HIGHER FANO MANIFOLDS AND RATIONAL SURFACES

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ABSTRACT. Let X be a Fano manifold of pseudo-index ≥ 3 such that $c_1(X)^2 - 2c_2(X)$ is nef. Irreducibility of some spaces of rational curves on X implies a general point of X is contained in a rational surface.

1. INTRODUCTION

One consequence of the bend-and-break lemma is uniruledness of Fano manifolds, [MM86]. In characteristic 0, Fano manifolds are even rationally connected, [KMM92], [Cam92]. We prove an analogous theorem with rational curves replaced by rational surfaces for Fano manifolds satisfying positivity of the second graded piece of the Chern character.

Definition 1.1. A Fano manifold is 2-*Fano* if $\mathrm{ch}_2(T_X)$ is nef, where $\mathrm{ch}_2(T_X)$ is the second graded piece of the Chern character $\frac{1}{2}(c_1(T_X)^2 - 2c_2(T_X))$. In other words, $\deg(\mathrm{ch}_2(T_X)|_S)$ is nonnegative for every surface S in X.

There is a space $\overline{\mathcal{M}}_{0,0}(X)$ parametrizing genus 0 *stable maps* to X. These are isomorphism classes of pairs (C, f) of a proper, at-worst-nodal, arithmetic genus 0 curve C and a morphism $f : C \to X$ such that every irreducible component of C on which f is constant intersects at least three other components. To be precise, $\overline{\mathcal{M}}_{0,0}(X)$ is an algebraic stack with finite diagonal (it is Deligne-Mumford in characteristic 0). It is proper and contains an open substack that is isomorphic to the Hilbert scheme of smooth, embedded rational curves in X, cf. [FP97].

Let $\mathcal{M}$ be a positive-dimensional, irreducible component of $\overline{\mathcal{M}}_{0,0}(X)$ whose general point parametrizes a stable map with irreducible domain, i.e., a morphism from $\mathbb{P}^1$ to X. Denote by M the coarse moduli space of $\mathcal{M}$. Denote by Δ the locally principal closed substack of $\overline{\mathcal{M}}_{0,0}(X)$ parametrizing stable maps with reducible domain. The closed substack $\mathcal{M} \cap \Delta$ is a Cartier divisor. The question we consider is uniruledness of M.

Theorem 1.2. *If X is 2-Fano, every point of M parametrizing a free curve and contained in a proper curve in $M - M \cap \Delta$ is contained in a rational curve in M.*

If a general point of M parametrizes a birational, free curve and is contained in a proper curve in $M - M \cap \Delta$, then a general point of X is contained in a rational surface.

For a proper, smooth curve C in $M - M \cap \Delta$ the deformation theory of $\mathrm{Hom}(C, M)$ is fairly simple and gives a lower bound on the dimension of $\mathrm{Hom}(C, M)$, cf. Lemma 2.2. Using the hypothesis that X is 2-Fano, this lower bound can be made arbitrarily large. Then the bend-and-break approach of [MM86] implies M contains a rational curve through every point of C.

Fortunately there are nice sufficient conditions for $M - M \cap \Delta$ to contain many proper curves. Recall the *pseudo-index* of X is defined to be the minimal $c_1(T_X)$-degree of any rational curve in X.

Proposition 1.3. *If the pseudo-index of X is ≥ 3 and every irreducible component of $\mathcal{M} \cap \Delta$ is an irreducible component of Δ, then $M - M \cap \Delta$ is a union of proper curves.*

If the pseudo-index of X is ≥ 3, then every free morphism deforms to a birational, free morphism.

The proof uses a contraction of the locally principal closed subspace Δ in $\overline{\mathcal{M}}_{0,0}(X)$ discovered independently by several groups: Coskun, Harris and one of us, [CHS05]; Adam Parker, [Par05]; and Mustaţă and Mustaţă, unpublished (but see [MM04]).

