RATIONAL POINTS OF RATIONALLY SIMPLY CONNECTED VARIETIES

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Abstract. These are notes prepared for a series of lectures at the conference *Variétés rationnellement connexes: aspects géométriques et arithmétiques* of the Société Mathématique de France held in Strasbourg, France in May 2008.

Contents

1. Introduction

The goal of these notes is to present some new results proved jointly with A. J. de Jong and Xuhua He. First, an algebraic fibration over a surface has a rational section if the fiber is "rationally simply connected" and if the *elementary obstruction* vanishes. Second, this implies the split, geometric case of a conjecture of Serre, "Conjecture II" in [Ser02, p. 137]: for a connected, simply connected, semisimple algebraic group, every principal bundle for the group over a surface has a rational section. Many others have worked towards the resolution of Serre's "Conjecture II" in the geometric case and in the general case: Merkurjev and Suslin; E. Bayer and

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R. Parimala; Chernousov; and P. Gille. These results are summarized in [CTGP04, Theorem 1.2(v)]. Because of these many results, the full "Conjecture II" in the geometric case reduces to the split, geometric case, so that "Conjecture II" is now settled in the geometric case.

These notes closely follow our article [dJHS08]. But the arguments here are a bit simpler, and the hypotheses are considerably stronger (yet still verified in the application to Serre's conjecture).

These notes accompany lectures delivered at the conference *Variétés rationnellement connexes: aspects géométriques et arithmétiques* of the Société Mathématique de France held in Strasbourg, France in May 2008. In addition to the new results, the lectures also presented the proof of the Kollár-Miyaoka-Mori conjecture proved by Tom Graber, Joe Harris and the author in characteristic 0 and by A. J. de Jong and the author in arbitrary characteristic. But as there are already several expositions of that work, I will only review the main statement.

Overview of the proof. Given a smooth, projective surface S over an algebraically closed field k, there always exists a Lefschetz pencil of divisors on S. The generic fiber C of this pencil is a smooth, projective, geometrically integral curve over the function field $\kappa = k(t)$. Given a projective, flat morphism $f : X \to S$ whose geometric generic fiber is integral and rationally connected, the fiber product $X_\kappa := C \times_S X$ is a projective κ-scheme together with a projective, flat morphism of κ-schemes $\pi : X_\kappa \to C$ whose geometric generic fiber is integral and rationally connected. Since the generic of π equals the generic fiber of f, rational sections of f are really the same as rational sections of π. So it suffices to prove that π has a section.

And the morphism π has one advantage over f: the base change morphism

$$\pi \otimes \mathrm{Id} : X_\kappa \otimes_\kappa \overline{\kappa} \to C \otimes_\kappa \overline{\kappa}$$

does have a section by Theorem 2.1. By Grothendieck's work on the Hilbert scheme there exists a κ-scheme $\mathrm{Sections}(X/C/\kappa)$ parameterizing families of sections of π. The goal is to prove $\mathrm{Sections}(X/C/\kappa)$ has a κ-point, but we at least know it has a $\overline{\kappa}$-point. As with all Hilbert schemes, this is really a countable union of quasi-projective κ-schemes, $\sqcup_e \mathrm{Sections}^e(X/C/\kappa)$, where $\mathrm{Sections}^e(X/C/\kappa)$ is the open and closed subscheme parameterizing sections which have degree e with respect to some π-relatively ample invertible sheaf $\mathcal{L}$.

The basic idea is to try to prove that $\mathrm{Sections}^e(X/C/\kappa)$ has some naturally defined closed κ-subscheme which is geometrically integral and geometrically rationally connected. Then we can apply Theorem 2.1 to this closed subscheme to produce a κ-point of $\mathrm{Sections}^e(X/C/\kappa)$, which is the same as a section of π.

Of course there is an obstruction to rational connectedness of $\mathrm{Sections}^e(X/C/\kappa)$: the Abel map

$$\alpha : \mathrm{Sections}^e(X/C/\kappa) \to \mathrm{Pic}^e_{C/\kappa}$$

sending each section of π to the pullback of $\mathcal{L}$ by this section. Since there are no rational curves in the Abelian variety $\mathrm{Pic}^e_{C/\kappa}$, every rationally connected subvariety of $\mathrm{Sections}^e(X/C/\kappa)$ is contained in a fiber of α. So the idea is to prove that for e sufficiently positive, some irreducible component of the generic fiber of α is geometrically integral and geometrically rationally connected. Of course this is the same as proving that there exists an irreducible component Z_e of $\mathrm{Sections}^e(X/C/\kappa)$ such that

$$\alpha|_{Z_e} : Z_e \to \mathrm{Pic}^e_{C/\kappa}$$

is dominant with integral and rationally connected geometric generic fiber. Observe that this would be enough to conclude the existence of a section of π: there are κ-points of $\mathrm{Pic}^e_{C/\kappa}$, e.g., coming from the basepoints of the Lefschetz pencil, and the fiber of $\alpha|_{Z_e}$ over these κ-points is then a geometrically integral and rationally connected variety defined over $\kappa = k(t)$. Such a variety has a κ-point by Theorem 2.1.

There are some issues. First of all if we change $\mathcal{L}$ then the Abel map α changes. For instance, if we replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$ with $n > 1$, then the original Abel map is composed with the "multiplication by n" morphism on the Picard scheme. Because this is a finite map of degree > 1, the geometric generic fiber of the new Abel map will not be integral. So it is crucial to work with the correct invertible sheaf $\mathcal{L}$. If the geometric generic fiber of f has Picard group isomorphic to $\mathbb{Z}$ (rationally connected varieties always have discrete Picard group), then this obstruction is equivalent to the well known *elementary obstruction* of Colliot-Thélène and Sansuc. We impose vanishing of the elementary obstruction in a somewhat hidden manner through existence properties for "lines" in the generic fiber, i.e., curves of $\mathcal{L}$-degree 1. Observe that there are no curves of $\mathcal{L}^{\otimes n}$-degree 1, which indicates the connection with the elementary obstruction.

A second, weightier issue is that $\mathrm{Sections}^e(X/C/\kappa)$ typically is not proper. So it is extremely unlikely any interesting subvarieties are rationally connected. Fortunately it suffices to prove there is a component Z_e as above for a compactification $\Sigma^e(X/C/\kappa)$ of $\mathrm{Sections}^e(X/C/\kappa)$. The compactification we use here comes from Kontsevich's moduli space of stable maps. But there is a third problem: this space will usually have more than one irreducible component. Some of these components have bad properties because the generic point parameterizes an obstructed section. So we restrict attention to those irreducible components which parameterize unobstructed sections, specifically what we call "(g)-free sections" where g is the genus of C. Still there may be more than one irreducible component Z parameterizing (g)-free sections.

We cannot fix this for any particular integer e: for any particular integer $e = \epsilon$ there may well be more than one irreducible component Z of $\Sigma^\epsilon(X/C/\kappa)$ parameterizing (g)-free sections. However the problem gets better as e becomes more positive. There is a standard way of producing new sections from old: attach vertical rational curves to the section curve and deform this reducible curve to get an irreducible curve which is again a section. If the original section curve and vertical curves are sufficiently free, then the reducible curve does deform and the deformations are again unobstructed. In particular the new section is parameterized by a smooth point of $\Sigma^{e'}(X/C/\kappa)$ for some $e' > e$. Of course there are many ways of attaching and deforming, so we choose the simplest possible: attach vertical "lines", i.e., curves whose $\mathcal{L}$-degree equals 1. We use the somewhat colorful name "porcupine" to denote a reducible curve obtained from a (g)-free section by attaching free lines in fiber of π. Using these porcupines, we produce a sequence $(Z_e)_{e\geq\epsilon}$ of irreducible components Z_e of $\Sigma^e(X/C/\kappa)$. Of course this presupposes the existence of many free lines to attach to our original section, and that leads to our first technical hypothesis: every point of every geometric fiber X_t of π is contained in free lines in X_t, and every line in X_t is free. Moreover, we will demand that the parameter space for lines in X_t containing a fixed point is itself integral and rationally connected.

Now the sequence $(Z_e)_{e\geq\epsilon}$ is still not unique. But if we assume that the parameter space for chains of lines in X_t containing two fixed, general points is also nonempty, integral and rationally connected, then the sequence is "asymptotically unique": for every other choice of starting integer $e=\epsilon'$ and for every sequence W of $\Sigma^{\epsilon'}(X/C/\kappa)$ parameterizing (g)-free sections, the sequence $(Z_e)_{e\geq\epsilon}$ and $(W_e)_{e\geq\epsilon'}$ become equal for all $e \gg 0$. This implies a Galois invariance property for the sequence $(Z_e)_{e\geq\epsilon}$. Therefore to prove the existence of a sequence $(Z_e)_{e\geq\epsilon}$ of components Z_e of $\Sigma^e(X/C/\kappa)$ such that

$$\alpha|_{Z_e} : Z_e \to \mathrm{Pic}^e(X/C/\kappa)$$

is dominant with integral and rationally connected geometric generic fiber, it suffices to prove the existence of a sequence of components $(Z_{e,\overline{\kappa}})_{e\geq\epsilon}$ of components $Z_{e,\overline{\kappa}}$ of the base-change $\Sigma^e(X_{\overline{\kappa}}/C_{\overline{\kappa}}/\overline{\kappa})$. So we are reduced to working over the algebraically closed field $\overline{k(t)}$.

The hypotheses above imply that there exists a sequence $(Z_e)_{e\geq\epsilon}$ such that each $\alpha|_{Z_e}$ is dominant with integral geometric generic fiber. But we need an additional hypothesis to prove that the geometric generic fiber is rationally connected: the existence of a "2-twisting scroll" in the geometric generic fiber of f. By carefully analyzing how the parameter spaces $\Sigma^e(X/C/\kappa)$ change under the porcupine operation mentioned above, we are able to show that these hypotheses do imply that the geometric generic fiber of $\alpha|_{Z_e}$ is rationally connected for all $e \gg 0$.

This analysis is quite intricate. In fact there are just a small number of simple, geometric ideas involved. But there is also a large amount of notation and bookkeeping. We tried to at least choose memorable names to ease the notation: "porcupines", "quills" of the porcupine, "pens" to keep porcupines together, etc. Still the amount of notation and the large number of small, technical lemmas both remain serious obstacles to understanding the main arguments. To help I have added quite a bit of "discussion" to Sections 6 and 7.

After the proof of the main theorem, there is still the issue of verifying the hypotheses for the varieties relevant to Serre's "Conjecture II": projective homogeneous spaces for semisimple algebraic groups. Some hypotheses are straightforward to verify, and we explain these completely in Part 2. But the main hypothesis, existence of a 2-twisting scroll, is a nontrivial result due to our coauthor Xuhua He. He's proof is elegant and easy to follow. But it involves a substantial fraction of the theory of root systems, so we have chosen to leave He's theorem as a "black box".

Finally in Part 3 we explain how Serre's "Conjecture II", as well as de Jong's "Period-Index Theorem", each reduce to existence of sections of fibrations over surfaces whose geometric generic fiber is a homogeneous space.

Index of some frequent notations.

(1) $f : X \to S$. See Corollary 8.1. f is a projective, flat morphism to a surface S over an algebraically closed field.

(2) $X/C/\kappa$ and $\mathcal{L}$, g, K and Y. See Notation 2.4. $X/C/\kappa$ denotes a projective, flat morphism $\pi : X \to C$ where C is a κ-curve of genus g. $\mathcal{L}$ is a π-ample invertible sheaf on X. K is the algebraic closure $K = \overline{\kappa(C)}$. Y is the geometric generic fiber of π.

(3) $\mathrm{Pic}^e_{C/\kappa}$. The Picard scheme parameterizing families of degree e invertible sheaves on C.

(4) δ; $t_1, \dots, t_\delta$. See the discussion following Definition 3.2. δ is a nonnegative integer and $t_1, \dots, t_\delta$ are points of C.

(5) $P \to Q$. This is notation for a quasi-projective morphism, particularly when describing general constructions which apply to any quasi-projective morphism.

(6) $\mathrm{Sec}^e(X/C/\kappa)$ and σ. See Theorem 3.1. The universal space of sections of $\pi : X \to C$ which have degree e with respect to $\mathcal{L}$.

(7) α. See Section 3. α always denotes an Abel map whose target is $\mathrm{Pic}^e_{C/\kappa}$ for some integer e.

(8) γ, n, β. See Section 3. γ is the arithmetic genus of a nodal curve, n is the number of marked points on the curve, and β is a curve class on some quasi-projective scheme P, i.e., a homomorphism $\mathrm{Pic}(P) \to \mathbb{Z}$.

(9) $\overline{\mathcal{M}}_{\gamma,n}(P,\beta)$ and $\overline{\mathrm{M}}_{\gamma,n}(P,\beta)$. See Section 3; $\overline{\mathcal{M}}_{\gamma,n}(P,\beta)$ is a Deligne-Mumford stack parameterizing genus γ, n-pointed stable maps to P with curve class β, and $\overline{\mathrm{M}}_{\gamma,n}(P,\beta)$ is the coarse moduli space of $\overline{\mathcal{M}}_{\gamma,n}(P,\beta)$.

(10) $\mathrm{ev}_{\gamma,n,\beta} : \overline{\mathrm{M}}_{\gamma,n}(P,\beta) \to P^n$. See Section 3. The evaluation map associating to a stable map from an n-pointed curve to P the n-tuple of images of the n marked points in P.

(11) $\Sigma^e(X/C/\kappa)$. See Definition 3.2. The coarse moduli space of stable porcupines of $X/C/\kappa$ which have degree e with respect to $\mathcal{L}$; this is projective and contains the quasi-projective scheme $\mathrm{Sec}^e(X/C/\kappa)$ as an open subscheme.

(12) $h : C' \to X$, $\sigma_0 : C \to X$, $D = \underline{t}_1 + \cdots + \underline{t}_\delta$, and $C'_1, \dots, C'_\delta$. See the discussion following Definition 3.2. h is a stable section, σ_0 is the unique section such that $\sigma_0(C)$ is contained in $h(C')$, D is the divisor of nodes of $h(C')$ contained in $\sigma_0(C)$, and C'_i is the maximal vertical subcurve of C' whose image under h contains $\sigma_0(t_i)$.

(13) Z_e. See Definition 4.8 and Definition 5.6. Z_e is an irreducible component of $\Sigma^e(X/C/\kappa)$.

(14) $\overline{\mathrm{M}}_{\gamma,n}(P/Q,e)$. See Definition 3.4. The space of stable maps relative to a morphism $f : P \to Q$, i.e., stable maps to P whose image is contained in a fiber of f and which have degree e with respect to an f-ample invertible sheaf.

(15) $f_{\gamma,n,\beta} : \overline{\mathrm{M}}_{\gamma,n}(P/Q,e) \to Q$. See the discussion following Definition 3.4. The map associating to a stable map into a fiber of f the point of Q which is the image of the stable map under f.

(16) $\mathrm{ev}_{\gamma,n,\beta} : \overline{\mathrm{M}}_{\gamma,n}(P/Q,e) \to P \times_Q \cdots \times_Q P$. See the discussion following Definition 3.4. The evaluation map as in item 10.

(17) R and ρ. See Definition 3.5. R is a *scroll*[1] for $X/C/\kappa$, i.e., a closed subscheme R of X such that the projection $\pi|_R : R \to C$ is smooth and surjective with geometric fibers being lines. The morphism $\pi|_R$ is usually denote by ρ.

[1]*scroll* Engl. = *surface réglée* Fr.

