \setminus \{\langle u,v\rangle\}.$$

The v-*divisor* Δ_v is the closed subscheme of $\mathbf{D}_{\widehat{\mathcal{O}}}$ with defining ideal $\langle v \rangle$,

$$\Delta_v := \text{Spec } (\widehat{\mathcal{O}} [\![v]\!] / \langle v \rangle).$$

Similarly, define Δ_v^* to be the intersection of Δ_v with $\mathbf{D}^*_{\widehat{\mathcal{O}}}$. Also for every integer N the u^N-*divisor* Δ_{u^N} is the closed subscheme of $\mathbf{D}_{\widehat{\mathcal{O}}}$ with defining ideal $\langle u^N \rangle$,

$$\Delta_{u^N} = \text{Spec } (\widehat{\mathcal{O}} [\![v]\!] / \langle u^N \rangle).$$

Similarly, define $\Delta^*_{u^N}$ to be the intersection of Δ_{u^N} with $\mathbf{D}^*_{\widehat{\mathcal{O}}}$.

Observe that Δ_v equals Spec $\widehat{\mathcal{O}}$ and Δ_v^* equals Spec Frac$(\widehat{\mathcal{O}})$ as $\widehat{\mathcal{O}}$-schemes. Also Δ_{u^N} equals Spec $(\widehat{\mathcal{O}}/u^N) [\![v]\!]$ and $\Delta^*_{u^N}$ equals Spec $(\widehat{\mathcal{O}}/u^N) (\!(v)\!)$ as $\widehat{\mathcal{O}}$-schemes. In particular, the restriction of $\widehat{s}$ to Frac$(\widehat{\mathcal{O}})$ gives an $\widehat{\mathcal{O}}$-morphism from Δ_v^* to $\mathcal{Y}$. And the restriction of $\widehat{t}$ to $\widehat{\mathcal{O}}/u^N$ gives an $\widehat{\mathcal{O}}$-morphism from $\Delta^*_{u^N}$ to $\mathcal{Y}$ (which happens to be constant in v).

defn-psiR

Definition 4.1. Let $\widehat{s}$ be a $\widehat{\mathcal{O}}$-point of $\mathcal{Y}$ and let $\widehat{t}_N$ be a $\widehat{\mathcal{O}}/u^N$-point of $\mathcal{Y}$. An N-*jet specialization* of $\widehat{t}_N$ to $\widehat{s}$ is an $\widehat{\mathcal{O}}$-morphism

$$r : \mathbf{D}^*_{\widehat{\mathcal{O}}} \to \mathcal{Y}$$

such that $r|_{\Delta_v^*}$ equals the restriction of $\widehat{s}$ to Spec Frac$(\widehat{\mathcal{O}})$ and $r|_{\Delta^*_{u^N}}$ equals the base change of $\widehat{t}_N$. When t_N equals the restriction of $\widehat{t}$ to $\widehat{\mathcal{O}}/u^N$, we say that $\widehat{t}$ N-*jet specializes* to $\widehat{s}$.

The $\widehat{\mathcal{O}}$-point $\widehat{t}$ *directly pseudo specializes* to $\widehat{s}$ if for every nonnegative integer N, $\widehat{t}$ N-jet specializes to $\widehat{s}$. The $\widehat{\mathcal{O}}$-point $\widehat{t}$ *pseudo specializes* to $\widehat{s}$ if there exists a sequence of $\widehat{\mathcal{O}}$-points $\widehat{t} = \widehat{t}_0, \dots, \widehat{t}_n = \widehat{s}$ such that for every $i = 1, \dots, n$, $\widehat{t}_{i-1}$ directly pseudo specializes to $\widehat{t}_i$. Elements $\widehat{s}$ and $\widehat{t}$ are *pseudo R-equivalent*, resp. *directly pseudo R-equivalent*, if each element pseudo specializes to the other, resp. directly pseudo specializes to the other. If every pair of $\widehat{\mathcal{O}}$-points of $\mathcal{Y}$ are pseudo R-equivalent, then $\mathcal{Y}$ is *pseudo R-connected*.