Section 3 gives some examples where the hypotheses of Theorem 1.2 and Proposition 1.3 hold. Section 4 shows Theorem 1.2 is sharp in two ways. First, there are Fano manifolds that are not 2-Fano where the components $\mathcal{M}$ are not uniruled. Second, there are 2-Fano manifolds where the components $\mathcal{M}$ are uniruled but not rationally connected. Finally Section 5 speculates on sufficient conditions for the components $\mathcal{M}$ to be rationally connected.

2. Proof of the theorem

For every point x, denote by $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)_{\mathrm{nc}}$ the open subscheme of $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ parametrizing nonconstant morphisms.

Lemma 2.1. *The dimension of every irreducible component of $Hom(\mathbb{P}^1, X, 0 \mapsto x)_{nc}$ is at least as large as the pseudo-index of X.*

Proof. This follows from [Kol96, Theorem II.1.2, Corollary II.1.6]. □

Proof of Proposition 1.3. First of all, the statement that a general deformation of a free morphism is birational is very similar to [Kol96, Thm. II.3.14]. Let $\mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1 \xrightarrow{h} X$ be a factorization of a free morphism $f = h \circ g$ where g has degree $m \geq 2$. Since $g^*h^*T_X$ is ample, also h^*T_X is ample. Thus h is free. So the dimension of $\overline{\mathcal{M}}_{0,0}(X)$ at $[h]$ is the "expected dimension", namely $\deg(h^*c_1(T_X)) + \dim(X) - 3$. The dimension of the stack of m-fold covers of deformations of h is

$$\deg(h^*c_1(T_X)) + \dim(X) - 3 + 2(m-1).$$

On the other hand, the dimension of $\overline{\mathcal{M}}_{0,0}(X)$ at $[f]$ is

$$\deg(f^*c_1(T_X)) + \dim(X) - 3 = m\deg(h^*c_1(T_X)) + \dim(X) - 3.$$

The difference of the first dimension from the second dimension is,

$$(m-1)(\deg(h^*c_1(T_X)) - 2).$$

Thus, if $\deg(h^*c_1(T_X)) \geq 3$ then a general deformation of f is not an m-fold cover of a deformation of h.

Next is the existence of proper curves in $M - M \cap \Delta$. Let $f : X \hookrightarrow \mathbb{P}^r$ be a plurianticanonical embedding. Denote by $\overline{\mathcal{M}}_{0,0}(f) : \overline{\mathcal{M}}_{0,0}(X) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r)$ the associated embedding. Denote by $\phi : \overline{\mathrm{M}}_{0,0}(\mathbb{P}^r) \to Y$ the contraction of the boundary constructed in [CHS05]. All that we will use about ϕ is the following.

(i) Every connected component of Y is a projective scheme.
(ii) The restriction of ϕ to the open subset $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r) - \Delta$ is an open immersion.

(iii) For each pair of integers $i \leq j$, the composition of ϕ with

$$\Delta_{i,j} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, i) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, j) \twoheadrightarrow \Delta \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r)$$

factors through projection

$$\pi_j : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, i) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, j) \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, j).$$

Denote by N the image of M in Y. Since the restriction of ϕ to $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r) - \Delta$ is an open immersion, the restriction of $\phi \circ \overline{\mathcal{M}}_{0,0}(f)$ to $M - \Delta$ is an immersion. Since $\mathcal{M}_{\text{free}}$ is dense in $\mathcal{M}$, M has pure dimension equal to the expected dimension, and $\mathcal{M} \cap \Delta$ is a Cartier divisor. Therefore $\dim(N)$ equals $\dim(M)$ and $\dim(\mathcal{M} \cap \Delta)$ equals $\dim(M) - 1$.

Denote $\Delta_{i,j} \cap \overline{\mathcal{M}}_{0,0}(X)$ by $\Delta_{X,i,j}$. Denote the restriction of π_j by $\pi_{X,j} : \Delta_{X,i,j} \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, j)$. By Lemma 2.1, every irreducible component of every fiber of $\pi_{X,j}$ has dimension ≥ 1, i.e., the difference of the pseudo-index and $\dim(\mathrm{Aut}(\mathbb{P}^1, 0))$. Therefore, for every irreducible component Δ' of Δ, the dimension of $\phi(\overline{\mathcal{M}}_{0,0}(f)(\Delta'))$ is strictly less than the dimension of Δ'. By hypothesis, every irreducible component Δ' of $\mathcal{M} \cap \Delta$ is an irreducible component of Δ. Since $\dim(\Delta')$ equals $\dim(\mathcal{M}) - 1$, the image of Δ' in N has dimension $\leq \dim(N) - 2$.