(18) (R, L). See Definition 3.6. An m-twisting scroll for $X/C/\kappa$, i.e., a scroll R for $X/C/\kappa$ together with a Cartier divisor class L on R satisfying various properties.
(19) $\text{Chn}_2(P/Q, n)$. See Definition 3.7. The moduli space of 2-pointed chains $(C', p_1, q_n) = ((L_1, p_1, q_1), \ldots, (L_n, p_n, q_n))$ of n lines in fibers of $P \to Q$.
(20) $\text{ev}_{0,2,n} : \text{Chn}_2(P/Q, n) \to P \times_Q P$. The evaluation map, see item 16.
(21) $\Phi : \overline{\text{M}}_{0,1}(X/C, 1) \to \overline{\text{M}}_{0,0}(X/C, 1)$. See Section 4. The forgetful morphism associating to a pointed line (L, p) the line L.
(22) (g) as in "(g)-free". See Definition 4.7. (g) is the maximum of 1 and $2g-1$.
(23) $(Z_e)_{e \geq \epsilon}$. See Definition 4.8. A sequence for all integer $e \geq \epsilon$ of an irreducible component Z_e of $\Sigma^e(X/C/\kappa)$.
(24) $\text{Porc}^{e,\delta}(X/C/\kappa)$. See Definition 5.1 and Proposition 5.2. The parameter space for *porcupines*[2]. A porcupine is a special stable map $h : C' \to X$. The section component C_0 is the *body*[3]. The vertical components are the *quills*[4]. The integer e denotes the total degree of the porcupine and δ denotes the number of quills.
(25) $T_{P/Q}$ or T_f. The *vertical tangent bundle* associated to a morphism $f : P \to Q$, i.e., the dual of the sheaf of relative differentials of f (usually only applied when the sheaf of relative differentials is locally free).
(26) $\Phi_{\text{body}} : \text{Porc}^{e,\delta}(X/C/\kappa) \to \text{Porc}^{e-\delta,0}(X/C/\kappa)$. See Lemma 5.3. The morphism associating to each porcupine $h : C' \to X$ the body $\sigma_o : C \to X$ of the porcupine.
(27) $\Phi'_{\text{body}} : \text{Porc}^{e,\delta}(X/C/\kappa) \to \text{Porc}^{e-\delta,0}(X/C/\kappa) \times_\kappa C_\delta$. See Lemma 5.3. The morphism associating to each porcupine $h : C' \to X$ the *extended body* (σ_0, D), i.e., the body σ_0 together with the attachment divisor $D = \underline{t}_1 + \cdots + \underline{t}_\delta$, see item 12.
(28) $X_{(t_1,\ldots,t_\delta)}$. See Notation 6.1. Given distinct closed points $t_1, \ldots, t_\delta$ of C, notation for the fiber product of the corresponding fibers of π, $X_{t_1} \times_\kappa \cdots \times_\kappa X_{t_\delta}$.
(29) $\text{Porc}^{e,\delta}(X/C/\kappa)_Z$. Given a sequence $(Z_e)_{e \geq \epsilon}$ as in item 23, the intersection of $\text{Porc}^{e,\delta}(X/C/\kappa)$ and Z_e inside $\Sigma^e(X/C/\kappa)$.
(30) $\text{Chn}_2(X/C, n)_{(t_1,\ldots,t_\delta)}$. See Notation 6.1. $t_1, \ldots, t_\delta$ is as in item 28. And $\text{Chn}_2(X/C/\kappa, n)_{(t_1,\ldots,t_\delta)}$ is the fiber product $\text{Chn}_2(X_{t_1}/\kappa, n) \times_\kappa \cdots \times_\kappa \text{Chn}_2(X_{t_\delta}/\kappa, n)$.
(31) O and $\text{Chn}_2^O(X/C, n)_{(t_1,\ldots,t_\delta)}$. See Notation 6.1. O is a dense open subset of $X_{(t_1,\ldots,t_\delta)}$. And $\text{Chn}_2^O(X/C, n)_{(t_1,\ldots,t_\delta)}$ is the open subscheme of $\text{Chn}_2(X/C, n)_{(t_1,\ldots,t_\delta)}$ parameterizing chains such that every associated sequence of marked points or nodes in $X_{(t_1,\ldots,t_\delta)}$ is contained in the dense open O.
(32) $\mathcal{O}_C(\Gamma)$. An invertible sheaf on C, especially in general constructions that apply to any invertible sheaf on C.
(33) $\zeta : C \to \overline{\text{M}}_{0,1}(X/C, 1)$ or $\tau : C \to \overline{\text{M}}_{0,1}(X/C, 1)$, R and σ_0. See Definition 7.2 and Lemma 7.3. ζ is a section of the projection $\pi_{0,1,1} : \overline{\text{M}}_{0,1}(X/C, 1) \to C$, see item 15, and sometimes so is τ when more than one such section needs to be used. A section of $\pi_{0,1,1}$ determines a scroll R, see item 17,

[2] *porcupine* Engl. = *porc-épic* Fr.
[3] *body* Engl. = *corps* Fr.
[4] *quill* Engl. = *piquant* Fr.

together with a section σ_0 of $\rho : R \to C$. The section (or more generally a porcupine whose body equals the section) is said to be *penned*[5] by the scroll R, and R is called a *pen*[6] for the section.

(34) $N_{P/Q}$ or N_i. The normal sheaf of a closed immersion $i : P \to Q$.

(35) $\overline{\mathrm{Porc}^{e,\delta}}(X/C/\kappa)$. See Proposition 7.8. The closure in $\Sigma^e(X/C/\kappa)$ of the locally closed subscheme $\mathrm{Porc}^{e,\delta}(X/C/\kappa)$.

(36) $\alpha^{-1}(\mathcal{O}_C(\Gamma))$. The fiber of the Abel map α over a point $\mathcal{O}_C(\Gamma)$ of $\mathrm{Pic}_{C/\kappa}$.

(37) G, P, Q and R, B, $R_u(B)$, T, Φ, I. See Section 9. G is a semisimple algebraic group. P, Q and R are all parabolic subgroups of G. B is a Borel subgroup of G. $R_u(B)$ is the unipotent radical of B. T is a maximal torus of G. Φ is a root system and I is the set of simple roots.

(38) $\mathcal{G}/T$, $\mathcal{T}/T$, $\mathcal{X}_{\mathcal{T}}$. See Section 10. T is a scheme. $\mathcal{G}$ is a reductive group scheme over T. And $\mathcal{T}$ is a $\mathcal{G}$-torsor over T. Given a T-scheme $\mathcal{X}$ together with an action of $\mathcal{G}$ as T-schemes, $\mathcal{X}_{\mathcal{T}}$ is the corresponding twist of $\mathcal{X}$.

(39) $\mathrm{Br}(K)$. See Section 11. $\mathrm{Br}(K)$ is the Brauer group of a field K.

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Part 1. Rationally simply connected fibrations

2. The Kollár-Miyaoka-Mori conjecture

Let k be an algebraically closed field. A smooth, connected k-scheme X is *rationally connected*, resp. *separably rationally connected*, if there exists an integral k-scheme M and a k-morphism

$$h : M \times_k \mathbb{P}^1_k \to X, \quad (m, t) \mapsto h(m, t)$$

such that the induced morphism

$$h_{0,\infty} : M \to X \times_k X, \quad m \mapsto (h(m, 0), h(m, \infty))$$

is dominant, resp. dominant and separable. Roughly this says that every pair of points in X is contained in the image of a morphism with domain $\mathbb{P}^1_k$, i.e., every pair is connected by a rational curve. More generally for an integral k-scheme X which is possibly singular, we will sometimes say that X is *rationally connected*, resp. *separably rationally connected*, if there exists a projective birational morphism $\tilde{X} \to X$ such that the smooth locus of $\tilde{X}$ is rationally connected, resp. separably rationally connected.

Among smooth, projective varieties, every unirational variety is rationally connected. The converse is unknown, but it is expected to be false. Rational connectedness satisfies many nice properties which are unknown for unirationality. One of these properties, which Fano conjectured fails for unirationality, is that the total space of an algebraic fibration of smooth, projective varieties is separably rationally connected if the base and general fiber are both separably rationally connected

[5] *penned* Engl. = *enclos* Fr.
[6] *pen* Engl. = *enclos* Fr.

(there are counterexamples showing one cannot replace "separably rationally connected" by "rationally connected"). This follows from a stronger result, originally conjectured by Kollár-Miyaoka-Mori [Kol96, Conjecture IV.6.1.1].

Theorem 2.1. [GHS03], [dJS03] *Let k be an algebraically closed field. Let C be a smooth, projective k-curve. Let $\pi : X \to C$ be a projective, flat morphism. Assume that the geometric generic fiber Y of π is normal and that the smooth locus of Y is separably rationally connected. Then there exists a k-morphism $\sigma : C \to X$ such that $\pi \circ \sigma = Id_C$, i.e., σ is a section of π.*

This has a number of consequences which are discussed elsewhere, cf. [Deb03]. Two of these, which Kollár-Miyaoka-Mori deduced from their conjecture, are quite useful in what comes later.

Corollary 2.2. [KMM92] *With hypotheses as above, let $t_1, \dots, t_\delta$ be distinct k-points of C such that the fiber X_{t_j} of π is smooth for $j = 1, \dots, \delta$. For each $j = 1, \dots, \delta$, let p_j be a k-point of X_{t_j}. There exists a section $\sigma : C \to X$ of π such that $\sigma(t_j)$ equals p_j for every $j = 1, \dots, \delta$.*

Corollary 2.3. [KMM92] *Assume that char(k) equals 0. Let $f : P \to Q$ be a surjective morphism of integral, projective k-schemes. Assume that Q is rationally connected and assume that the geometric generic fiber of f is integral and rationally connected. Then also P is rationally connected.*

Assume now that char(k) equals 0. By resolution of singularities we may assume X is smooth. And by generic smoothness there are at most finitely many singular fibers of π. Thus, by Corollary 2.2, π has *many* sections. In particular, except in the trivial case that π is an isomorphism, the k-scheme parameterizing sections of π has components whose dimensions become arbitrarily large. Under additional hypotheses on π we can prove that these components "eventually become as rationally connected as possible". The precise statement is given below.

For the applications it is important to also consider the case when the ground field is not algebraically closed.

Notation 2.4. Let κ denote a characteristic 0 field (possibly not algebraically closed). Let C denote a smooth, projective, geometrically connected κ-curve. Denote the genus of C by g. Let X denote a smooth, projective κ-scheme. Let $\pi : X \to C$ denote a surjective morphisms of κ-schemes whose geometric generic fiber is irreducible. Denote by K the algebraically closed field $K = \overline{\kappa(C)}$, and denote by Y the geometric generic fiber of π which is a smooth, projective, connected K-scheme. Finally, let $\mathcal{L}$ denote an invertible sheaf on X which is π-ample. And denote by $\mathcal{L}_Y$ the base-change of $\mathcal{L}$ to Y.

3. Sections, stable sections and Abel maps

The proof of the main theorem uses several different kinds of curves in X, some of which are reducible. This section introduces these different types of curves and reviews the basics of the parameter spaces for these curves. Of course the most important curves are section curves: images of sections of π. But the parameter spaces for such curves are not proper. They are open subsets of proper moduli spaces, the moduli spaces of "stable sections". The main result of this section is that the Abel map extends to the moduli space of stable sections and that the image of a stable section under the Abel map can be understood by a simple analysis of the components of the stable section.

The notations here are as in Notation 2.4. Let S be a κ-scheme. A *family of sections of* $\pi : X \to C$ *parameterized by* S is a morphism of C-schemes

$$\tau : S \times_\kappa C \to X.$$

For an integer e, the family of sections has *degree* e if the invertible sheaf $\tau^*\mathcal{L}$ on $S \times_\kappa C$ has relative degree e over S, i.e., for every geometric point s of S the base-change of $\tau^*\mathcal{L}$ to C_s has degree e. A pair (S, τ) as above is *universal* if for every κ-scheme S' and for every family of degree e sections of π parameterized by S,

$$\tau' : S' \times_\kappa C \to X,$$

there exists a unique κ-morphism $f : S' \to S$ such that τ' equals $\tau \circ (f, \mathrm{Id}_C)$.

Theorem 3.1 (Grothendieck)**.** [Gro62, Part IV.4.c, p. 221-19] *For every integer e there exists a* universal pair $(\mathit{Sec}^e(X/C/\kappa), \sigma)$ *of a* κ*-scheme and a family of degree* e *sections of* π *parameterized by* $\mathit{Sec}^e(X/C/\kappa)$,

$$\sigma : \mathit{Sec}^e(X/C/\kappa) \times_\kappa C \to X.$$

Moreover $\mathit{Sec}^e(X/C/\kappa)$ *is a quasi-projective* κ*-scheme.*

Invertible sheaves on C of degree e are parameterized by the Picard scheme $\mathrm{Pic}^e_{C/\kappa}$. Thus, associated to the invertible sheaf $\sigma^*\mathcal{L}$ there is a morphism of κ-schemes

$$\alpha : \mathrm{Sec}^e(X/C/\kappa) \to \mathrm{Pic}^e_{C/\kappa}.$$

This morphism is the *Abel map* associated to $\mathcal{L}$.

Stable maps, stabilization and evaluation morphisms. The κ-scheme $\mathrm{Sec}^e(X/C/\kappa)$ is quasi-projective. It is rarely projective. It is convenient to work with a projective scheme. Fortunately there exists a projective scheme $\Sigma^e(X/C/\kappa)$ containing $\mathrm{Sec}^e(X/C/\kappa)$ as an open subscheme: the coarse moduli (algebraic) space for "stable sections". This comes from a more fundamental scheme: the coarse moduli space for "stable maps". An excellent reference for stable maps is the article of Fulton and Pandharipande, cf. [FP97]. Here is a very brief summary. For every quasi-projective κ-scheme P, for nonnegative integers γ, n, and for a group homomorphism

$$\beta : \mathrm{Pic}(P) \to \mathbb{Z}, \quad L \mapsto \langle L, \beta \rangle,$$

hereafter known as a "curve class", a *family of* n*-pointed, genus* γ*, stable maps to* P *of class* β parameterized by a κ-scheme S is a datum

$$(\rho : \mathcal{C} \to S, (\tau_i : S \to \mathcal{C})_{i=1,\dots,n}, h : \mathcal{C} \to P)$$

of a proper, flat morphism $\rho : \mathcal{C} \to S$, n section morphisms $\tau_i : S \to \mathcal{C}$ of $\rho : \mathcal{C} \to S$, and a κ-morphism $h : \mathcal{C} \to P$ such that for every geometric point s of S,

(i) the fiber C_s of ρ over s is a connected, at-worst-nodal curve of genus γ,
(ii) the images $\tau_i(s)$ are pairwise distinct and all contained in the smooth locus of C_s,
(iii) the induced homomorphism

$$\mathrm{Pic}(P) \to \mathbb{Z}, \quad \mathcal{L} \mapsto \deg_{C_s}(h^*\mathcal{L}|_{C_s})$$

equals β, and

(iv) the group of automorphisms of C_s fixing each point $\tau_i(s)$ and commuting with $h|_{C_s}$ is finite.

Such families are the objects of a category $\overline{\mathcal{M}}_{\gamma,n}(P,\beta)$ which is fibered in groupoids over the category of κ-schemes S: the morphisms in $\overline{\mathcal{M}}_{\gamma,n}(P,\beta)$ as well as the clivage (i.e., the "pullbacks") are the evident ones, cf. [Kon95]. The category $\overline{\mathcal{M}}_{\gamma,n}(P,\beta)$ is a separated, finite type Deligne-Mumford stack over κ which is proper if P is proper, cf. [Kon95]. The coarse moduli (algebraic) space, $\overline{\mathrm{M}}_{\gamma,n}(P,\beta)$, of the Deligne-Mumford stack $\overline{\mathcal{M}}_{\gamma,n}(P,\beta)$ is a quasi-projective κ-scheme which is proper if P is proper, cf. [FP97].

"Stabilization" is another useful feature of stable maps. Assume that the triple (γ,n,β) is "stable" in the sense that either β is nonzero on some ample invertible sheaf or else (γ,n) is different from $(0,0)$, $(0,1)$, $(0,2)$, and $(1,0)$. For every family of maps satisfying (i), (ii) and (iii) but not necessarily (iv) (such families are called "prestable"), there exists a stable family

$$(\rho_{\mathrm{stab}} : \mathcal{C}_{\mathrm{stab}} \to S, (\tau_{i,\mathrm{stab}} : S \to \mathcal{C}_{\mathrm{stab}})_{i=1,\dots,n}, h_{\mathrm{stab}} : \mathcal{C}_{\mathrm{stab}} \to P)$$

and a proper, surjective morphism $u : \mathcal{C} \to \mathcal{C}_{\mathrm{stab}}$ compatible with h and the maps τ_i (i.e., h equals $h_{\mathrm{stab}} \circ u$ and each $\tau_{i,\mathrm{stab}}$ equals $u \circ \tau_i$) and such that for every geometric point s of S and every connected, closed subcurve B of C_s both of the following hold.

(i) The subcurve B is contracted (to a point) by u if and only if it is an *unstable tree*: the arithmetic genus of B equals 0, B is contracted by h, and B contains < 3 special points, i.e., marked points $\tau_i(s)$ and intersections points of D with $\overline{C_s - D}$.

(ii) If B contains no subcurve which is an unstable tree then $u|_B : B \to u(B)$ is a birational, unramified morphism which identifies a pair of distinct points of B if and only if the points are contained in a common unstable tree of C_s.