Basic results. Although it is defined in terms of $\widehat{\mathcal{O}}$-schemes, in fact pseudo R-specialization depends only on the generic fiber, i.e., the fiber over Frac$(\widehat{\mathcal{O}})$.

lem-model

Lemma 4.2. *Let $f : \mathcal{Y}' \to \mathcal{Y}$ be a morphism of flat, quasi-projective $\mathcal{O}$-schemes. Let $\widehat{s}'$ and $\widehat{t}'$ be $\widehat{\mathcal{O}}$-points of $\mathcal{Y}'$ mapping to $\widehat{s}$ and $\widehat{t}$.*

- (i) *Every N-jet specialization of $\widehat{t}'$ to $\widehat{s}'$ in $\mathcal{Y}'$ maps to an N-jet specialization of $\widehat{t}$ to $\widehat{s}$ in $\mathcal{Y}$. Thus if $\widehat{t}'$ directly pseudo specializes to $\widehat{s}'$ in $\mathcal{Y}'$, then $\widehat{t}$ directly pseudo specializes to $\widehat{s}$ in $\mathcal{Y}$.*
- (ii) *Assume that f is projective and is an isomorphism on Frac$(\widehat{\mathcal{O}})$-fibers. Then there exists a nonnegative integer c such that for every nonnegative integer N, every $(N+c)$-jet specialization of $\widehat{t}$ to $\widehat{s}$ in $\mathcal{Y}$ is the image of an N-jet specialization of $\widehat{t}'$ to $\widehat{s}'$ in $\mathcal{Y}'$. Thus if $\widehat{t}$ directly pseudo specializes to $\widehat{s}$ in $\mathcal{Y}$, then $\widehat{t}'$ directly pseudo specializes to $\widehat{s}'$ in $\mathcal{Y}'$.*
- (iii) *If $\widehat{t}$ and $\widehat{s}$ are directly R-equivalent, then they are directly pseudo R-equivalent.*

Proof. Item (i) is obvious. Next assume the hypotheses in Item (ii). Let

$$r : \mathbf{D}^*_{\widehat{\mathcal{O}}} \to \mathcal{Y}$$

be an M-jet specialization of $\widehat{t}$ to $\widehat{s}$. By the valuative criterion of properness there exists a unique $\widehat{\mathcal{O}}$-morphism

$$r' : \mathbf{D}^*_{\widehat{\mathcal{O}}} \to \mathcal{Y}'$$

such that $f \circ r'$ equals r. Since f is an isomorphism on $\text{Frac}(\widehat{\mathcal{O}})$-fibers and since $r|_{\Delta^*_v}$ equals $\widehat{s}$, also $r'|_{\Delta^*_v}$ equals $\widehat{s}'$. It remains to prove that there exists an integer c such that for every nonnegative integer N, $r'|_{\Delta^*_{u^N}}$ equals the restriction of $\widehat{t}'$ if $r|_{\Delta^*_{u^{N+c}}}$ equals the restriction of $\widehat{t}$.

This is easiest to see in coordinates. Let V be an open affine in $\mathcal{Y}$ containing $\widehat{t}$ and let V' be an open affine in $f^{-1}(V)$ containing $\widehat{t}'$. Let $y_1, \dots, y_l$ be generators for $\Gamma(V', \mathcal{O}_{V'})$ as a $\widehat{\mathcal{O}}$-algebra such that the ideal of $\widehat{t}'$ is $\langle y_1, \dots, y_l \rangle$. There exists an integer c such that $u^c y_i$ is in $\Gamma(V, \mathcal{O}_V)$ for every $i = 1, \dots, l$. And then $u^c y_i$ is in the ideal of $\widehat{t}$ for every i. Let τ be a $\widehat{\mathcal{O}}$-point of V'. If $(f \circ \tau)^*(u^c y_i)$ is divisible by u^{N+c}, then $\tau^*(y_i)$ is divisible by u^{N+c}. Thus if $f \circ \tau$ agrees with $\widehat{t}$ to order $N + c$, then τ agrees with $\widehat{t}'$ to order N. Applying this to the germ of r' at $\Delta^*_{u^N}$ shows that $r'|_{\Delta^*_{u^N}}$ equals the restriction of $\widehat{t}'$ if $r|_{\Delta^*_{u^{N+c}}}$ equals the restriction of $\widehat{t}$.