Since every connected component of Y is projective, also N is projective. Because $\dim(\mathrm{Image}(\Delta')) \leq \dim(N) - 2$, a general intersection of N with $\dim(N) - 1$ hyperplanes containing a point of $N - \mathrm{Image}(\Delta')$ is a complete curve that does not intersect $\mathrm{Image}(\Delta')$. Because there are only finitely many irreducible components of $\mathcal{M} \cap \Delta$, a general intersection of N with $\dim(N) - 1$ hyperplanes containing a point of $N - \mathrm{Image}(\mathcal{M} \cap \Delta)$ is a complete curve that does not intersect $\mathrm{Image}(\mathcal{M} \cap \Delta)$. The inverse image of this curve in $\mathcal{M} - \mathcal{M} \cap \Delta$ is a complete curve containing a given point of $\mathcal{M} - \mathcal{M} \cap \Delta$. To be completely precise, this curve is actually a proper, 1-dimensional stack. But every such stack is the image of a finite 1-morphism whose domain is a proper, smooth curve. □

Let C be a smooth, proper, connected curve and let $\zeta : C \to \overline{\mathcal{M}}_{0,0}(X) - \Delta$ be a nonconstant 1-morphism whose general point parametrizes a free curve of $(-K_X)$-degree e. This is equivalent to a pair $(\pi : \Sigma \to C, F : \Sigma \to X)$ of a ruled surface Σ over C and a morphism $F : \Sigma \to X$ mapping fibers of π to curves in X of $(-K_X)$-degree e.

Lemma 2.2. *The dimension at $[\zeta]$ of $Hom(C, \overline{\mathcal{M}}_{0,0}(X), \zeta|_B)$ is at least,*

$$deg(ch_2(T_X)|_{F(\Sigma)}) + \frac{1}{2e} deg(c_1(T_X)^2|_{F(\Sigma)}) - (e + dim(X) - 3)(g(C) - 1 + \#(B)).$$

Proof. The argument will use Riemann-Roch on Σ. Before beginning, we recall one useful fact about the Chow ring of the ruled surface Σ. Tsen's theorem implies existence of a section of Σ, i.e., a morphism $\sigma : C \to \Sigma$ such that $\pi \circ \sigma = \mathrm{Id}_C$. Every element in the Picard group of Σ has the form

$$\alpha = e[\sigma(C)] + \pi^* D$$

for a unique integer e and a unique divisor class D on C. On the one hand,

$$-e(\omega_\pi \cdot \alpha) = -e^2(\omega_\pi \cdot [\sigma(C)]) + 2e\deg(D).$$

On the other hand,

$$\alpha \cdot \alpha = e^2([\sigma(C)] \cdot [\sigma(C)]) + 2e\deg(D).$$

Finally, by the adjunction formula

$$\omega_\pi \cdot [\sigma(C)] = -[\sigma(C)] \cdot [\sigma(C)].$$

Altogether this gives the following equation for every divisor class α,

$$-e(\omega_\pi \cdot \alpha) = \alpha \cdot \alpha. \tag{1}$$

The particular case $\alpha = \omega_\pi$ gives,

$$\omega_\pi \cdot \omega_\pi = 0.$$

To resume the proof, the deformation theory of $\mathrm{Hom}(C, \overline{\mathcal{M}}_{0,0}(X), \zeta_B)$ at $[\zeta]$ is the same as the theory of deformations of the finite morphism

$$(\pi, F) : \Sigma \to C \times X$$

holding C, X and $(\pi, F)|_{\pi^{-1}B}$ fixed. The deformation theory of such morphisms is controlled by the normal sheaf $\mathcal{N}$ of (π, F),

$$\mathcal{N} := \mathrm{Coker}(T_\Sigma \xrightarrow{d(\pi,F)} \pi^* T_C \oplus F^* T_X).$$