The stable family together with the proper morphism u is a *stabilization* of the original prestable family. The stabilization is unique up to unique isomorphism, and every stable family is its own stabilization. Stabilization is compatible with pullbacks. The main application of stabilization is the following. Let $f : P \to Q$ be a κ-morphism of quasi-projective κ-schemes. For each family of stable maps to P

$$(\rho : \mathcal{C} \to S, (\tau_i : S \to \mathcal{C})_{i=1,\dots,n}, h : \mathcal{C} \to P)$$

there is an induced family of prestable maps to Q,

$$(\rho : \mathcal{C} \to S, (\tau_i : S \to \mathcal{C})_{i=1,\dots,n}, f \circ h : \mathcal{C} \to Q).$$

So if $f_*\beta := \beta \circ f^*$ is nonzero on some ample invertible $\mathcal{O}_Q$-module or if (γ,n) is different from $(0,0)$, $(0,1)$,$(0,2)$ and $(1,0)$ then stabilization determines a 1-morphism of Deligne-Mumford stacks

$$f_* : \overline{\mathcal{M}}_{\gamma,n}(P,\beta) \to \overline{\mathcal{M}}_{\gamma,n}(Q,f_*\beta)$$

which in turn determines a morphism of coarse moduli spaces

$$f_* : \overline{\mathrm{M}}_{\gamma,n}(P,\beta) \to \overline{\mathrm{M}}_{\gamma,n}(Q,f_*\beta).$$

It is sometimes also useful that the morphism u is *rational*, i.e., $\mathcal{O}_{\mathcal{C}_{\mathrm{stab}}} \to u_*\mathcal{O}_{\mathcal{C}}$ is an isomorphism and $R^q u_*\mathcal{O}_{\mathcal{C}}$ is zero for all $q > 0$.

Also useful are the associated "evaluation morphisms" of Kontsevich stacks. Associated to every family of stable maps there is an induced morphism

$$(h \circ \tau_1, \ldots, h \circ \tau_n) : S \to P \times_\kappa \cdots \times_\kappa P.$$

This is compatible with pullbacks, hence determines a 1-morphism of stacks over κ

$$\mathrm{ev}_{\gamma,n,\beta} : \overline{\mathcal{M}}_{\gamma,n}(P,\beta) \to P^n,$$

and also a κ-morphism

$$\mathrm{ev}_{\gamma,n,\beta} : \overline{\mathrm{M}}_{\gamma,n}(P,\beta) \to P^n.$$

These morphisms are called *evaluation morphisms.* The convention is to drop the subscript and denote the morphism by ev whenever confusion is unlikely.

Definition 3.2. The *space of degree e stable sections of* π, $\Sigma^e(X/C/\kappa)$, is the fiber of the stabilization morphism

$$\pi_* : \overline{\mathrm{M}}_{g,0}(X,e) \to \overline{\mathrm{M}}_{g,0}(C,[C])$$

over the κ-point corresponding to the identity stable map $\mathrm{Id}_C : C \to C$.

Here $[C]$ denotes the *fundamental curve class*,

$$\mathrm{Pic}(C) \to \mathbb{Z}, \quad L \mapsto \langle L, [C] \rangle := \deg_C(L).$$

And $\overline{\mathrm{M}}_{g,0}(X,e)$ denotes the disjoint union

$$\overline{\mathrm{M}}_{g,0}(X,e) = \bigsqcup_\beta \overline{\mathrm{M}}_{g,0}(X,\beta)$$

where β ranges over all curve classes in X such that $\pi_*\beta$ equals $[C]$, i.e., $\langle \pi^*L, \beta \rangle = \langle L, [C] \rangle$, and such that $\langle \mathcal{L}, \beta \rangle$ equals e. In fact there are at most finitely many such β for which $\overline{\mathrm{M}}_{g,0}(X,e)$ is nonempty: finiteness follows from boundedness of the Hilbert scheme $\mathrm{Hilb}^{(e+f)t+1-g}_{X/\kappa}$, where the Hilbert polynomial is with respect to any ample invertible sheaf of the form $\mathcal{L} \otimes \pi^*\mathcal{N}$ and where f is the degree of $\mathcal{N}$. Since $\overline{\mathrm{M}}_{g,0}(X,e)$ is a disjoint union of finitely many projective κ-schemes, $\overline{\mathrm{M}}_{g,0}(X,e)$ is a projective κ-scheme.

Stated simply $\Sigma^e(X/C/\kappa)$ is a scheme which is a coarse moduli space parameterizing stable maps of degree e with respect to $\mathcal{L}$, $h : C' \to X$, such that the stabilization of $\pi \circ h : C' \to C$ is an isomorphism to C. Since $\overline{\mathcal{M}}_{g,0}(X,e)$ is a projective κ-scheme, also the fiber $\Sigma^e(X/C/\kappa)$ is a projective κ-scheme. The universal section

$$\sigma : \mathrm{Sec}^e(X/C/\kappa) \times_\kappa C \to X$$

is a family of stable sections parameterized by $\mathrm{Sec}^e(X/C/\kappa)$. Thus there is an associated κ-morphism

$$\mathrm{Sec}^e(X/C/\kappa) \to \Sigma^e(X/C/\kappa).$$

It is straightforward to check that this is an open immersion.

In fact every stable section parameterized by $\Sigma^e(X/C/\kappa)$ can be understood in terms of an "honest" section. Let k be an algebraically closed extension of κ and let $h : C' \to X$ be a stable map corresponding to a k-point of $\Sigma^e(X/C/\kappa)$. There exists a unique irreducible component C'_0 of C' such that

$$\pi \circ h : C'_0 \to C \otimes_\kappa k$$

is an isomorphism, i_0. Thus there exists a unique section $\sigma_0 : C \otimes_\kappa k \to X$ such that

$$h|_{C'_0} : C'_0 \to X$$

equals $\sigma_0 \circ i_0$. Denote $\deg(\sigma_0^*\mathcal{L})$ by e_0. The irreducible component C'_0 meets the rest of C' in finitely many k-points $p_1, \dots, p_\delta$. Denote the image points in $C \otimes_\kappa k$ by $t_j = i_0(p_j)$. For each $j = 1, \dots, \delta$, there exists a maximal connected subcurve C'_j of C' intersecting C'_0 precisely in p_j. These subcurves give a decomposition

$$C' = C'_0 \cup C'_1 \cup \dots \cup C'_\delta.$$

And for every $j = 1, \dots, \delta$, the datum

$$(C'_j, p_j, u|_{C'_j} : C'_j \to X_{t_j})$$

is a genus 0, 1-pointed, stable map to X_{t_j} of some positive degree $e_j = \deg_{C'_j}(h^*\mathcal{L}|_{C'_j})$. The sum $e_0+e_1+\dots+e_\delta$ equals e. Thus $\Sigma^e(X/C/\kappa)$ can be understood as parameterizing degree-e sections of π together with reducible curves obtained by attaching to a section σ_0 of lower degree e_0 a collection of genus 0 curves in fibers of π to bring the total degree up to e.

Extension of the Abel map. The Abel map extends to $\Sigma^e(X/C/\kappa)$. For a family of stable maps parameterized by a κ-scheme S,

$$(\rho : \mathcal{C} \to S, h : \mathcal{C} \to X)$$

the *det* construction of [KM76] defines an invertible sheaf on $S \times_\kappa C$,

$$\det(R(\rho, \pi \circ h_*(h^*\mathcal{L})).$$

If the family of stable maps is

$$(\mathrm{pr}_S : S \times_\kappa C \to S, \tau : S \times_\kappa C \to X)$$

this invertible sheaf is precisely $\tau^*\mathcal{L}$. Thus the *det* sheaf on $\Sigma^e(X/C/\kappa) \times_\kappa C$ extends $\sigma^*\mathcal{L}$ on $\mathrm{Sec}^e(X/C/\kappa) \times_\kappa C$. In other words, the Abel map α extends to a morphism

$$\alpha : \Sigma^e(X/C/\kappa) \to \mathrm{Pic}^e_{C/\kappa}.$$

And for a stable map $h : C' \to X$ as described in the last paragraph, the *div* construction of [KM76] implies that the *det* sheaf on $C \otimes_\kappa k$ equals

$$\det(R(\pi \circ h)_*(h^*\mathcal{L})) \cong \sigma_0^*\mathcal{L}(e_1 \cdot \underline{t_1} + \dots + e_\delta \cdot \underline{t_\delta}),$$

where $\underline{t_1}, \dots, \underline{t_\delta}$ are the effective Cartier divisors on C associated to the closed points $t_1, \dots, t_\delta$.

Definition 3.3. Two points of $\Sigma^e(X/C/\kappa)$ are *Abel equivalent* if they are contained in the same fiber of the Abel map α. Since there are no rational curves in an Abelian variety, every pair of points of $\Sigma^e(X/C/\kappa)$ which are rationally chain connected are also Abel equivalent.

From the discussion above it is clear that as soon as $\delta \geq g$, varying the attachment points of the stable section over all $(t_1, \dots, t_\delta) \in C^\delta$ varies the invertible sheaf $\sigma_0^*\mathcal{L}(e_1 \cdot \underline{t_1} + \dots + e_\delta \cdot \underline{t_\delta})$ over a dense open subset of $\mathrm{Pic}^e_{C/\kappa}$. See [HT08, Theorem 20] for a variation on this theme. Since there are no rational curves in $\mathrm{Pic}^e_{C/\kappa}$, an irreducible component of $\Sigma^e(X/C/\kappa)$ will not be rationally connected so long as the Abel map is nonconstant on the component (and the Abel map will be nonconstant for all "interesting" irreducible components).

What we might hope, and what often holds, is that a general fiber of α is rationally connected, i.e., for an "interesting" irreducible component Z_e of $\Sigma^e(X/C/\kappa)$ the fiber of

$$\alpha|_{Z_e} : Z_e \to \mathrm{Pic}^e_{C/\kappa}$$

over the geometric generic point of $\mathrm{Pic}^e_{C/\kappa}$ is rationally connected. In other words, every general pair of Abel equivalent points are rationally connected. In the next section we state the main theorem in this direction.

Kontsevich stacks relative to a morphism. Of course all of these notions have analogues relative to a morphism. Let $f : P \to Q$ be a morphism of quasi-projective κ-schemes and let $\mathcal{L}$ be an f-relatively ample invertible sheaf.

Definition 3.4. For integers γ, n and e, the *Kontsevich stack relative to the morphism f* $\overline{\mathcal{M}}_{\gamma,n}(P/Q,e)$ is the disjoint union

$$\overline{\mathcal{M}}_{\gamma,n}(P/Q,e) := \bigsqcup_{\beta} \overline{\mathcal{M}}_{\gamma,n}(P,\beta)$$

where β ranges over all curve classes such that $f_*\beta$ equals 0 and such that $\langle \mathcal{L}, \beta \rangle$ equals e.

Using boundedness of the Hilbert scheme once more there are only finitely many classes β as above for which $\overline{\mathcal{M}}_{\gamma,n}(P,\beta)$ is nonempty. Hence $\overline{\mathcal{M}}_{\gamma,n}(P/Q,e)$ is a finite type, separated Deligne-Mumford stack with a quasi-projective coarse moduli space $\overline{\mathrm{M}}_{\gamma,n}(P/Q,e)$. And if P and Q are proper, then $\overline{\mathcal{M}}_{\gamma,n}(P/Q,e)$ and $\overline{\mathrm{M}}_{\gamma,n}(P/Q,e)$ are also proper.

For every family of stable maps in $\overline{\mathcal{M}}_{\gamma,n}(P/Q,e)$,

$$(\rho : \mathcal{C} \to S, (\sigma_i : S \to \mathcal{C})_{i=1,\dots,n}, h : \mathcal{C} \to P),$$

the morphism $f \circ h : \mathcal{C} \to Q$ is constant on fibers C_s of ρ since $f(h(C_s))$ has class 0 in the quasi-projective variety Q. Thus $f \circ h$ factors as $h_C \circ \rho$ for a unique κ-morphism $h_C : S \to Q$. Moreover h_C is compatible with pullbacks and thus defines a 1-morphism

$$f_{\gamma,n,e} : \overline{\mathcal{M}}_{\gamma,n}(P/Q,e) \to Q$$

which induces a morphism of κ-schemes,

$$f_{\gamma,n,e} : \overline{\mathrm{M}}_{\gamma,n}(X/C,e) \to C.$$

This construction is compatible with base-change: for a morphism $Q' \to Q$ of quasi-projective κ-schemes, $\overline{\mathcal{M}}_{\gamma,n}(P/Q,e) \times_Q Q'$ is canonically equivalent to $\overline{\mathcal{M}}_{\gamma,n}(P'/Q',e)$ where P' is the fiber product $P \times_Q Q'$. Similarly $\overline{\mathrm{M}}_{\gamma,n}(P/Q,e) \times_Q Q'$ equals $\overline{\mathrm{M}}_{\gamma,n}(P'/Q',e)$. In particular, the fiber of $f_{\gamma,n,e}$ over a point t of Q is simply the Kontsevich stack of the fiber, $\overline{\mathcal{M}}_{\gamma,n}(P_t/\kappa(t),e)$. This is the sense in which $\overline{\mathcal{M}}_{\gamma,n}(P/Q,e)$ is "relative" to the morphism f.

The evaluation morphism to $P^n = X \times_\kappa \cdots \times_\kappa X$ factors as a morphism of Q-schemes

$$\mathrm{ev}_{\gamma,n,e} : \overline{\mathrm{M}}_{\gamma,n}(P/Q,e) \to P \times_Q \cdots \times_Q P.$$

As usual, this evaluation morphism will be denoted by ev whenever confusion is unlikely.

There are two quite important example of this. When $\mathcal{L}$ is f-relatively very ample, then $\overline{\mathcal{M}}_{0,0}(P/Q,1)$ is nothing other than the parameter space for lines in fibers of p. Similarly $\overline{\mathcal{M}}_{0,1}(P/Q,1)$ is the parameter space for pointed lines in fibers of p.

Definition 3.5. Let κ, C and X be as in Notation 2.4 and assume that $\mathcal{L}$ is π-relatively very ample. A *scroll* [7] *for* $X/C/\kappa$ is a closed subscheme R of X such that

$$\pi|_R : R \to C$$

is smooth and the geometric fibers are rational curves whose $\mathcal{L}$-degree equals 1. Equivalently a scroll is a section of the morphism

$$\pi_{0,0,1} : \overline{\mathcal{M}}_{0,0}(X/C,1) \to C.$$

There is a special type of scroll which will be particularly important in Section 7.

Definition 3.6. Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that $\mathcal{L}$ is π-relatively very ample. Let m be a positive integer. An *m-twisting scroll* for $X/C/\kappa$ is a pair (R,L) where R is a scroll for $X/C/\kappa$ and L is a Cartier divisor class on R satisfying all of the following.

(1) The invertible sheaf $\mathcal{O}_R(L)$ has degree 1 on all geometric fibers R_t, it is globally generated, and it is non-special, i.e., $h^1(R,\mathcal{O}_R(L))$ equals 0.
(2) The normal bundle $N_{R/X}$ is globally generated and $h^1(R,N_{R/X})$ equals 0.
(3) For every invertible sheaf $\mathcal{O}_C(\Gamma)$ on C of degree $\leq m$, $h^1(R,N_{R/X}(-L)\otimes_{\mathcal{O}_R}\pi^*\mathcal{O}_C(-\Gamma))$ equals 0.

There is a second important example of this. Again for the morphism $f : P \to Q$ assume that $\mathcal{L}$ is f-relatively very ample. Then the fiber product of the forgetful morphisms

$$\overline{\mathcal{M}}_{0,1}(P/Q,1) \times_{\overline{\mathcal{M}}_{0,0}(P/Q,1)} \overline{\mathcal{M}}_{0,1}(P/Q,1)$$

is a parameter space for 2-pointed lines (L,p,q) in fibers of f.