For (iii), first consider the special case when $\mathcal{Y}$ equals $\mathbb{P}^1_{\widehat{\mathcal{O}}}$, i.e., Proj $\widehat{\mathcal{O}}[Y_0, Y_1]$. Also assume that $\widehat{s}$, resp. $\widehat{t}$, equals $[1, 0]$, resp. $[0, 1]$. For every nonnegative integer N, consider the morphism

$$r : \mathbf{D}^*_{\widehat{\mathcal{O}}} \to \mathbb{P}^1_{\widehat{\mathcal{O}}}, \quad r^*[Y_0, Y_1] = [v, u^N].$$

This is clearly an N-jet specialization of $\widehat{t}$ to $\widehat{s}$. Thus $\widehat{t}$ directly pseudo specializes to $\widehat{s}$. By symmetry, $\widehat{t}$ and $\widehat{s}$ are directly pseudo R-equivalent.

Next consider the case when $\mathcal{Y}$ is an $\widehat{\mathcal{O}}$-curve P whose generic fiber $P_{\text{Frac}(\widehat{\mathcal{O}})}$ is isomorphic to $\mathbb{P}^1_{\text{Frac}(\widehat{\mathcal{O}})}$. Any two distinct rational points of $\mathbb{P}^1_{\text{Frac}(\widehat{\mathcal{O}})}$ are projectively equivalent to $[1, 0]$ and $[0, 1]$. So by the previous paragraph and by (ii), any two $\widehat{\mathcal{O}}$-points of P are directly pseudo R-equivalent.

Finally consider the case of general $\mathcal{Y}$. Let $\widehat{s}$ and $\widehat{t}$ be $\widehat{\mathcal{O}}$-points which are directly R-equivalent, i.e., which are the images under a $\widehat{\mathcal{O}}$-morphism

$$f : P \to \mathcal{Y}$$

of $\widehat{\mathcal{O}}$-points of P, where P is a $\widehat{\mathcal{O}}$-curve as in the previous paragraph. By the previous paragraph, every pair of $\widehat{\mathcal{O}}$-points of P are directly pseudo R-equivalent. Thus, by (i), $\widehat{s}$ and $\widehat{t}$ are directly pseudo R-equivalent in $\mathcal{Y}$. □

To prove the relation between pseudo specialization and Brauer equivalence, it is useful to prove a lemma about extending Brauer classes. We believe this is well-known; we include the proof to illustrate the technique.

lem-unramified

Lemma 4.3. *Let $\widehat{t}$ be an $\widehat{\mathcal{O}}$-point of $\mathcal{Y}$. Let α be a Brauer class on the generic fiber $\mathcal{Y}_{Frac(\widehat{\mathcal{O}})}$ such that the pullback $\widehat{t}^*\alpha$ on Frac$(\widehat{\mathcal{O}})$ extends to all of $\widehat{\mathcal{O}}$. Then there exists a projective, birational morphism $f : \mathcal{Y}' \to \mathcal{Y}$, an $\widehat{\mathcal{O}}$-point $\widehat{t}'$ of $\mathcal{Y}'$, and an open subscheme U of $\mathcal{Y}'$ containing the generic fiber and containing Image$(\widehat{t}')$, such that α extends to all of U.*