To be precise, the Zariski tangent space to $\mathrm{Hom}(C, \overline{\mathcal{M}}_{0,0}(X), \zeta|_B)$ at $[\zeta]$ is isomorphic to

$$H^0(C, \mathcal{N}(-\pi^{-1}B))$$

and

$$\dim_{[\zeta]} \mathrm{Hom}(C, \overline{\mathcal{M}}_{0,0}(X), \zeta_B) \geq h^0(C, \mathcal{N}(-\pi^{-1}B)) - h^1(C, \mathcal{N}(-\pi^{-1}B)).$$

For an excellent description of the infinitesimal theory of Hilbert schemes leading to this formula we recommend [Kol96, Chapter I].

By the Leray spectral sequence

$$h^2(\Sigma, \mathcal{N}(-\pi^{-1}B)) = h^1(C, R^1\pi_*\mathcal{N}(-B)).$$

Because a general point of C parametrizes a free curve, the restriction of $\mathcal{N}$ to a general fiber of π is generated by global sections; thus it has no higher cohomology. Therefore $R^1\pi_*\mathcal{N}$ is a torsion sheaf on the curve C, which also has no higher cohomology:

$$h^1(C, R^1\pi_*\mathcal{N}(-B)) = 0.$$

Because $h^2(\Sigma, \mathcal{N}(-\pi^{-1}B)) = 0$,

$$h^0(\mathcal{N}(-\pi^{-1}B)) - h^1(\mathcal{N}(-\pi^{-1}B)) = \chi(\Sigma, \mathcal{N}(-\pi^{-1}B)).$$

Riemann-Roch gives

$$\chi(\Sigma, \mathcal{N}(-\pi^{-1}B)) = \deg(\mathrm{ch}(\mathcal{N}) \cdot \pi^*\mathrm{ch}(\mathcal{O}_C(-B)) \cdot \mathrm{Todd}(\Sigma)).$$

There are a couple of observations. First,

$$\mathrm{ch}(\mathcal{N}) = \mathrm{ch}(F^*T_X) - \mathrm{ch}(T_\pi) = F^*\mathrm{ch}(T_X) - (1 - \omega_\pi + \omega_\pi^2/2).$$

Second, since the Todd class is multiplicative,

$$\mathrm{Todd}(\Sigma) = \mathrm{Todd}(\pi)\pi^*\mathrm{Todd}(C) = (1 - \omega_\pi/2 + \omega_\pi^2/12)(1 - \pi^*\omega_C/2).$$

Combined with the contribution from $-\pi^{-1}B$, this gives,

$$\pi^*\mathrm{ch}(\mathcal{O}_C(-B))\mathrm{Todd}(\Sigma) = (1 - \omega_\pi/2 + \omega_\pi^2/12)(1 - \pi^*\omega_C/2 - \pi^*B)$$

which reduces to

$$\pi^*\mathrm{ch}(\mathcal{O}_C(-B))\mathrm{Todd}(\Sigma) = 1 - (\omega_\pi + \pi^*\omega_C + 2\pi^*B)/2 - (g(C) - 1 + \#B)[\mathrm{point}]$$

because $\omega_\pi^2 = 0$. Therefore $\chi(\Sigma, \mathcal{N}(-\pi^{-1}B))$ equals

$$\deg(\mathrm{ch}_2(T_X)|_{F(\Sigma)}) - \deg(\omega_\pi \cdot g^* c_1(T_X)) - (g(C) - 1 + \#B)(e + \dim(X) - 3).$$

Finally, applying Equation 1 to $\alpha = F^* c_1(T_X)$ gives

$$-\omega_\pi \cdot F^* c_1(T_X) = \frac{1}{2e} F^* c_1(T_X)^2.$$

$\square$

Proof of Theorem 1.2. Every proper curve in $M - M \cap \Delta$ is the image of a non-constant 1-morphism $\zeta : C \to \mathcal{M} - \mathcal{M} \cap \Delta$ from a smooth curve C. The induced morphism $\mathrm{Hom}(C, \mathcal{M} - \mathcal{M} \cap \Delta) \to \mathrm{Hom}(C, M)$ is finite. By Lemma 2.2, $\dim(\mathrm{Hom}(C, M; \zeta|_B))$ behaves as if M is smooth along the image of ζ and the anticanonical degree of $\zeta(C)$ equals

$$\deg(\mathrm{ch}_2(T_X)|_{F(\Sigma)}) + \frac{1}{2e} \deg(c_1(T_X)^2|_{F(\Sigma)}).$$