Definition 3.7. For each integer $n \geq 1$, the space of *chains of n lines* $\mathrm{Chn}_2(P/Q,n)$ is the n-fold fiber product

$$\left(\overline{\mathcal{M}}_{0,1}(P/Q,1) \times_{\overline{\mathcal{M}}_{0,0}(P/Q,1)} \overline{\mathcal{M}}_{0,1}(P/Q,1)\right) \times_{\mathrm{ev}_{0,1,1}\circ\mathrm{pr}_2,P,\mathrm{ev}_{0,1,1}\circ\mathrm{pr}_1} \cdots$$
$$\times_{\mathrm{ev}_{0,1,1}\circ\mathrm{pr}_2,P,\mathrm{ev}_{0,1,1}\circ\mathrm{pr}_1} \left(\overline{\mathcal{M}}_{0,1}(P/Q,1) \times_{\overline{\mathcal{M}}_{0,0}(P/Q,1)} \overline{\mathcal{M}}_{0,1}(P/Q,1)\right).$$

This parameterizes *chains* (C',p_1,q_n) in fibers of f,

$$(C',p_1,q_n) = ((L_1,p_1,q_1),(L_2,p_2,q_2),\dots,(L_n,p_n,q_n))$$

where each (L_i,p_i,q_i) is a 2-pointed line in a fiber of f and satisfying the "matching condition" that q_i equals p_{i+1} for each $i = 1,\dots,n-1$. The points p_1 and q_n are *boundary points* of the chain (since there is no matching condition on these points). The *evaluation morphism* is the morphism of Q-schemes,

$$\mathrm{ev}_{0,2,n} : \mathrm{Chn}_2(P/Q,n) \to P \times_Q P, \quad (C',p_1,q_n) \mapsto (p_1,q_n).$$

Of course the main case is when $f : P \to Q$ is a morphism $\pi : X \to C$ as in Notation 2.4. Note also that this construction is compatible with base-change by $Q' \to Q$ in the same sense that all relative Kontsevich spaces are compatible with base-change.

[7] *scroll* Engl. = *surface réglée*

4. Rational connectedness of fibers of Abel maps

As always the notations are as in Notation 2.4. The main theorem has several strong hypotheses on X, $\mathcal{L}$, the morphism π and the geometric generic fiber Y defined over $K = \overline{\kappa(C)}$. One can prove a version of the theorem with weaker hypotheses. This is done in [dJHS08]. But then the proof is even more technical. In the application to Serre's conjecture the strong forms of these hypotheses are all valid.

Hypothesis 4.1. The morphism π is smooth and projective.

Hypothesis 4.2. The invertible sheaf $\mathcal{L}$ is π-very ample.

The remaining hypotheses are insensitive to replacing $\mathcal{L}$ by $\mathcal{L} \otimes_{\mathcal{O}_X} \pi^*\mathcal{O}_C(\Gamma)$ for an invertible sheaf $\mathcal{O}_C(\Gamma)$ on C. Thus in what follows we will assume that $\mathcal{L}$ is very ample. We consider X as an embedded subscheme of projective space via the complete linear system of $\mathcal{L}$. In particular, the relative Kontsevich stack

$$\overline{\mathrm{M}}_{0,0}(X/C, 1) \to C$$

is canonically equivalent to the relative Hilbert scheme whose fiber over a point t of C parameterizes lines L contained in the fiber X_t. And

$$\overline{\mathrm{M}}_{0,1}(X/C, 1) \to C$$

is canonically equivalent to the relative flag Hilbert scheme whose fiber over a point t of C parameterizes pairs (L, p) of a line L contained in the fiber X_t and a point p contained in L. In particular, $\overline{\mathrm{M}}_{0,0}(X/C, 1)$, resp. $\overline{\mathrm{M}}_{0,1}(X/C, 1)$ equals the coarse moduli space $\overline{\mathrm{M}}_{0,0}(X/C, 1)$, resp. $\overline{\mathrm{M}}_{0,1}(X/C, 1)$.

There is a forgetful morphism (over C)

$$\rho : \overline{\mathrm{M}}_{0,1}(X/C, 1) \to \overline{\mathrm{M}}_{0,0}(X/C, 1), \quad (L, p) \mapsto L.$$

And, as discussed at the end of the previous section, there is an evaluation morphism (over C)

$$\mathrm{ev}_{0,1,1} : \overline{\mathrm{M}}_{0,1}(X/C, 1) \to X, \quad (L, p) \mapsto p.$$

The standard convention is to drop the subscript and denote the evaluation morphism by ev.

Hypothesis 4.3. The morphism $\mathrm{ev} = \mathrm{ev}_{0,1,1}$ above is smooth and surjective with rationally connected geometric generic fiber.

Hypothesis 4.3, in fact just smoothness of ev, implies that every line L in every fiber X_t of π is a *free line* in X_t, i.e., the normal bundle of L in X_t is globally generated, cf. [Kol96, Corollary IV.3.5.4, Proposition IV.3.10].

The remaining hypotheses involve the geometric generic fiber Y of π over the field $K = \overline{\kappa(C)}$, as well as the pullback $\mathcal{L}_Y$ of $\mathcal{L}$ to Y.

Hypothesis 4.4. For some positive integer n_0 the evaluation morphism for chains of n_0 lines

$$\mathrm{ev}_{0,2,n_0} : \mathrm{Chn}_2(Y/K, n_0) \to Y \times_K Y$$

is surjective and the geometric generic fiber is integral and rationally connected.

Under Hypotheses 4.2 and 4.3 $\mathrm{ev}_{0,2,n}$ is dominant for all $n \gg 0$ if and only if Y is irreducible and $\mathcal{L}_Y$ is a generator of $\mathrm{Pic}(Y)$. If the Picard number equals 1, the argument proving ev is dominant appears, for instance, in [Kol96, Corollary IV.4.14]. We essentially repeat this argument in the first half of the proof of Lemma 9.7. And the opposite direction follows from [Kol96, Proposition IV.3.13.3].

The hypothesis that the fibers of $\mathrm{ev}_{0,1,1}$ and $\mathrm{ev}_{0,2,n}$ are rationally connected is strong. The fiber of $\mathrm{ev}_{0,1,1}$ is analogous to the space of continuous paths in a path connected CW complex with one basepoint fixed. Rational connectedness of this fiber is analogous to path connectedness of the space of such paths. Of course path connectedness of this space of paths is obvious by retracting the path to the fixed basepoint. But rational connectedness of the fiber of $\mathrm{ev}_{0,1,1}$ is nonetheless a strong condition. Similarly, the fiber of $\mathrm{ev}_{0,2,e}$ is analogous to the space of continuous paths in a path connected CW complex with both endpoints fixed. Rational connectedness of this fiber is analogous to path connectedness of the space of such paths. Path connectedness of this space of paths is exactly simple connectedness of the CW complex. In this way we consider Hypotheses 4.3 and 4.4 to be a version of "rational simple connectedness".

The final hypothesis, Hypothesis 4.5, is technical but essential.

Hypothesis 4.5. There exists a morphism $\zeta : \mathbb{P}^1_K \to \overline{\mathrm{M}}_{0,1}(Y/K, 1)$ such that the pullbacks of the vertical tangent bundles $\zeta^* T_\Phi$, $\zeta^* T_{\mathrm{ev}_{0,1,1}}$ and $\zeta^* \mathrm{ev}^*_{0,1,1} T_{Y/K}$ are ample, where the vertical tangent bundle is the dual of the locally free sheaf of relative differentials and where Φ and $\mathrm{ev}_{0,1,1}$ are the morphisms

$$\Phi : \overline{\mathrm{M}}_{0,1}(Y/K, 1) \to \overline{\mathrm{M}}_{0,0}(Y/K, 1), \quad (L, p) \mapsto L,$$

$$\mathrm{ev}_{0,1,1} : \overline{\mathrm{M}}_{0,1}(Y/K, 1) \to Y, \quad (L, p) \mapsto p.$$

Here is an explanation of why Hypothesis 4.5 is essential for Theorem 4.9. For a general hypersurface Y in $\mathbb{P}^n$ of degree d Hypothesis 4.4 hold if and only if $n \geq d^2 - 1$. And Hypothesis 4.5 holds if and only if $n \geq d^2$. So for $n = d^2 - 1$ Hypothesis 4.4 holds yet Hypothesis 4.5 fails. Moreover when $X/C/\kappa$ is a one-parameter family of degree d hypersurfaces in $\mathbb{P}^n$ coming from any sufficiently general rational curve of degree $\geq d$ in the linear system $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, we claim that Theorem 4.9 holds if and only if $n \geq d^2$. Thus Hypothesis 4.5 is essential. The "if" direction of the claim is discussed in [dJHS08] and uses joint work with de Jong and with Harris on rational simple connectedness of hypersurfaces. Via Corollary 8.1 this gives a new proof of the Tsen-Lang theorem, cf. [Lan52], [Tse36]. On the other hand, the Tsen-Lang theorem is sharp: for $n < d^2$ there are explicit families for which Corollary 8.1 fails. Thus also Theorem 4.9 fails for these families. This gives the "only if" direction of the claim.

Assuming Hypotheses4.1 – 4.4, one can conclude something close to Hypothesis 4.5. There exists a morphism $\zeta_{\mathrm{ev}} : \mathbb{P}^1_K \to \overline{\mathrm{M}}_{0,1}(Y, 1)$ such that $\zeta^*_{\mathrm{ev}} T_{\mathrm{ev}}$ and $\zeta^*_{\mathrm{ev}} \mathrm{ev}^* T_{Y/K}$ are both ample, and there also exists a morphism $\zeta_\rho : \mathbb{P}^1_K \to \overline{\mathrm{M}}_{0,1}(Y, 1)$ such that $\zeta^*_\rho T_\rho$ and $\zeta^*_\rho \mathrm{ev}^* T_{Y/K}$ are both ample. But it is essential to have a single morphism ζ such that $\zeta^* T_{\mathrm{ev}}$ and $\zeta^* T_\rho$ are *simultaneously* ample. As just discussed, there is no such morphism when Y is a general degree d hypersurface in $\mathbb{P}^n_K$ with $n < d^2$, even though Hypothesis 4.4 does hold.

As will be proved in Corollary 7.5, Hypothesis 4.5 is essentially equivalent to the existence of a 2-*twisting* scroll for the constant rationally connected fibration $\mathbb{P}^1_K \times_k Y$ over the curve $\mathbb{P}^1_K$: a pair (R, D) consisting of a scroll R for $\mathbb{P}^1_K \times_K Y/\mathbb{P}^1_K$ and a Cartier divisor class D on R such that

(1) $\mathcal{O}_R(D)$ is globally generated and D has relative degree 1 over $\mathbb{P}^1$,
(2) the normal sheaf $N_{R/\mathbb{P}^1\times Y}$ is globally generated, and
(3) $h^1(R, N_{R/\mathbb{P}^1\times Y}(-D) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(-2))$ equals 0.

Existence of a 2-twisting scroll in Y is basically an "infinitesimal homotopy" condition. Consider the reducible curve D' which is the union of a section curve in R of divisor class D and two fibers of the projection. Since the scroll is free, D' is unobstructed, in fact a *porcupine*[8] as will be defined in Definition 5.1. (Moreover, a 2-twisting scroll in Y is free if and only if every such D' is a porcupine.) The porcupine D' is a "horn" in R analogous to the horn in $[0,1] \times [0,1]$ coming from the union $(\{0\} \times [0,1]) \cup ([0,1] \times \{0\}) \cup (\{1\} \times [0,1])$. Of course for a path connected CW complex, every continuous map from the horn in $[0,1] \times [0,1]$ extends to a continuous map from all of $[0,1] \times [0,1]$, i.e., every horn can be filled. The closed immersion of the scroll $R \hookrightarrow \mathbb{P}^1_K \times_K Y$ "fills" the horn $D' \hookrightarrow \mathbb{P}^1_K \times_K Y$. Moreover, the conditions (1), (2) and (3) guarantee that for every infinitesimal deformation of the horn D' in $\mathbb{P}^1_K \times_K Y$ over an arbitrary Artin ring, there exists a deformation of R in $\mathbb{P}^1_K \times_K Y$ over that same Artin ring such that the deformation of D' is a Cartier divisor on the deformation of R, i.e., every infinitesimal deformation of the horn can still be filled. Essentially the conditions (1), (2) and (3) are exactly the conditions necessary to deduce that all deformations of the horn are unobstructed and every deformation of the horn is filled. Again, the analogous topological result is trivial for path connected CW complexes. Nonetheless this is a quite nontrivial hypothesis in the algebraic category, and it is independent of Hypothesis 4.4.

These are all the hypotheses. However, before stating the theorem there are some definitions. The first two definitions quantify the positivity of the normal sheaf of a section. This is important for understanding the deformation theory of sections and stable sections.

Definition 4.6. Let $d \geq 0$ be an integer. For an algebraically closed extension field k of κ, a *d-free section* of π defined over k is a section $\sigma : C_k \to X_k$ defined over k such that for one (and hence every sufficiently general) effective Cartier divisor D of C_k of degree d,

$$h^1(C_k, \sigma^* N_{\sigma(C_k)/X_k}(-D)) \text{ equals } 0.$$

Here C_k equals $C \otimes_\kappa k$ and X_k equals $X \otimes_\kappa k$.

In particular, when $d \geq \max(2g, 1)$ the first cohomology vanishes if and only if the map on global sections

$$H^0(C_k, \sigma^* N_{\sigma(C_k)/X_k}) \to H^0(C_k, \sigma^* N_{\sigma(C_k)/X_k} \otimes_{\mathcal{O}_{C_k}} \mathcal{O}_D)$$

is surjective. So we make a special definition for this case.

Definition 4.7. A section is (g)-*free* if it is d-free for some $d \geq \max(2g, 1)$, i.e., for $g \geq 1$ it is $2g$-free and for $g = 0$ it is 1-free.

[8] *porcupine* Engl. = *porc-épic* Fr.

The main theorem has to do with the existence of a special sequence of irreducible components of $\Sigma^e(X/C/\kappa)$. Since there will be some intermediate lemmas it is convenient to make this a definition.

Definition 4.8. Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. And assume that Hypotheses 4.1, 4.2, and 4.3 hold. Let ϵ be an integer. An *Abel sequence* for $X/C/\kappa$ starting at ϵ is a sequence $(Z_e)_{e\geq\epsilon}$ for each integer $e \geq \epsilon$ of an irreducible component Z_e of $\Sigma^e(X/C/\kappa)$ which is geometrically irreducible (sometimes called *absolutely irreducible*) and satisfying all of the following.

(i) For every $e \geq \epsilon$ a general point of Z_e parameterizes a (g)-free section of π.

(ii) For every $e \geq \epsilon$ the Abel map

$$\alpha|_{Z_e} : Z_e \to \mathrm{Pic}^e_{C/\kappa}$$

is surjective and the geometric generic fiber is integral and rationally connected, i.e., a general pair of Abel equivalent points in Z_e is rationally connected.

(iii) For every algebraically closed field extension k of κ, for every (g)-free section $\sigma_0 : C \otimes_\kappa k \to X$ of π defined over k and having some degree e_0, there exists an integer $\delta_0 \geq \epsilon - e_0$ such that for all integers $\delta \geq \delta_0$, every stable section obtained by attaching δ lines in fibers of π to $\sigma_0(C)$ gives a k-point of $\Sigma^{e_0+\delta}(X/C/\kappa)$ lying in $Z_{e_0+\delta}$ (and lying in no other irreducible component).

A *pseudo Abel sequence* is a sequence $(Z_e)_{e\geq\epsilon}$ as above where (ii) is replaced by the weaker condition that $\alpha|_{Z_e}$ is surjective and the geometric generic fiber is integral (but not necessarily rationally connected).

Theorem 4.9. *Notations are as in Notation 2.4; in particular κ is characteristic 0 but not necessarily algebraically closed. If Hypotheses 4.1 – 4.5 hold then there exists an Abel sequence for $X/C/\kappa$. If only Hypotheses 4.1 – 4.4 hold then there exists a pseudo Abel sequence for $X/C/\kappa$.*

Galois invariance. The main goal of Part 1 is to prove that there exists an Abel sequence for $X/C/\kappa$. This is proved at the end of Section 7. There are several intermediate steps and reductions. The first reduction is to the case that κ is algebraically closed. This uses a Galois invariance result coming from condition (iii) of Definition 4.8. But first there is a trivial result: given a sequence of irreducible components $(Z_e)_{e\geq\epsilon}$ which are geometrically irreducible, one can check whether this is an Abel sequence (or pseudo Abel sequence) after base-change to any extension field, in particular after base-change to an algebraic closure. In this sense the property of being an Abel sequence is *geometric*. Let κ'/κ be a field extension. Denote by $X_{\kappa'}/C_{\kappa'}/\kappa'$ the base-change of $X/C/\kappa$ over κ'. In particular, $\Sigma^e(X_{\kappa'}/C_{\kappa'}/\kappa')$ is canonically isomorphic to $\Sigma^e(X/C/\kappa) \otimes_\kappa \kappa'$ as κ'-schemes. The proof of the following result is an exercise in base-change, which is left to the reader.