Proof. There is a first reduction. Denote by $p : \mathcal{Y} \to \operatorname{Spec} \widehat{\mathcal{O}}$ the "structure morphism". Consider $\beta = \alpha - p^*\widehat{t}^*\alpha$. Since $p^*\widehat{t}^*\alpha$ is defined on all of $\mathcal{Y}$, for a morphism f and open set U as above, $f^*\alpha$ extends to all of U if and only if $f^*\beta$ extends to all of U. So, up to replacing α by β, we may assume that $\widehat{t}^*\alpha$ equals 0.

Since $\mathcal{Y}_{\operatorname{Frac}(\widehat{\mathcal{O}})}$ is smooth and quasi-projective, there exists an integer n and a Brauer-Severi variety

$$\pi : \mathcal{P} \to \mathcal{Y}_{\operatorname{Frac}(\widehat{\mathcal{O}})}$$

with geometric fibers $\mathbb{P}^{n-1}$ such that the associated Brauer class is α. Since $\widehat{t}^*\alpha$ equals 0, $\widehat{t}^*\mathcal{P}$ is isomorphic to $\mathbb{P}^{n-1}_{\operatorname{Frac}(\widehat{\mathcal{O}})}$.

Denote by ω_π the relative dualizing sheaf. The associated sheaf $\omega_\pi^\vee$ is π-relatively very ample and $\pi_*(\omega_\pi^\vee)$ is locally free of rank $\binom{2n}{n}$. Since Brauer classes on a regular scheme satisfy glueing for Zariski open covers, it suffices to prove the existence of f and U after replacing $\mathcal{Y}$ by any open affine neighborhood of $\widehat{t}$. So without loss of generality assume that $\mathcal{Y}$ is an affine neighborhood on which $\pi_*(\omega_\pi^\vee)$ is trivialized. More precisely, choose a free basis $s_1, \dots, s_M$ for $H^0(\mathbb{P}^{n-1}_{\mathbb{Z}}, \omega^\vee_{\mathbb{P}^{n-1}_{\mathbb{Z}}})$ and then choose a trivialization

$$\phi : \mathcal{O}^{\oplus M}_{\mathcal{Y}_{\operatorname{Frac}(\widehat{\mathcal{O}})}} \to \pi_*(\omega_\pi^\vee)$$

whose pullback by $\widehat{t}$ has components $(s_1, \dots, s_M)$. Since $\omega_\pi^\vee$ is π-relatively very ample, this trivialization induces a closed immersion of $\mathcal{Y}$-schemes,

$$\iota : \mathcal{P} \hookrightarrow \mathbb{P}^{M-1}_{\mathcal{Y}_{\operatorname{Frac}(\widehat{\mathcal{O}})}}.$$

Denote by $h(d)$ the numerical polynomial

$$h(d) := \binom{n(d+1)}{n}.$$

The closed immersion ι determines a $\widehat{\mathcal{O}}$-morphism to the Hilbert scheme

$$\widetilde{\iota} : \mathcal{Y}_{\operatorname{Frac}(\widehat{\mathcal{O}})} \to \operatorname{Hilb}^{h(d)}_{\mathbb{P}^{M-1}_{\mathcal{Y}}/\mathcal{Y}}.$$

As $\operatorname{Hilb}^{h(d)}_{\mathbb{P}^{M-1}_{\mathcal{Y}}/\mathcal{Y}}$ is projective over $\mathcal{Y}$, there exists a projective, birational morphism

$$f : \mathcal{Y}' \to \mathcal{Y}$$

such that the pullback of $\mathcal{P}$ over $\mathcal{Y}'$ extends to a $\mathcal{Y}'$-flat closed subscheme