Because X is 2-Fano, this degree is positive. Therefore the usual bend-and-break argument applies, cf. [Kol96, Theorem II.5.8]. $\square$

3. Examples

We list below a number of Fano manifolds satisfying the hypotheses of Theorem 1.2 and Proposition 1.3. For further work towards classifying such manifolds, please see the note [dJS05]. But first there are a couple of observations about the conclusion of Theorem 1.2, i.e., existence of rational surfaces containing a general point. First of all, this is a birational property. Therefore it is reasonable to focus on *minimal* Fano manifolds, for instance those with Picard number 1.

The second observation has to do with specialization for flat families. Let R be a DVR whose residue field is algebraically closed of characteristic 0. Let $\mathcal{X} \to \mathrm{Spec}\ R$ be a projective smooth morphism. Denote the closed fiber by X_0 and the geometric generic fiber by $X_{\overline{\eta}}$. Even assuming a general point of $X_{\overline{\eta}}$ is contained in a rational surface, it is not obvious that a general point of X_0 is contained in a rational surface. After all rational surfaces can specialize to non-rational surfaces. As we will explain, a general point of X_0 is contained in a rational surface if Theorem 1.2 applies to $X_{\overline{\eta}}$, even if X_0 itself does not satisfy the hypotheses of Theorem 1.2. This is one advantage of our approach to studying rational surfaces on Fano manifolds.

Here is the argument. Form the relative Kontsevich space

$$\overline{\mathcal{M}}_{0,0}(\mathcal{X}/\mathrm{Spec}\ R) \to \mathrm{Spec}\ R.$$

The closed fiber is $\overline{\mathcal{M}}_{0,0}(X_0)$ and the geometric generic fiber is $\overline{\mathcal{M}}_{0,0}(X_{\overline{\eta}})$. Consider those irreducible components $\mathcal{M}$ of $\overline{\mathcal{M}}_{0,0}(\mathcal{X}/\mathrm{Spec}\ R)$ dominating Spec R, i.e., such that $\mathcal{M} \to \mathrm{Spec}\ R$ is flat. In particular, the moduli point of each free curve C in X_0 is contained in a unique irreducible component $\mathcal{M}$ dominating Spec R (in fact $\mathcal{M}$ is smooth over Spec R at $[C]$). If the geometric generic fiber of $\mathcal{M}$ is uniruled then also the closed fiber is uniruled. Therefore, if Theorem 1.2 applies to the geometric generic fiber $X_{\overline{\eta}}$, then C is contained in a rational surface in X_0 (assuming C is a free, *embedded* curve). Since this also applies to deformations of C, a general point of X_0 is contained in a rational surface. Therefore it is reasonable to focus on Fano manifolds $X_{\overline{\eta}}$ which are *general in moduli*.

We know of two infinite sets of families of 2-Fano manifolds of Picard number 1. For each of these, the hypotheses of Theorem 1.2 and Proposition 1.3 hold for every member of the family which is general in moduli (in the sense above). First, the Grassmannian $\text{Grass}(k,n)$ of k-dimensional subspaces of a fixed n-dimensional vector space is always Fano, and it is 2-Fano if and only if $k=1$, $n=2k$ or $n=2k+1$ (normalizing so that $2k \leq n$). Second, a smooth complete intersection of type $(d_1,\dots,d_c)$ in an n-dimensional weighted projective space is Fano if and only if $d_1+\cdots+d_c \leq n$. And it is 2-Fano if and only if $d_1^2+\cdots+d_c^2 \leq n+1$.