Lemma 4.10. *Notations are as in Notation 2.4. Let κ'/κ be a field extension. Let $(Z_e)_{e\geq\epsilon}$ be a sequence of irreducible components of $\Sigma^e(X/C/\kappa)$ which are geometrically irreducible. Then the base-change $Z_e \otimes_\kappa \kappa'$ is an irreducible component of $\Sigma^e(X_{\kappa'}/C_{\kappa'}/\kappa')$ which is geometrically irreducible. Moreover $(Z_e \otimes_\kappa \kappa')_{e\geq\epsilon}$ is an Abel sequence for $X_{\kappa'}/C_{\kappa'}/\kappa'$, resp. a pseudo Abel sequence for $X_{\kappa'}/C_{\kappa'}/\kappa'$, if and only if $(Z_e)_{e\geq\epsilon}$ is an Abel sequence for $X/C/\kappa$, resp. a pseudo Abel sequence for $X/C/\kappa$.*

Denote by k the algebraic closure of κ and denote by $\mathrm{Gal}(\kappa)$ the Galois group. Let T be a finite type κ-scheme. There is an induced action of $\mathrm{Gal}(\kappa)$ on $T \otimes_\kappa k$ covering the action of $\mathrm{Gal}(\kappa)$ on $\mathrm{Spec}\, k$. This is an action by isomorphisms of schemes, hence by homeomorphisms. In particular, there is an induced action of $\mathrm{Gal}(\kappa)$ on the set of irreducible components of $T \otimes_\kappa k$. An elementary result is that every irreducible component of $T \otimes_\kappa k$ dominates an irreducible component of T. And the induced map from the set of irreducible components of $T \otimes_\kappa k$ to the set of irreducible components of T is a surjective set map whose fibers are exactly the Galois orbits. Moreover since T is finite type over κ also $T \otimes_\kappa k$ is finite type over k, and hence both sets of irreducible components are finite. In particular the stabilizer group of each irreducible component of $T \otimes_\kappa k$ has finite index in $\mathrm{Gal}(\kappa)$. So there is a bijection between the set of irreducible components of T and the set of (finite) Galois orbits of irreducible components of $T \otimes_\kappa k$, and this restricts to a bijection between the subset of irreducible components of T which are geometrically integral and the subset of Galois invariant irreducible components of $T \otimes_\kappa k$. The relevant case here is $T = \Sigma^e(X/C/\kappa)$ and $T \otimes_\kappa k = \Sigma^e(X_k/C_k/k)$.

Lemma 4.11. *Notations are as in Notation 2.4. Denote by k the algebraic closure of κ. Assume that there exists a pseudo Abel sequence $(Z_{e,k})_{e\geq\epsilon}$ for $X_k/C_k/k$.*

- (i) *The pseudo Abel sequence is "asymptotically unique": for every pseudo Abel sequence $(Z'_{e,k})_{e\geq\epsilon'}$ for $X_k/C_k/k$ there exists an integer $e'' \geq \max(e, e')$ such that $Z_{e,k}$ equals $Z'_{e,k}$ for every $e \geq \epsilon''$. More generally, for every finite collection of pseudo Abel sequences for $X_k/C_k/k$, there exists an integer ϵ'' such that the all the sequences agree for every integer $e \geq \epsilon''$.*
- (ii) *There exists a finite index subgroup of Gal(κ) preserving all the components $Z_{e,k}$, $e \geq \epsilon$. There exists an integer $\epsilon'' \geq \epsilon$ such that for every $e \geq \epsilon''$, $Z_{e,k}$ is Galois invariant.*
- (iii) *There exists a pseudo Abel sequence for $X/C/\kappa$ and the pseudo Abel sequence is "asymptotically unique" in the sense of (i).*
- (iv) *If there exists an Abel sequence for $X_k/C_k/k$, then there exists an Abel sequence for $X/C/\kappa$.*

Proof. **(i).** Let σ_0 be a (g)-free closed point of $Z_{\epsilon,k}$. By (iii) applied to the sequence $(Z_{e,k})_e$ and to the sequence $(Z'_{e,k})_e$, there exists an integer δ_0 such that for every $\delta \geq \delta_0$, $\epsilon + \delta \geq \max(\epsilon, \epsilon')$ and both $Z_{\epsilon+\delta,k}$ and $Z'_{\epsilon+\delta}$ satisfy (iii) for σ_0 and δ. But the component satisfying (iii) is unique. Thus $Z_{\epsilon+\delta,k}$ equals $Z'_{\epsilon+\delta,k}$ for all $\delta \geq \delta_0$. So the result holds for $\epsilon'' = \epsilon + \delta_0$. And by induction on the number of pseudo Abel sequences, the same holds for every finite collection of pseudo Abel sequences.

(ii). Now let σ_0 be a closed point of $\Sigma^e(X/C/\kappa)$ whose base change $\sigma_{0,k}$ in $\Sigma^e(X_k/C_k/k)$ is a (g)-free closed point of $Z_{\epsilon,k}$. Since $\Sigma^e(X/C/\kappa)$ is finite type over κ, the residue field of σ_0 is a finite extension of κ. Hence there is a finite index subgroup H of $\mathrm{Gal}(\kappa)$ fixing $\sigma_{0,k}$. Thus H is a finite index subgroup of $\mathrm{Gal}(\kappa)$ fixing $Z_{\epsilon,k}$. Iterating this argument, for every integer $\epsilon' \geq \epsilon$ there exists a finite index subgroup $H' \leq H$ fixing $Z_{e,k}$ for every $e = \epsilon, \dots, \epsilon'$.

Let δ_0 be an integer as in (iii) such that for every integer $\delta \geq \delta_0$, $Z_{\epsilon+\delta,k}$ is the unique component containing every curve coming from $\sigma_{0,k}$ by attaching δ free lines. Then H preserves $Z_{\epsilon+\delta}$. Thus choosing $H' \leq H$ to be a finite subgroup as in the previous paragraph fixing Z_e for each $e = \epsilon, \dots, \epsilon + \delta_0 - 1$, then H' fixes Z_e for every $e \geq \epsilon$.

For each ϕ in $\mathrm{Gal}(\kappa)$, the conjugate sequence $({}^{\phi}\ Z_{e,k})_{e\geq\epsilon}$ is also a pseudo Abel sequence. Since H' has finite index in $\mathrm{Gal}(\kappa)$, this gives only finitely many pseudo Abel sequences. Thus by (i) there exists an integer $\epsilon''\geq\epsilon$ such that for every $e\geq\epsilon''$ and for every ϕ in $\mathrm{Gal}(\kappa)$, ${}^{\phi}\ Z_{e,k}$ equals $Z_{e,k}$. Therefore $(Z_{e,k})_{e\geq\epsilon''}$ is Galois invariant.

(iii) and (iv). For every integer $e\geq\epsilon''$, let Z_e denote the image of $Z_{e,k}$ in $\Sigma^e(X/C/\kappa)$. This is an irreducible component of $\Sigma^e(X/C/\kappa)$. Since $Z_{e,k}$ is Galois invariant, Z_e is geometrically irreducible and $Z_{e,k}$ equals $Z_e\otimes_\kappa k$. By Lemma 4.10, $(Z_e)_{e\geq\epsilon''}$ is an Abel sequence for $X/C/\kappa$, resp. a pseudo Abel sequence for $X/C/\kappa$, if and only if $(Z_{e,k})_{e\geq\epsilon''}$ is an Abel sequence for $X_k/C_k/k$, resp. a pseudo Abel sequence for $X_k/C_k/k$. $\square$

Because of the lemma it suffices to prove Theorem 4.9 for algebraically closed, characteristic 0 fields κ. Moreover, for an extension of algebraically closed fields k/κ, by a similar argument to that above, Theorem 4.9 for $X_k/C_k/k$ is equivalent to Theorem 4.9 for $X/C/\kappa$: there is a natural bijection between the set of irreducible components of $\Sigma^e(X/C/\kappa)$ and the set of irreducible components of $\Sigma^e(X_k/C_k/k)$. Thus it suffices to pass to "sufficiently large" algebraically closed fields, e.g., uncountable fields. This will be technically convenient later although, of course, the interested reader can easily formulate all of our arguments over any characteristic 0 field. From this point on we will assume that κ is an **uncountable and algebraically closed** field of characteristic 0.

5. The sequence of components

As always, notations are as in Notation 2.4. And from here on we will assume that κ is **uncountable and algebraically closed**. Throughout this section we assume Hypotheses 4.1 and 4.2 hold. In each result we will specify any additional hypotheses which are needed.

Because of Lemma 4.11, when a pseudo Abel sequence exists for $X/C/\kappa$ it is asymptotically unique. Moreover condition (iii) in Definition 4.8 is a prescription for constructing this sequence. First we need to prove that there exists at least one (g)-free section, which we do in Lemma 5.4. Then we will attach lines in fibers to get stable sections as in (iii). These stable sections will be points of the components Z_e we are trying to construct.

Because the stable sections in (iii) come up so often, we give them a special name, "porcupines", which is meant to capture some aspect of the curve: many lines pointing vertically out of the body like the quills of a porcupines. But also it should remind us that although these curves are better behaved than some of their spinier cousins (more complicated vertical "spines" are difficult to work with, particularly because of stack-theoretic issues they entail), nonetheless even porcupines should be treated with some delicacy. Because we are trying to avoid some technicalities, the definition of porcupine here is stronger than the definition in [dJHS08]. But the basic idea is the same. We trust this will cause no confusion.

Definition 5.1. A *porcupine*[9] in X is a stable section $h : C' \to X$ of π whose associated section $\sigma_0 : C \to X$ – called the *body*[10] of the porcupine – is (g)-free and whose vertical components $h|_{C'_j} : C'_j \to X_{t_j}$ – called the *quills*[11] of the porcupine – are each isomorphisms to a line in X_{t_j}. The *attachment divisor* of the porcupine is the divisor $D = \underline{t}_1 + \cdots + \underline{t}_\delta$ of points in C mapping under σ_0 to nodes of $h(C')$. The pair (σ_0, D) is the *extended body* of the porcupine.

It is quite convenient that the vertical components are all lines: the extended Abel map is easier to understand and specializations of families of porcupines are easier to bound. But the main point is that both the body and quills have positive normal bundle. By standard deformation theory this implies nice "transversality" properties of the Kontsevich space in the neighborhood of each porcupine.

Proposition 5.2. (i) *There is a unique open subscheme $\mathit{Porc}^e(X/C/\kappa)$ of $\Sigma^e(X/C/\kappa)$ whose κ-points parameterize porcupines, this open subscheme is smooth, and this open subscheme is contained in the "fine moduli locus" of $\Sigma^e(X/C/\kappa)$, i.e., the open set where the* 1*-morphism from the stack to the coarse moduli space is an equivalence.*

(ii) *The closed subscheme $\mathit{Porc}^{e,\geq 1}(X/C/\kappa)$ of $\mathit{Porc}^e(X/C/\kappa)$ parameterizing porcupines with at least* 1 *quill is a simple normal crossings divisor (if it is nonempty).*

(iii) *For every integer $\delta \geq 1$, the stratum $\mathit{Porc}^{e,\geq\delta}(X/C/\kappa)$ of the simple normal crossing divisor $\mathit{Porc}^{e,\geq 1}(X/C/\kappa)$ having codimension δ in $\mathit{Porc}^e(X/C/\kappa)$ equals the reduced closed subscheme whose κ-points parameterize porcupines with at least δ quills.*

(iv) *Thus the open subscheme $\mathit{Porc}^{e,\delta}(X/C/\kappa)$ of $\mathit{Porc}^{e,\geq\delta}(X/C/\kappa)$ parameterizing porcupines with precisely δ nodes is a smooth, locally closed subscheme of $\mathit{Porc}^e(X/C/\kappa)$ of pure codimension δ (if it is nonempty). In particular, $\mathit{Porc}^{e,0}(X/C/\kappa)$ is a dense, open subscheme of $\mathit{Porc}^e(X/C/\kappa)$ which equals the open subscheme of $\mathit{Sec}^e(X/C/\kappa)$ parameterizing (g)-free sections.*

Proof. This follows by standard deformation theory, so we will only give a few indications of the proof. A porcupine is a special case of the more general notion of a *comb*, cf. [Kol96, Definition II.7.7]. The basic deformation theory of combs is worked out in [Kol96, Section II.7] and other references, e.g., [Sta09, Section 3.2]. The key ideas in the proof of the proposition are [Kol96, Theorem II.7.9] and [Sta09, Proposition 3.14, Lemma 3.17]. For a porcupine $h' : C' \to X$ with body $\sigma_0 : C \to X$, attachment points $t_1, \dots, t_\delta$ and quills $C'_1, \dots, C'_\delta$, there are exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{N} \longrightarrow N_{C'/X} \longrightarrow \bigoplus_{i=1}^{\delta} T_{p_i}\sigma_0(C) \otimes_\kappa T_{p_i}C'_i \longrightarrow 0,$$

$$0 \longrightarrow \bigoplus_{i=1}^{\delta} N_{C'_i/X}(-p_i) \longrightarrow \mathcal{N} \longrightarrow N_{\sigma_0(C)/X} \longrightarrow 0.$$

Since the body is (g)-free, the last term of the second sequence has vanishing h^1. Since the quills are free curves, the first term of the second exact sequence has vanishing h^1. Thus by the long exact sequence of cohomology, the middle terms of the second exact sequence has vanishing h^1. Since this is also the first term of the

[9] *porcupine* Engl. = *porc-épic* Fr.
[10] *body* Engl. = *corps* Fr.
[11] *quill* Engl. = *piquant* Fr.

first exact sequence, it follows that $h^1(C', N_{C'/X})$ is zero and the map of Zariski tangent spaces

$$H^0(C', N_{C'/X}) \to \bigoplus_{i=1}^{\delta} T_{p_i}\sigma_0(C') \otimes_\kappa T_{p_i}C'_i$$

is surjective. Thus the Kontsevich space is smooth at $(C', h : C' \to X)$ and the map from the formal neighborhood of the Kontsevich space to the versal deformation space of the nodes of C' is smooth. Thus the boundary stratification is a simple normal crossings variety near $(C', h : C' \to X)$. $\square$

There is a morphism

$$\Phi_{\text{body}} : \text{Porc}^{e,\delta}(X/C/\kappa) \to \text{Porc}^{e-\delta,0}(X/C/\kappa)$$

associating to each porcupine $(h : C' \to X)$ with precisely δ quills the (g)-free section $(\sigma_0 : C \to X)$. Denoting by C_δ the δ-fold symmetric power of C over κ, there is a refined version of this morphism

$$\Phi'_{\text{body}} : \text{Porc}^{e,\delta}(X/C/\kappa) \to \text{Porc}^{e-\delta,0}(X/C/\kappa) \times_\kappa C_\delta$$

associating to each porcupine $(h : C' \to X)$ the extended body: σ_0 together with the attachment divisor $D = \underline{t}_1 + \cdots + \underline{t}_\delta$ in C. Of course the image of $\text{pr}_{C_\delta} \circ \Phi'_{\text{body}}$ in C_δ is contained in the dense open subset C^0_δ parameterizing reduced divisors, i.e., unordered δ-tuples $\{t_1, \dots, t_\delta\}$ of distinct points of C.

Lemma 5.3. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that Hypotheses 4.1, 4.2 and 4.3 hold. Let e be an integer and let δ be a nonnegative integer. The morphism*

$$\Phi'_{body} : Porc^{e,\delta}(X/C/\kappa) \to Porc^{e-\delta,0}(X/C/\kappa) \times_\kappa C^o_\delta$$

is smooth, surjective and projective. And the geometric fibers are integral and rationally connected.

Proof. As a morphism of schemes over C_δ, we can understand Φ'_{body} in terms of a fiber product diagram. The evaluation morphism of κ-schemes

$$\text{ev}_{0,1,1} : \overline{\text{M}}_{0,1}(X/C, 1) \to X$$

induces a morphism of the associated δ-fold symmetric powers of these κ-schemes,

$$(\text{ev}_{0,1,1})_\delta : (\overline{\text{M}}_{0,1}(X/C, 1))_\delta \to X_\delta.$$

Similarly there is a morphism

$$(\pi)_\delta : X_\delta \to C_\delta.$$

None of these symmetric powers nor morphisms of symmetric powers need be smooth, but they become smooth when restricted to certain open subsets. Denote by X^o_δ the inverse image $(\pi)_\delta^{-1}(C^o_\delta)$, and denote by $(\overline{\text{M}}_{0,1}(X/C, 1))^o_\delta$ the inverse image $(\text{ev}_{0,1,1})_\delta^{-1}(X^o_\delta)$. Assuming Hypotheses 4.1 – 4.3 hold so that $\text{ev}_{0,1,1}$ is smooth, surjective and projective with integral and rationally connected generic fibers, then also

$$(\text{ev}_{0,1,1})^o_\delta : (\overline{\text{M}}_{0,1}(X/C, 1))^o_\delta \to X^o_\delta$$

is a smooth, surjective, projective morphism of smooth, quasi-projective C^o_δ-schemes. And each geometric fiber is a fiber product of δ base-changes of geometric fibers of $\text{ev}_{0,1,1}$. Thus the geometric generic fiber of $(\text{ev}_{0,1,1})^o_\delta$ is also integral and rationally connected.