$$\mathcal{P}' \hookrightarrow \mathbb{P}^{M-1}_{\mathcal{Y}'}.$$

There exists an open subscheme U of $\mathcal{Y}'$ such that for every $\mathcal{Y}'$-scheme T, the pullback of $\mathcal{P}'$ to T is smooth over T if and only if T factors through U; for the projection $\pi' : \mathcal{P}' \to \mathcal{Y}'$, U is the complement of the image under π' of the singular locus of π'. By construction, $\widetilde{\iota} \circ \widehat{t}|_{\operatorname{Frac}(\widehat{\mathcal{O}})}$ is the constant morphism corresponding to the anticanonical embedding of $\mathbb{P}^{n-1}$ in $\mathbb{P}^{M-1}$ by the basis $[s_1, \dots, s_M]$. Thus the restriction of $\mathcal{P}'$ to the strict transform $\widehat{t}'$ of $\widehat{t}$ in $\mathcal{Y}'$ equals the constant family $[s_1, \dots, s_M](\mathbb{P}^{n-1})$ over the dense open subscheme $\operatorname{Spec} \operatorname{Frac}(\widehat{\mathcal{O}})$ of $\operatorname{Spec} \widehat{\mathcal{O}}$. By separatedness of the Hilbert scheme, the restriction of $\mathcal{P}'$ over all of $\widehat{t}'(\operatorname{Spec} \widehat{\mathcal{O}})$ equals the constant family. In particular, $\widehat{t}'(\operatorname{Spec} \widehat{\mathcal{O}})$ is contained in U.

The restriction of $\mathcal{P}'$ over U is a Brauer-Severi scheme whose associated Brauer class extends α. $\square$

As promised, pseudo specialization implies Brauer equivalence.

prop-Brauereq

Proposition 4.4. *Assume that $\widehat{\mathcal{O}}$ is Henselian. If $\widehat{t}$ pseudo specializes to $\widehat{s}$, then the pullback maps,*

$$\widehat{t}^*, \widehat{s}^* : Br(\mathcal{Y}_{Frac(\widehat{\mathcal{O}})}) \to Br(Frac(\widehat{\mathcal{O}}))$$

are equal.

Proof. Of course it suffices to verify this when $\widehat{t}$ directly pseudo specializes to $\widehat{s}$. The goal is to prove that the difference $\widehat{t}^* - \widehat{s}^*$ equals the zero map, i.e., the kernel is the entire Brauer group. Of course the kernel contains the pullback $p^*\mathrm{Br}(\mathrm{Frac}(\widehat{\mathcal{O}}))$ since $\widehat{t}^*p^* = (p \circ \widehat{t})^* = \mathrm{Id}^*$ and likewise for $\widehat{s}^*p^*$. And the sum of $p^*\mathrm{Br}(\mathrm{Frac}(\widehat{\mathcal{O}}))$ and $\mathrm{Ker}(\widehat{t}^*)$ is the entire Brauer group. Thus it suffices to prove that the kernel of $\widehat{t}^*$ is contained in the kernel of $\widehat{s}^*$.

To this end, let α be a Brauer class on $\mathcal{Y}_{\mathrm{Frac}(\widehat{\mathcal{O}})}$ such that $\widehat{t}^*\alpha$ equals 0. By Lemma 4.3, there exists a projective, birational morphism $f : \mathcal{Y}' \to \mathcal{Y}$ and an open subscheme U of $\mathcal{Y}'$ containing both the generic fiber and the strict transform $\widehat{t}'$ of $\widehat{t}$ such that α extends to U. Denote by $\widehat{s}'$ the strict transform of $\widehat{s}$. By Lemma 4.2(ii), $\widehat{t}'$ directly pseudo specializes to $\widehat{s}'$. In particular, there exists a 1-jet specialization

$$r : \mathbf{D}^*_{\widehat{\mathcal{O}}} \to \mathcal{Y}'$$

of $\widehat{t}'$ to $\widehat{s}'$. Since U contains both the generic fiber and the image of $\widehat{t}'$, U also contains the image of r. Consider $r^*\alpha$. By purity for Brauer classes, cf. [Gro68, Théorème 6.1(b), Corollaire 6.2], $r^*\alpha$ extends to a Brauer class γ defined on all of $\mathbf{D}_{\widehat{\mathcal{O}}}$.