For Grassmannians, and more generally for any projective homogeneous space, Kim and Pandharipande have proved that every connected component of $\overline{\mathcal{M}}_{0,0}(X)$ is irreducible, [KP01]. Second, by the same technique as in [HRS04], for a complete intersection of type $(d_1,\dots,d_c)$ in $\mathbb{P}^n$ which is *general in moduli*, the connected components of $\overline{\mathcal{M}}_{0,0}(X)$ are irreducible provided

$$\sum_{i=1}^{c} d_i \leq \frac{n+c+1}{2}.$$

This inequality is implied by the previous inequality $\sum d_i^2 \leq n+1$. There do exist complete intersections for which the connected components are reducible, e.g., any Fano hypersurface of degree $d \geq 4$ having a conical hyperplane section. However, the argument above proves that a general point of every 2-Fano complete intersection is contained in a rational surface.

Two operations producing new 2-Fano manifolds are worthy of comment, although they do not produce minimal 2-Fano manifolds. First, every product of 2-Fano manifolds is 2-Fano. Second, let X be a smooth Fano manifold and let L be a nef invertible sheaf. The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_X \oplus L^\vee)$ is Fano if and only if $c_1(T_X)-c_1(L)$ is ample. And then it is 2-Fano if and only if $\text{ch}_2(T_X)+\frac{1}{2}c_1(L)^2$ is nef. Notice $\text{ch}_2(T_X)$ need not be nef, i.e., X need not be 2-Fano.

Other operations on Fano manifolds produce 2-Fano manifolds only under strong hypotheses. For instance, a projective bundle $\mathbb{P}(E)$ of fiber dimension ≥ 2 over a Fano manifold is Fano if E satisfies a weak version of semistability. However, if $\mathbb{P}(E)$ is 2-Fano then E satisfies a very strong version of semistability: the pullback to *every* curve in X is semistable. If X is $\mathbb{P}^n$ this condition implies $\mathbb{P}(E)$ is $\mathbb{P}^m \times \mathbb{P}^n$ for some m.

4. The theorem is sharp

The theorem is sharp in two ways. First, let X be a general cubic hypersurface in $\mathbb{P}^5$. This is Fano, but it is not 2-Fano. By the main theorem of [dJS04], there are infinitely many non-uniruled irreducible components $\mathcal{M}$ of $\overline{\mathcal{M}}_{0,0}(X)$ satisfying the hypotheses of Theorem 1.2.

Second, let Y be the $\mathbb{P}^1$-bundle over X, $Y=\mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_{\mathbb{P}^5}(-2)|_X)$. By the construction in the last section, Y is 2-Fano. Associated to the projection $\pi : Y \to X$, there is a 1-morphism $\overline{\mathcal{M}}_{0,0}(\pi) : \overline{\mathcal{M}}_{0,0}(Y) \to \overline{\mathcal{M}}_{0,0}(X)$. Let $\mathcal{N}$ be an irreducible component of $\overline{\mathcal{M}}_{0,0}(Y)$ containing a very free curve. It is easy to prove the restriction to $\mathcal{N}$ of the contraction ϕ maps the Cartier divisor $\mathcal{N}\cap\Delta$ to a subvariety of codimension ≥ 2. (However it is not true that every component of $\mathcal{N}\cap\Delta$ is a component of Δ.) Thus Theorem 1.2 implies N is uniruled. In fact, the restriction of $\overline{\mathcal{M}}_{0,0}(\pi)$ to $\mathcal{N}$ is birational to a projective bundle over the image component $\mathcal{M}$ of $\overline{\mathcal{M}}_{0,0}(X)$. Choosing $\mathcal{N}$ appropriately, $\mathcal{M}$ is one of the infinitely many non-uniruled irreducible components of $\overline{\mathcal{M}}_{0,0}(X)$. Therefore N is not rationally connected, and the MRC quotient of N is precisely M.

5. Speculation

For the counterexample Y in the previous section, $\mathrm{ch}_2(T_Y)$ is nef. But it is not "positive". It has intersection number 0 with the surface $\pi^{-1}B$ for every curve B in X. If X is a Fano manifold such that $\mathrm{ch}_2(T_X)$ has positive intersection number with every surface, is $\mathcal{M}$ rationally connected? We know no counterexample.

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