There is another relevant evaluation morphism,

$$\mathrm{ev}^o_{g,\{\delta\},e-\delta} : \mathrm{Porc}^{e-\delta,0}(X/C/\kappa)\times_\kappa C^o_\delta \to X^o_\delta,\ \ (\sigma_0, \{t_1, \dots, t_\delta\}) \mapsto \{\sigma_0(t_1), \dots, \sigma_0(t_\delta)\}.$$

Again assuming Hypotheses 4.1 – 4.3 hold then the fiber product of $(\mathrm{ev}_{0,1,1})^o_\delta$ and $\mathrm{ev}^o_{g,\{\delta\},e-\delta}$,

$$(\overline{\mathrm{M}}_{0,1}(X/C,1))^o_\delta \times_{X^o_\delta} (\mathrm{Porc}^{e-\delta,0}(X/C/\kappa) \times_\kappa C^o_\delta) \xrightarrow{\mathrm{pr}_2} \mathrm{Porc}^{e-\delta,0}(X/C/\kappa) \times_\kappa C^o_\delta$$

is equivalent to Φ'_{body} as morphisms to $\mathrm{Porc}^{e-\delta,0}(X/C/\kappa) \times_\kappa C^o_\delta$. Thus since $(\mathrm{ev}_{0,1,1})^o_\delta$ is smooth, surjective and projective with integral and rationally connected generic fibers, the same is true of Φ'_{body}. □

Geometrically the way we construct (g)-free sections is to find a family of sections which is "sufficiently moving", and then take the general member. The next lemma explains why this works.

Lemma 5.4. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Let δ denote the integer $\max(1, 2g)$. Let $t_1, \dots, t_\delta$ be distinct κ-points of C. Let S be an irreducible, finite type κ-scheme. And let*

$$\tau : S \times_\kappa C \to X, \quad (s,t) \mapsto \tau(s,t)$$

be a family of sections of π parameterized by S. If the associated κ-morphism

$$\tau_{t_1,\dots,t_\delta} : S \to X_{t_1} \times_\kappa \cdots \times_\kappa X_{t_\delta}, \quad s \mapsto (\tau(s,t_1), \dots, \tau(s,t_\delta))$$

is dominant, then a general point of S parameterizes a (g)-free section.

Conversely, if $\sigma_0 : C \to X$ is a (g)-free section, then for a general point collection $t_1, \dots, t_\delta$ of κ-points of C and for the universal section

$$\sigma : \mathit{Sec}(X/C/\kappa) \times_\kappa C \to X$$

the associated κ-morphism

$$\sigma_{t_1,\dots,t_\delta} : \mathit{Sec}(X/C/\kappa) \to X_{t_1} \times_\kappa \cdots \times_\kappa X_{t_\delta}$$

is smooth at $[\sigma_0]$.

Proof. There is a dense open subscheme of S which is smooth. After replacing S by this open subscheme, assume S is smooth. Then by generic smoothness, $\tau_{t_1,\dots,t_d}$ is smooth at a general point s of S. By the Jacobian criterion this implies that

$$H^0(C, \tau_s^* T_\pi) \to H^0(C, \tau_s^* T_\pi \otimes_{\mathcal{O}_C} \mathcal{O}_D)$$

is surjective, where $D = \underline{t}_1 + \cdots + \underline{t}_\delta$. Since $\delta = \max(1, 2g)$, this implies that

$$h^1(C, \tau_s^* T_\pi(-D)) \text{ equals } 0.$$

Thus τ_s is (g)-free.

Conversely, if σ_0 is (g)-free, then $\mathrm{Sec}(X/C/\kappa)$ is smooth at $[\sigma_0]$ and the derivative of $\sigma_{t_1,\dots,t_d}$ at $[\sigma_0]$ is precisely

$$H^0(C, \sigma_0^* T_\pi) \to H^0(C, \sigma_0^* T_\pi \otimes_{\mathcal{O}_C} \mathcal{O}_D)$$

which is surjective. Thus $\sigma_{t_1,\dots,t_d}$ is smooth at $[\sigma_0]$. □

Because of Lemma 5.4, there exists at least one (g)-free section. This is the "seed" from which our Abel sequence will grow.

Corollary 5.5. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume Hypotheses 4.1 – 4.4 hold. There exists an integer e_0 such that for every integer $e \geq e_0$, $Porc^{e,0}(X/C/\kappa)$ is nonempty, i.e., there exists a (g)-free section of degree e.*

Proof. By Hypothesis 4.4, the geometric generic fiber Y is rationally chain connected. By Hypothesis 4.1 it is also smooth and projective. Thus Y is rationally connected, cf. [Kol96, Theorem IV.3.10.3]. Denote by δ the integer $\max(2g, 1)$. By Corollary 2.2, for every κ-point $(p_1, \dots, p_\delta)$ of

$$X_{(t_1,\dots,t_\delta)} := X_{t_1} \times_\kappa \cdots \times_\kappa X_{t_\delta}$$

there exists a section $\sigma : C \to X$ with $\sigma(t_j) = p_j$ for each $j = 1, \dots, \delta$. Every such σ is a κ-point of one of the countably many irreducible components $(S_i)_{i\in\mathbb{Z}}$ of $\Sigma(X/C/\kappa)$. Denote by C_i the closed image of S_i in $X_{(t_1,\dots,t_\delta)}$. Then $X_{(t_1,\dots,t_\delta)}(k)$ equals $\cup_{i\in\mathbb{Z}} C_i(k)$. Since κ is uncountable and since $X_{(t_1,\dots,t_\delta)}$ is an irreducible κ-scheme, there exists $i \in \mathbb{Z}$ such that C_i equals $X_{(t_1,\dots,t_\delta)}$. By Lemma 5.4 a general κ-point of S_i parameterizes a (g)-free section of π.

Let e_0 be the degree of the section above. By Proposition 5.2 $\mathrm{Porc}^{e_0,0}(X/C/\kappa)$ is nonempty since it parameterizes that section. By Lemma 5.3 for every integer $e > e_0$,

$$\Phi'_{\text{body}} : \mathrm{Porc}^{e,e-e_0}(X/C/\kappa) \to \mathrm{Porc}^{e_0,0}(X/C/\kappa) \times_\kappa C_{e-e_0}$$

is dominant. In particular $\mathrm{Porc}^{e}(X/C/\kappa)$ is nonempty. Hence by Proposition 5.2 also $\mathrm{Porc}^{e,0}(X/C/\kappa)$ is nonempty. □

Let e_0 be as in Corollary 5.5. And let $Z = Z_{e_0}$ be the closure in $\Sigma^{e_0}(X/C/\kappa)$ of one of the components of the nonempty, dense open subset $\mathrm{Porc}^{e_0,0}(X/C/\kappa)$. Denote this component of $\mathrm{Porc}^{e_0,0}(X/C/\kappa)$ by $\mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z$.

For every integer $e \geq e_0$, by Lemma 5.3 the morphism

$$\Phi'_{\text{body}} : \mathrm{Porc}^{e,e-e_0}(X/C/\kappa) \to \mathrm{Porc}^{e_0,0}(X/C/\kappa) \times_\kappa C^o_{e-e_0}$$

is smooth, surjective and projective with integral geometric fibers. So for the integral, open and closed subscheme $\mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z \times_\kappa C^o_{e-e_0}$ of the target, the inverse image $Z_{e,e-e_0}$ in $\mathrm{Porc}^{e,e-e_0}(X/C/\kappa)$ is an integral, open and closed subscheme of $\mathrm{Porc}^{e,e-e_0}(X/C/\kappa)$. Since $\mathrm{Porc}^{e}(X/C/\kappa)$ is smooth, there is a unique connected component Z^o_e of $\mathrm{Porc}^{e}(X/C/\kappa)$ containing $Z_{e,e-e_0}$.

Definition 5.6. Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Let e_0 be an integer and let Z be an irreducible component of $\Sigma^{e_0}(X/C/\kappa)$ which parameterizes at least one (g)-free section. For every integer $e \geq e_0$, the *distinguished irreducible component* Z_e of $\Sigma^e(X/C/\kappa)$ associated to Z is the closure in $\Sigma^e(X/C/\kappa)$ of the connected component Z^o_e of $\mathrm{Porc}^e(X/C/\kappa)$ as constructed above. For every integer $e \geq e_0$ and for every integer $\delta \geq 0$, $\mathrm{Porc}^{e,\delta}(X/C/\kappa)_Z$ denotes the intersection of $\mathrm{Porc}^{e,\delta}(X/C/\kappa)$ and Z_e in $\Sigma^e(X/C/\kappa)$ (this intersection is a smooth, quasi-projective κ-scheme).

This is the sequence of irreducible components we work with. By Proposition 5.2, this sequence satisfies (i) from Definition 4.8. Next we prove this sequence satisfies the "integrality" part of (ii). In Section 6 we will verify (iii), which implies the existence of pseudo Abel sequences in Corollary 6.8. But the hardest step is proving the "rational connectedness" part of (iii), which is completed in Section 7 using Hypothesis 4.5.

Lemma 5.7. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that Hypotheses 4.1 – 4.4 hold and let e_0 and Z be as in Definition 5.6. For every integer $e \geq e_0+2g-1$, the restriction of the Abel map*

$$\alpha|_{Porc^{e,e-e_0}(X/C/\kappa)_Z} : Porc^{e,e-e_0}(X/C/\kappa)_Z \to Pic^e_{C/\kappa}$$

is smooth and dominant with integral geometric generic fiber.

Proof. A composition of smooth, dominant morphisms having integral geometric generic fibers is also a smooth, dominant morphism have integral geometric generic fiber. Thus it suffices to decompose $\alpha|_{\mathrm{Porc}^{e,e-e_0}(X/C/\kappa)_Z}$ as a composition of such morphisms.

For ease of notation denote $e-e_0$ by δ so that e equals $e_0+\delta$. Since e is $\geq e_0+2g-1$, also δ is $\geq 2g-1$. And when δ is $\geq 2g-1$, the usual Abel map

$$\gamma : C_\delta \to \mathrm{Pic}^\delta_{C/\kappa}$$

is smooth, surjective and projective with integral geometric fibers: it is a projective $\mathbb{P}^{\delta-g}$-bundle bundle by Abel's theorem and Jacobi inversion. The restricted Abel map $\alpha|_{\mathrm{Porc}^{e,\delta}(X/C/\kappa)_Z}$ factors as the composition of

$$\Phi'_{\mathrm{body}} : \mathrm{Porc}^{e_0+\delta,\delta}(X/C/\kappa)_Z \to \mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z \times_\kappa C_\delta$$

and the morphism

$$\beta : \mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z \times_\kappa C_\delta \to \mathrm{Pic}^{e_0+\delta}_{C/\kappa}$$

where $\beta([\sigma_0],[D])$ equals $\sigma_0^*\mathcal{L}(D)$. By Lemma 5.3, the first morphism is smooth and dominant with integral geometric generic fiber. Thus to prove that $\alpha|_{\mathrm{Porc}^{e,\delta}(X/C/\kappa)_Z}$ is smooth and dominant with integral geometric generic fiber, it suffices to prove the same of β.

The morphism β factors as the composition of a projective bundle,

$$(\mathrm{pr}_1, \gamma\circ\mathrm{pr}_2) : \mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z \times_\kappa C_\delta \to \mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z \times_\kappa \mathrm{Pic}^\delta_{C/\kappa},$$

an isomorphism

$$(\mathrm{pr}_1, \mathrm{pr}_2+\alpha\circ\mathrm{pr}_1) : \mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z \times_\kappa \mathrm{Pic}^\delta_{C/\kappa} \to \mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z \times_\kappa \mathrm{Pic}^\delta_{C/\kappa}$$

and the smooth, surjective projection

$$\mathrm{pr}_2 : \mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z \times_\kappa \mathrm{Pic}^\delta_{C/\kappa} \to \mathrm{Pic}^\delta_{C/\kappa}$$

whose geometric generic fiber is the base change of the integral scheme $\mathrm{Porc}^{e_0,0}(X/C/\kappa)_Z$. Since each of the factors is a smooth, dominant morphism whose geometric generic fiber is integral, the same is true of β. □

Lemma 5.8. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that Hypotheses 4.1 – 4.4 hold and let e_0 and Z be as in Definition 5.6. For every integer $e \geq e_0+2g-1$, the restriction of the Abel map*

$$\alpha|_Z : Z_e \to Pic^e_{C/\kappa}$$

is surjective and the fiber over the geometric generic point of $Pic^{e_0}_{C/\kappa}$ is irreducible.

Proof. By Lemma 5.7 the morphism

$$\alpha|_{\mathrm{Porc}^{e,e-e_0}(X/C/\kappa)_Z} : \mathrm{Porc}^{e,e-e_0}(X/C/\kappa)_Z \to \mathrm{Pic}^e_{C/\kappa}$$

is dominant and that the geometric generic fiber is irreducible. Also Z_e is irreducible and $\mathrm{Porc}^{e,e-e_0}(X/C/\kappa)_Z$ is contained in the smooth locus of Z_e. Thus also $\alpha|_{Z_e}$ is dominant and the geometric generic fiber is irreducible. Since Z_e is projective, the morphism $\alpha|_{Z_e}$ is projective. Every projective, dominant morphism is surjective. □

6. Rational connectedness of the boundary modulo the interior

The *boundary* of a Kontsevich (algebraic) space is the closed subspace parameterizing stable maps with singular domain. The open complement of the boundary is the *interior*. The boundary has a stratification where the deeper strata correspond to maps with many nodes. This stratification is studied in detail in an article of Behrend and Manin, [BM96]. The intersection of the boundary stratification with $\mathrm{Porc}^e(X/C/\kappa)$ is simply the stratification from Proposition 5.2. The interior of $\mathrm{Porc}^e(X/C/\kappa)$ is the open subscheme $\mathrm{Porc}^{e,0}(X/C/\kappa)$ parameterizing (g)-free sections.

In the last section we constructed the sequence $(Z_e)_{e\geq e_0}$. (Of course the "starting integer" ϵ will have to be chosen larger than e_0, as is already clear from Lemma 5.7.) If there exists an Abel sequence, resp. a pseudo Abel sequence, then eventually it equals $(Z_e)_{e\geq e_0}$. So the goal is to prove that $(Z_e)_{e\geq e_0}$ eventually becomes an Abel sequence, i.e., it satisfies (i), (ii) and (iii) of Definition 4.8.

The sequence $(Z_e)_{e\geq e_0}$ satisfies (i) of Definition 4.8 since each Z_e is the closure of the nonempty open $\mathrm{Porc}^{e,0}(X/C/\kappa)_Z$ parameterizing (g)-free sections. And Lemma 5.8 proves that $(Z_e)_{e\geq e_0}$ satisfies the "integrality" part of (ii). To complete (ii) we must prove that the geometric generic fiber of the Abel map is rationally connected. Of course the arguments of the previous section do not prove this since they do not use Hypothesis 4.5, and they do not use the "rational connectedness" in Hypothesis 4.4. But Lemma 5.7 does prove something noteworthy, which we explain below. This uses the notion of the *maximally rationally connected fibration* of a variety and its corresponding *MRC quotient*, cf. [Kol96, Section IV.5]. The MRC quotient is a measure of the failure of rational connectedness for a variety; a variety is rationally connected if and only if the MRC quotient is a point. To prove (ii), we need to prove that the MRC quotients of the geometric generic fibers of $\alpha|_{Z_e}$ eventually stabilize to a point.