Consider first the restriction of γ to $\Delta_{u^1} = \mathrm{Spec}\ (\widehat{\mathcal{O}}/u)\,[\![v]\!]$. Since $(\widehat{\mathcal{O}}/u)\,[\![v]\!]$ is Henselian, every Brauer class γ on $(\widehat{\mathcal{O}}/u)\,[\![v]\!]$ is the pullback of a Brauer class γ_0 on $\widehat{\mathcal{O}}/u$. But since $r|_{\Delta^*_{u^1}}$ equals the restriction of $\widehat{t}'$, and since α is trivial on $\widehat{t}'$, the restriction of γ to $(\widehat{\mathcal{O}}/u)\,(\!(v)\!)$ equals 0. Therefore γ_0 equals 0.

Consider next the restriction of γ to $\Delta_v = \mathrm{Spec}\ (\widehat{\mathcal{O}})$. The further restriction to the closed point is γ_0, which equals 0 by the previous paragraph. Since $\widehat{\mathcal{O}}$ is Henselian, a Brauer class on $\widehat{\mathcal{O}}$ which is trivial on the closed point is trivial on all of $\widehat{\mathcal{O}}$. Thus the restriction of γ to Δ_v equals 0. In other words, $\widehat{s}^*(\alpha)$ equals 0. Therefore the kernel of $\widehat{t}*$ is contained in the kernel of $\widehat{s}^*$. $\square$

Relation to weak approximation. Let k be an algebraically closed field. Let $\pi : \mathcal{X} \to B$ be a flat, proper k-morphism from an algebraic space $\mathcal{X}$ to a smooth, projective k-curve B. Let b be a k-point of B. Denote by $\widehat{\mathcal{O}}$ the completion of the stalk of $\mathcal{O}_B$ at b. And denote by $\mathcal{X}_{\widehat{\mathcal{O}}}$ the base change of $\mathcal{X}$ to $\widehat{\mathcal{O}}$. A $\widehat{\mathcal{O}}$-point $\widehat{s}$ of $\mathcal{X}_{\widehat{\mathcal{O}}}$ is *approximable* if for every integer N there exists a section s_N of π such that $s_N \cong \widehat{s}$ modulo $\mathfrak{m}^N$. If each s_N intersects the very free locus of π, then $\widehat{s}$ is *approximable by sections intersecting the very free locus.*

thm-second

Theorem 4.5. *Assume that $\mathcal{X}_{\widehat{\mathcal{O}}}$ is regular. Let $\widehat{s}$ and $\widehat{t}$ be $\widehat{\mathcal{O}}$-points of $\mathcal{X}_{\widehat{\mathcal{O}}}$. If $\widehat{t}$ pseudo specializes to $\widehat{s}$, and if $\widehat{s}$ is approximable by sections intersecting the very free locus, then also $\widehat{t}$ is approximable by sections intersecting the very free locus.*

Of course one can deduce an analogue using a larger number of closed points by the Weil restriction technique of de Jong, cf. [GHS03]. Sections of π are parameterized by an algebraic space which is locally of finite type over k. But $\widehat{\mathcal{O}}$-points of $\mathcal{X}_{\widehat{\mathcal{O}}}$ are not parameterized by an object which is locally of finite type over k. An important step in the proof of Theorem 4.5, and hence Theorem 1.3, is the introduction of a new parameter space which is locally of finite type over k and which serves a similar role to the set of $\widehat{\mathcal{O}}$-points of $\mathcal{X}_{\widehat{\mathcal{O}}}$. This is done in the next two sections.

[CONTINUE HERE: From this point on remains to be written / revised.]

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