By the factorization in Lemma 5.7, for all $e \geq e_0 + 2g - 1$ the MRC quotient of the geometric generic fiber of the morphism

$$\alpha|_{\mathrm{Porc}^{e,e-e_0}(X/C/\kappa)\cap Z_e} : \mathrm{Porc}^{e,e-e_0}(X/C/\kappa)_Z \to \mathrm{Pic}^e_{C/\kappa}$$

is a quotient of the MRC quotient of Z_{e_0}. Using this, one can show that the MRC quotients eventually stabilize. But if $g > 0$ then they do *not* stabilize to a point; in fact they stabilize to a variety which dominates $\mathrm{Pic}^0(C)$. This same argument would hold if we chose e_0 to be a larger integer. So the conclusion applies to the *δ-boundary stratum* $\mathrm{Porc}^{e,\geq\delta}(X/C/\kappa)_Z$ for each $\delta \geq 2g-1$: we cannot connect two general Abel equivalent points of the δ-boundary stratum by a rational curve which is *entirely* contained in that stratum. To rationally connect Abel equivalent points in the δ-boundary stratum we must use rational curves that pass out of that stratum, e.g., rational curves which intersect the interior. In this section, using the rational connectedness of fibers of $\mathrm{ev}_{0,\{\delta\},e-\delta}$ as in Hypothesis4.4, we prove that we can connect Abel equivalent points of $\mathrm{Pic}^{e,\delta}(X/C/\kappa)$ if δ is sufficiently large relative to e. Speaking loosely, "the boundary is rationally connected modulo the interior".

This section also proves that $(Z_e)_{e\geq e_0}$ satisfies (iii) of Definition 4.8. It is worth noting that Hypothesis 4.5 is never used in this section. Of course for this reason there must be something more in the proof of Theorem 4.9: as discussed there are counterexamples to Theorem 4.9 which satisfy Hypothesis 4.4 but not Hypothesis 4.5. In the next section we will use Hypothesis 4.5 to show that "the interior is rationally connected modulo the boundary", i.e., for a general pair of a point in the interior and an Abel equivalent point in the δ-boundary stratum, there exists a rational curve connecting the two points. This will prove (ii) and hence also Theorem 4.9.

Let δ be a positive integer. Let $t_1,\dots,t_\delta$ be distinct κ-points of C. The following notation and proposition are useful for showing that certain families of sections introduced later always parameterize some (g)-free sections. In one crucial step, Claim 6.6, in the proof of Proposition 6.5 an open set O will appear about which we *a priori* know nothing except that it is dense (in fact the point of the claim is that there does not exist an open set O as in that proof, but the contradiction proof of the claim requires working with such an O to conclude it cannot exist). Because of this, we have to introduce variants of all of our notations which are adapted to some variable open set O.

Notation 6.1. Denote by $X_{(t_1,\dots,t_\delta)}$ the product $X_{t_1}\times_k\cdots\times_k X_{t_\delta}$. And denote by $\mathrm{Chn}_2(X/C,n)_{(t_1,\dots,t_\delta)}$ the product

$$\mathrm{Chn}_2(X/C,n)_{(t_1,\dots,t_\delta)} := \mathrm{Chn}_2(X_{t_1}/\kappa,n)\times_\kappa\cdots\times_\kappa \mathrm{Chn}_2(X_{t_\delta}/\kappa,n).$$

Finally, for every dense open subset O of $X_{(t_1,\dots,t_\delta)}$, denote by $\mathrm{Chn}_2^O(X/C,n)_{(t_1,\dots,t_\delta)}$ the open subscheme of $\mathrm{Chn}_2(X/C,n)_{(t_1,\dots,t_\delta)}$ parameterizing sequences $((C'_j,p_{1,j},q_{n,j}))_{j=1,\dots,\delta}$ of chains of n lines

$$(C'_j,p_{1,j},q_{n,j}) = ((L_{1,j},p_{1,j},q_{1,j}),\dots,(L_{n,j},p_{n,j},q_{n,j}))\in \mathrm{Chn}_2(X_{t_j}/\kappa,n)$$

such that for every $i=1,\dots,n$, $(p_{i,1},\dots,p_{i,\delta})$ and $(q_{i,1},\dots,q_{i,\delta})$ are each contained in the open subset O of $X_{(t_1,\dots,t_d)}$: all sequences of marked points and all sequence of nodes of the chain (i.e., all sequences of "special points") are contained in the open set O.

Hypothesis 4.4 involves chains of lines of some fixed length n_0. The next proposition shows that the same property holds for chains of lines of length n for all $n\geq n_0$. The "O variant" will be useful in Claim 6.6

Proposition 6.2. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume Hypotheses 4.1 – 4.4 hold. Let n_0 be as in Hypothesis 4.4. For every $n\geq n_0$, $Chn_2(Y/K,n)$ is smooth and irreducible. And the morphism*

$$ev: Chn_2(Y/K,n)\to Y\times_K Y$$

is surjective with integral and rationally connected geometric generic fiber. Moreover for every $n\geq 2n_0$, for every positive integer δ, for every sequence $t_1,\dots,t_\delta$ of *distinct κ-points of C, and for every dense open subset O of $X_{(t_1,\dots,t_\delta)}$, the evaluation morphism*

$$ev\colon Chn_2^O(X/C,n)_{(t_1,\dots,t_\delta)} \to O\times_\kappa O,$$
$$((C_1,p_{1,1},q_{n,1}),\dots,(C_\delta,p_{1,\delta},q_{n,\delta}))\mapsto((p_{1,1},\dots,p_{1,\delta}),(q_{n,1},\dots,q_{n,\delta}))$$

is surjective.

Proof. The first statement is proved by induction on n. The base case $n=n_0$ follows from Hypothesis 4.4 Thus, by way of induction, assume $n>n_0$ and assume the result is known for $n-1$. The forgetful morphism

$$\mathrm{Chn}_2(Y/K,n)\to\mathrm{Chn}_2(Y/K,n-1),$$
$$((L_1,p_1,q_1),\dots,(L_{n-1},p_{n-1},q_{n-1}),(L_n,p_n,q_n))\mapsto((L_1,p_1,q_1),\dots,(L_{n-1},p_{n-1},q_{n-1}))$$

is a smooth, surjective, projective morphism with integral and rationally connected geometric fibers; it is a base-change of the evaluation morphism $\mathrm{ev}_{0,1,1}$ composed with projection from a $\mathbb{P}^1$-bundle. Since $\mathrm{Chn}_2(Y/K,n-1)$ is smooth and irreducible by the induction hypothesis, also $\mathrm{Chn}_2(Y/K,n)$ is smooth and irreducible.

Next let x and y be general K-points of Y. Denote by $F_{x,y}$ the variety parameterizing chains of n lines

$$(C',p_1,q_n)=((L_1,p_1,q_1),\dots,(L_{n-1},p_{n-1},q_{n-1}),(L_n,p_n,q_n))$$

with $p_1=x$ and $q_n=y$. To give such a chain, it is equivalent to give a line L_n in Y containing $q_n=y$, to give a point p_n in L_n, and then to give a chain of $(n-1)$ free lines

$$(C'',p_1,q_{n-1})=((L_1,p_1,q_1),\dots,(L_{n-1},p_{n-1},q_{n-1}))$$

with $p_1=x$ and $q_{n-1}=p_n$. By Hypothesis 4.3 the variety A parameterizing free lines L_n containing $q_n=y$ is integral and rationally connected. The variety B parameterizing pairs (L_n,p_n,q_n) is a $\mathbb{P}^1$-bundle over A, i.e., the $\mathbb{P}^1$ is simply L_n. Thus B is rationally connected. Moreover, since y is general, a general point p_n in L_n is also general (it is left to the reader to make a meaningful statement from this sentence). Thus, by the induction hypothesis, the variety parameterizing free chains (C'',p_1,q_{n-1}) with $p_1=x$ and $q_{n-1}=p_n$ is rationally connected. Thus $F_{x,y}$ is a rationally connected fibration over the rationally connected variety B. By Corollary 2.3, a rationally connected fibration over a rationally connected base has rationally connected total space. Thus $F_{x,y}$ is rationally connected. So the first part of the proposition is proved by induction on n.

The second part of the proposition is proved in a similar way. Let n_x and n_y be integers with $n_x,n_y\geq 0$. Let $x=(x_1,\dots,x_\delta)$ and $y=(y_1,\dots,y_\delta)$ be κ-points of O, i.e., (x,y) is a κ-point of $O\times_\kappa O$. Denote by $F_x(n_x)$, resp. $F_y(n_y)$, the subset of $\mathrm{Chn}_2(X/C,n_x)_{(t_1,\dots,t_\delta)}$, resp. of $\mathrm{Chn}_2(X/C,n_y)_{(t_1,\dots,t_\delta)}$, parameterizing chains

$$((C'_{x,1},p_{1,1},q_{n_x,1}),\dots,(C'_{x,\delta},p_{1,\delta},q_{n_x,\delta}))$$

with $(p_{1,1},\dots,p_{1,\delta})=(x_1,\dots,x_\delta)$, resp. parameterizing chains

$$((C'_{y,1},p_{n_x+1,1},q_{n,1}),\dots,(C'_{y,\delta},p_{n_x+1,\delta},q_{n,\delta}))$$

with $(q_{n,1},\dots,q_{n,\delta})=(y_1,\dots,y_\delta)$. And denote by $F_x^O(n_x)$, resp. $F_y^O(n_y)$, the intersection of $F_x(n_x)$ with $\mathrm{Chn}_2^O(X/C,n_x)$, resp. the intersection of $F_y(n_y)$ with $\mathrm{Chn}_2^O(X/C,n_y)$.

Since $\mathrm{Chn}_2^O(X/C, n_X)$ is an open subset of $\mathrm{Chn}_2(X/C, n_x)$, also $F_x^O(n_x)$ is an open subset of $F_x(n_x)$. Similarly $F_y^O(n_y)$ is an open subset of $F_y(n_y)$. Next we prove by induction on n_x, resp. n_y, that $F_x^O(n_x)$ is dense in $F_x(n_x)$, resp. $F_y^O(n_y)$ is dense in $F_y(n_Y)$. Here is the induction argument for $F_x^O(n_x)$, the argument for $F_y^O(n_Y)$ is the same. For the base case, $n_x = 1$ we must prove that $F_x^O(1)$ is dense in $F_x(1)$. Let $(L_{1,1}, \ldots, L_{1,\delta})$ be a sequence of lines such that each $L_{1,j}$ contains $p_{1,j} = x_j$. Then the following subscheme of of $X_{(t_1,\ldots,t_\delta)}$,

$$L_1 := L_{1,1} \times \cdots \times L_{1,\delta}$$

is an irreducible subvariety and parameterizes the choices for the points $(q_{1,1}, \ldots, q_{1,\delta})$ making up a chain

$$((L_{1,1}, p_{1,1}, q_{1,1}), \ldots, (L_{1,\delta}, p_{1,\delta}, q_{1,\delta})$$

in $F_x(1)$. The intersection of L_1 with O is an open subset which contains the point $(p_{1,1}, \ldots, p_{1,d})$. Since L_1 is irreducible, this open subset is dense in L_1. Thus $F_x^O(1)$ is dense in $F_x(1)$, establishing the base case. For $n_x > 1$, there is a forgetful morphism

$$\Phi : F_x(n_x) \to F_x(1)$$

which is smooth and surjective with integral geometric fibers. Since $F_x^O(1)$ is dense in $F_x(1)$, also $\Phi^{-1}(F_x^O(1))$ is dense in $F_x(n_x)$. And for a point

$$(L, p, q) := ((L_{1,1}, p_{1,1}, q_{1,1}), \ldots, (L_{1,\delta}, p_{1,\delta}, q_{1,\delta})$$

in $F_x(1)$, setting $x' := (q_{1,1}, \ldots, q_{1,\delta})$, the fiber $\Phi^{-1}(L, p, q)$ is canonically isomorphic to $F_{x'}(n_x)$. Assuming (L, p, q) is in $F_x^O(1)$, the intersection of $\Phi^{-1}(L, p, q)$ with $F_x^O(n_x)$ is canonically isomorphic to $F_{x'}^O(n_x - 1)$. By the induction hypothesis $F_{x'}^O(n_x - 1)$ is dense in $F_{x'}(n_x - 1)$. Hence also $F_x^O(n_x)$ is dense in $F_x(n_x)$.

Now assume that n_x and n_y are both $\geq n_0$ and that $n_x + n_y$ equals n. By Hypothesis 4.4 the subset of $X_{(t_1,\ldots,t_\delta)}$ parameterizing those κ-points $(q_{n_x,1}, \ldots, q_{n_x,\delta})$, resp. $(p_{n_x+1,1}, \ldots, p_{n_x+1,\delta})$, arising from chains in $F_x(n_x)$, resp. from chains in $F_y(n_y)$, contains a nonempty open subset of $X_{(t_1,\ldots,t_\delta)}$ (the proof is similar to the proof of the first part of the lemma). Since $F_x^O(n_x)$ is a dense open in $F_x(n_x)$ the set parameterizing $(q_{n_x,1}, \ldots, q_{n_x,\delta})$ arising from chains in $F_x^O(n_x)$ also contains a nonempty open subset of $X_{(t_1,\ldots,t_\delta)}$. Similarly the set parameterizing $(p_{n_x+1,1}, \ldots, p_{n_x+1,\delta})$ arising from chains in $F_y^O(n_y)$ contains a nonempty open subset of $X_{(t_1,\ldots,t_\delta)}$. Since $X_{(t_1,\ldots,t_\delta)}$ is irreducible, these nonempty open subsets intersect. Thus there exist chains in $F_x^O(n_x)$ and in $F_y^O(n_Y)$ such that $(q_{n_x,1}, \ldots, q_{n_x,\delta})$ equals $(p_{n_x+1,1}, \ldots, p_{n_x+1,\delta})$. By concatenating these chains, i.e., by forming the union $C_i = C_{x,i} \cup_{q_{n_x,i} \sim p_{n_x+1,i}} C_{y,i}$, there is a chain in $\mathrm{Chn}_2^O(X/C, n)_{(t_1,\ldots,t_\delta)}$ with $(p_{1,1}, \ldots, p_{1,\delta}) = (x_1, \ldots, x_\delta)$ and with $(q_{n,1}, \ldots, q_{n,\delta}) = (y_1, \ldots, y_\delta)$. $\square$

To prove the "rational connectedness" part of (ii) of Definition 4.8, we need to prove that Abel equivalent porcupines are rationally connected. Working with one porcupine already requires care. Getting two porcupines to "connect" is difficult; usually porcupines run wild having nothing to do with one another. But we can encourage porcupines to connect by putting them both in a common "pen". Here a "pen" is a scroll which contains a porcupine. The crucial point is that porcupines of the same degree which are penned in a common scroll are connected by a chain of rational curves if and only if they are Abel equivalent.

Definition 6.3. A porcupine $h : C' \to X$ is *penned*[12] by a scroll R for $X/C/\kappa$ (recall Definition 3.5) if $h(C')$ is contained in R. And then R is called a *pen*[13] for h.

Let e be an integer, and let δ_a, δ_b be nonnegative integers. Let $h_a : C'_a \to X$, resp. $h_b : C'_b \to X$, be a porcupine in $\mathrm{Porc}^{e,\delta_a}(X/C/\kappa)$, resp. in $\mathrm{Porc}^{e,\delta_b}(X/C/\kappa)$, with extended body $(\sigma_{0,a}, D_a)$, resp. $(\sigma_{0,b}, D_b)$. Assume that $\sigma_{0,a}$ and $\sigma_{0,b}$ are penned in a common scroll

$$\rho = \pi|_R : R \to C$$

for $X/C/\kappa$.

Lemma 6.4. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Let e be an integer and let h_a and h_b be porcupines whose bodies are penned in a common scroll R as above. The Abel images $\alpha(h_a)$ and $\alpha(h_b)$ in $Pic^e_{C/\kappa}$ are equal if and only if the Cartier divisors $\sigma_{0,a}(C) + \rho^* D_a$ and $\sigma_{0,b}(C) + \rho^* D_b$ are linearly equivalent in R. In this case, assuming Hypotheses 4.1 – 4.4 hold, there exists a connected chain of rational curves in $\Sigma^e(X/C/\kappa)$ containing $[h_a]$ and $[h_b]$ and whose nodes all parameterize porcupines in $Porc^e(X/C/\kappa)$ – in particular, smooth points of $\Sigma^e(X/C/\kappa)$.*

Proof. The Abel images are equal if and only if $\sigma^*_{0,a}\mathcal{L}(D_a)$ is isomorphic to $\sigma^*_{0,b}\mathcal{L}(D_b)$. Since R is a $\mathbb{P}^1$-bundle over C and $\mathcal{L}$ is a relative $\mathcal{O}(1)$-sheaf, these invertible sheaves are isomorphic if and only if the divisors $\sigma_{0,a}(C) + \rho^* D_a$ and $\sigma_{0,b}(C) + \rho^* D_b$ are linearly equivalent in R.

Assume these divisors are linearly equivalent and assume Hypotheses 4.1 − 4.4 hold. For every κ-point t in D_a, resp. in D_b, the variety parameterizing lines containing $\sigma_{0,a}(t)$, resp. $\sigma_{0,b}(t)$, is smooth, projective and rationally connected. Therefore there exists a morphism from $\mathbb{P}^1$ to $\mathrm{Porc}^{e,\delta_a}(X/C/\kappa)$, resp. to $\mathrm{Porc}^{e,\delta_b}(X/C/\kappa)$, sending 0 to h_a, resp. to h_b, sending every point of $\mathbb{P}^1$ to a porcupine with extended body $(\sigma_{0,a}, D_a)$, resp. $(\sigma_{0,b}, D_b)$, and mapping ∞ to a porcupine whose quill at each attachment point t is the line $R_t = \rho^{-1}(t)$. Speaking loosely, such a morphism from $\mathbb{P}^1$ is a "pivot" holding the body of the porcupine fixed and pivoting each quill into the pen.

The image under ∞ of each of these pivots is a Cartier divisor in R which equals $\sigma_{0,a}(C) + \rho^* D_a$, resp. $\sigma_{0,b}(C) + \rho^* D_b$. Since these divisors in R are linearly equivalent, there is a pencil of divisors in R containing these two divisors. The general member of this pencil is a stable section in $\Sigma^e(X/C/\kappa)$. Thus the pencil determines a rational map from the base $\mathbb{P}^1$ of the pencil to $\Sigma^e(X/C/\kappa)$. Since $\Sigma^e(X/C/\kappa)$ is a projective scheme (recall we work with the coarse moduli space rather than the Deligne-Mumford stack), this rational map extends to give a rational curve in $\Sigma^e(X/C/\kappa)$ containing the two new stable sections. The union of this rational curve with the rational curves from the previous paragraph is a rational curve as in the statement of the lemma. □

Please note from this proof that some of the rational curves in the chain may intersect the interior of Z_e. That is why these arguments only prove rational connectedness "modulo the interior". Of course this lemma cannot be applied directly

[12] *penned* Engl. = *enclos* Fr.
[13] *pen* Engl. = *enclos* Fr.

to prove rational connectedness modulo the interior, since a typical pair of porcupines are not penned by a common scroll. However Hypothesis 4.4 implies that any pair of porcupines is penned by a sequence of scrolls. In order to apply the lemma, we must first attach quills to each porcupine. Thus the formulation is quite technical.

Let e_a, e_b be integers and let δ_a, δ_b be integers $\geq 2g-1$. By Lemma 5.7, the Abel morphisms

$$\alpha_a : \mathrm{Porc}^{e_a+\delta_a,\delta_a}(X/C/\kappa) \to \mathrm{Pic}^{e_a+\delta_a}_{C/\kappa},$$

$$\alpha_b : \mathrm{Porc}^{e_b+\delta_b,\delta_b}(X/C/\kappa) \to \mathrm{Pic}^{e_b+\delta_b}_{C/\kappa}$$

are both smooth. Let I_a, resp. I_b, be a connected component of a nonempty fiber of α_a, resp. α_b, over a κ-point of $\mathrm{Pic}^{e_a+\delta_a}_{C/\kappa}$, resp. of $\mathrm{Pic}^{e_b+\delta_b}_{C/\kappa}$.

Proposition 6.5. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that Hypotheses 4.1 – 4.4 hold. Let e_a, e_b be integers, let δ_a, δ_b be integers $\geq 2g-1$. And let I_a and I_b be as above. Let n_0 be as in Hypothesis 4.4 and let n be an integer $\geq 2n_0$. There exists an integer $e_1 \geq \max(e_a+\delta_a, e_b+\delta_b)$ and a dense open subset U of $I_a \times_\kappa I_b$ with the following property. For every $e \geq e_1$, for every pair of porcupines $([h_a, h_b])$ in U with bodies $\sigma_{0,a}$ and $\sigma_{0,b}$, and for every degree e Cartier divisor class Γ on C, there exists a sequence of porcupines $h_1, \dots, h_{n-1}$ in $Porc^e(X/C/\kappa)$ with bodies $\sigma_{0,1}, \dots, \sigma_{0,n}$ and there exists a sequence of scrolls $R_1, \dots, R_n$ satisfying all of the following properties.*

- (i) *The Abel image $\alpha([h_i])$ equals $\mathcal{O}_C(\Gamma)$ for every $i = 1, \dots, n-1$.*
- (ii) *The sections $\sigma_{0,a}$ and $\sigma_{0,1}$ are both penned in R_1.*
- (iii) *For $i = 2, \dots, n-1$, the sections $\sigma_{0,i-1}$ and $\sigma_{0,i}$ are both penned in R_i.*
- (iv) *And $\sigma_{0,n-1}$ and $\sigma_{0,b}$ are both penned in R_n.*

Proof. We will also need to use the attaching divisors of these porcupines. Thus denote the extended bodies of h_a, resp. h_b, h_i by $(\sigma_{0,a}, D_a)$, resp. $(\sigma_{0,b}, D_b)$, $(\sigma_{0,i}, D_i)$. For a general element $([h_a], [h_b])$ in $I_a \times_\kappa I_b$, for a general κ-point t of C, the fiber of

$$\mathrm{ev} : \mathrm{Chn}_2(X/C, n) \to X \times_C X$$

over $(\sigma_{0,a}(t), \sigma_{0,b}(t))$ is smooth, projective and rationally connected by Hypothesis 4.4. Thus by Theorem 2.1 there exists a section

$$\tau : C \to \mathrm{Chn}_2(X/C, n)$$

such that $\mathrm{ev} \circ \tau$ equals $(\sigma_{0,a}, \sigma_{0,b})$. And by Corollary 2.2 the section τ can be chosen so that $\tau(t_j)$ is a specified general point in the fiber of

$$\mathrm{ev} : \mathrm{Chn}_2(X_{t_j}/\kappa, n) \to X_{t_j} \times_\kappa X_{t_j}$$

over $(\sigma_{0,a}(t_j), \sigma_{0,b}(t_j))$ for every $j = 1, \dots, d$.

It is convenient to make that last sentence precise. There exists a smooth, connected κ-scheme T and a C-morphism

$$\tau : T \times_\kappa C \to \mathrm{Chn}_2(X/C, n)$$

such that the induced morphism

$$\mathrm{ev}_{0,2,n} \circ \tau : T \times_\kappa C \to X \times_C X$$

is equivalent to a morphism

$$T \to \mathrm{Sec}(X/C/\kappa) \times_\kappa \mathrm{Sec}(X/C/\kappa)$$

mapping the geometric generic point of T to the pair of bodies $([\sigma_{0,a}], [\sigma_{0,b}])$ of the geometric generic point $([h_a], [h_b])$ of $I_a \times_\kappa I_b$ and such that the induced morphism

$$\tau_{(t_1,\ldots,t_\delta)} : T \to \mathrm{Chn}_2(X_{t_1}/\kappa, n) \times_\kappa \cdots \times_\kappa \mathrm{Chn}_2(X_{t_\delta}/\kappa, n)$$

is dominant.

For every $i = 1, \ldots, n$ denote by

$$P_i, Q_i : \mathrm{Chn}_2(X/C, n) \to X$$

the morphism which sends each chain

$$((L_1, p_1, q_1), \ldots, (L_n, p_n, q_n))$$

to the point p_i, resp. q_i. By construction Q_i equals P_{i+1} for $i = 1, \ldots, n-1$.

Claim 6.6. *For every $i = 1, \ldots, n$, the induced morphisms*

$$P_i \circ \tau, Q_i \circ \tau : T \times_\kappa C \to X$$

are equivalent to morphisms

$$P_i \circ \tau, Q_i \circ \tau : T \to \mathit{Sec}(X/C/\kappa)$$

each mapping the geometric generic point of T to a (g)-free section.

By Lemma 5.4, to prove Claim 6.6, it suffices to prove for a general choice of Δ that the induced morphism

$$(P_i \circ \tau)_{t_1,\ldots,t_\delta} : T \to X_{(t_1,\ldots,t_\delta)} := X_{t_1} \times_\kappa \cdots \times_\kappa X_{t_\delta}$$

has dense image, and similarly for $(Q_i \circ \tau)_{t_1,\ldots,t_\delta}$. If the image were not dense, the complement of the closure would be a dense open subset O of $X_{(t_1,\ldots,t_\delta)}$. Thus to prove Claim 6.6, it suffices to prove that for Δ general, for every dense open subset O of $X_{(t_1,\ldots,t_\delta)}$, the image of each morphism

$$(P_i \circ \tau)_{t_1,\ldots,t_\delta} : T \to X_{(t_1,\ldots,t_\delta)}$$

intersects O, and similarly for $(Q_i \circ \tau)_{t_1,\ldots,t_\delta}$.

If Δ is general, then for the pair of bodies $([\sigma_{0,a}], [\sigma_{0,b}])$ of a general point $([h_a], [h_b])$ of $I_a \times_\kappa I_b$, both

$$\sigma_{0,a}(t) := (\sigma_{0,a}(t_1), \ldots, \sigma_{0,a}(t_\delta)) \text{ and } \sigma_{0,b}(t) := (\sigma_{0,b}(t_1), \ldots, \sigma_{0,b}(t_\delta))$$

are contained in O. By Proposition 6.2 the fiber $\mathrm{Chn}_2^O(X/C, n)_{\sigma_{0,a}(t),\sigma_{0,b}(t)}$ of

$$\mathrm{ev}_{0,2,n} : \mathrm{Chn}_2^O(X/C, n)_{(t_1,\ldots,t_\delta)} \to O \times_\kappa O,$$

over $(\sigma_{0,a}(t), \sigma_{0,a}(t))$ is nonempty and hence dense in the corresponding fiber of

$$\mathrm{ev}_{0,2,n} : \mathrm{Chn}_2(X/C, n)_{(t_1,\ldots,t_\delta)} \to X_{(t_1,\ldots,t_\delta)} \times_\kappa X_{(t_1,\ldots,t_\delta)}.$$

Since the morphism

$$\tau_{(t_1,\ldots,t_\delta)} : T \to \mathrm{Chn}_2(X_{t_1}/\kappa, n) \times_\kappa \cdots \times_\kappa \mathrm{Chn}_2(X_{t_\delta}/\kappa, n)$$

is dominant, the image intersects the dense open subset $\mathrm{Chn}_2^O(X/C, n)_{\sigma_{0,a}(t),\sigma_{0,b}(t)}$. So a general point u of T parameterizes a section

$$\tau_u : C \to \overline{\mathrm{Chn}}_2(X/C, n)$$

such that $(\tau_u(t_1), \ldots, \tau_u(t_\delta))$ is in $\mathrm{Chn}_2^O(X/C, n)_{\sigma_{0,a}(t), \sigma_{0,b}(t)}$. By the definition of $\mathrm{Chn}_2^O(X/C, n)$, for every $i = 1, \ldots, n$, the associated point $(P_i \circ \tau_u(t_1), \ldots, P_i \circ \tau_u(t_\delta))$ is in O and similarly for Q_i. This completes the proof of Claim 6.6.

Let e_1 be an integer which is at least as large as $e_a + \delta_a$, $e_b + \delta_b$, $\deg_{\mathcal{L}}(p_i \circ \tau_u) + 2g - 1$ and $\deg_{\mathcal{L}}(q_i \circ \tau_u) + 2g - 1$ for every $i = 1, \ldots, n-1$. For every Cartier divisor class Γ of degree e, for every $i = 1, \ldots, n-1$, there exists an effective divisor D_{P_i}, resp. D_{Q_i}, on C such that

$$(P_i \circ \tau_u)^* \mathcal{L}(D_{P_i}) \cong \mathcal{O}_C(\Gamma), \text{ resp.}$$

$$(Q_i \circ \tau_u)^* \mathcal{L}(D_{Q_i}) \cong \mathcal{O}_C(\Gamma).$$

Moreover, for general choices, D_{P_i} and D_{Q_i} are reduced. Thus any choice of quills attached to $(P_i \circ \tau_u)(C)$, resp. $(Q_i \circ \tau_u)(C)$, at $(P_i \circ \tau_u)(D_{P_i})$, resp. at $(Q_i \circ \tau_u)(D_{Q_i})$, gives a degree e porcupine. This defines the porcupines $h_1, \ldots, h_{n-1}$.

Finally, for every $i = 1, \ldots, n$, the projection

$$L_i : \overline{\mathrm{Chn}}_2(X/C, n) \to \overline{\mathrm{M}}_{0,0}(X/C, 1)$$

sending each comb to the i^{th} line in the comb defines a section

$$L_i \circ \tau_u : C \to \overline{\mathrm{M}}_{0,0}(X/C, 1),$$

or equivalently a scroll R_i. The conditions that p_i, q_i are contained in L_i and that q_i equals p_{i+1} translate into conditions (i)–(iv) of the proposition for this choice of $h_1, \ldots, h_{n-1}$ and $R_1, \ldots, R_n$. □

This proposition is useful because of the next corollary.

Corollary 6.7. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume Hypotheses 4.1 – 4.4 hold. For every integer e there exists a positive integer $\delta_0(e) \geq 2g - 1$ such that for every integer $\delta \geq \delta_0(e)$ and for every κ-point $[\mathcal{O}_C(\Gamma)]$ in $Pic^{e+\delta}_{C/\kappa}$, there is a dense open subset U of the 2-fold self product*

$$\left(Porc^{e+\delta,\delta}(X/C/\kappa) \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])\right) \times_\kappa \left(Porc^{e+\delta,\delta}(X/C/\kappa) \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])\right)$$

with the following property. For every κ-point $([h_a], [h_b])$ in U, there exists a chain of rational curves in $\Sigma^{e+\delta}(X/C/\kappa)$ containing $[h_a]$ and $[h_b]$ and whose nodes all lie in $Porc^{e+\delta}(X/C/\kappa)$. In particular, the nodes are all smooth points of $\Sigma_{e+\delta}(X/C/\kappa)$. So the scheme $Porc^{e+\delta,\delta}(X/C/\kappa) \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$ is contained in a single irreducible component of $\Sigma^{e+\delta}(X/C/\kappa)$.

Proof. This follows from Proposition 6.5 and Lemma 6.4. □

Corollary 6.8. *As always the notation is as in Notation 2.4. Assume that Hypotheses 4.1 – 4.4 hold. Let e_0 and Z be as in Definition 5.6, which exists by Corollary 5.5. And let $(Z_e)_{e \geq e_0}$ be the associated sequence of irreducible components Z_e of $\Sigma^e(X/C/\kappa)$. Then $(Z_e)_{e \geq e_0 + 2g - 1}$ is a pseudo Abel sequence for $X/C/\kappa$; in particular condition (iii) of Definition 4.8 holds. Thus there exists a pseudo Abel sequence for $X/C/\kappa$.*

Proof. First of all, by Lemma 5.4, there exists e_0 and Z as in Definition 5.6. By construction $(Z_e)_{e\geq e_0}$ satisfies condition (i) of Definition 4.8. By Lemma 5.7, $(Z_e)_{e\geq e_0+2g-1}$ satisfies the modified condition (ii) for a pseudo Abel sequence. So it only remains to prove that $(Z_e)_{e\geq e_0+2g-1}$ satisfies condition (iii).

Let e_1 be any integer such that $\mathrm{Porc}^{e_2}(X/C/\kappa)$ is nonempty, and let W be an irreducible component of $\Sigma^{e_1}(X/C/\kappa)$ which intersects $\mathrm{Porc}^{e_1}(X/C/\kappa)$. By the same construction as in Definition 5.6, for every integer $e \geq e_1$, there is a unique irreducible component W_e of $\Sigma^e(X/C/\kappa)$ containing $\mathrm{Porc}^{e,e-e_1}(X/C/\kappa)\cap W_e$. By Proposition 6.5, for $e \gg 0$, there is a single irreducible component of $\Sigma^e(X/C/\kappa)$ containing both $\mathrm{Porc}^{e,e-e_0}(X/C/\kappa)_Z$ and $\mathrm{Porc}^{e,e-e_1}(X/C/\kappa)_W$. But the unique irreducible component of $\Sigma^e(X/C/\kappa)$ containing $\mathrm{Porc}^{e,e-e_0}(X/C/\kappa)_Z$ is Z_e and the unique irreducible component of $\Sigma^e(X/C/\kappa)$ containing $\mathrm{Porc}^{e,e-e_1}(X/C/\kappa)_W$ is W_e. Therefore, for all $e \gg 0$, Z_e equals W_e. □

7. Rational connectedness of the interior modulo the boundary

Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume Hypotheses 4.1 – 4.4 hold. By Corollary 6.8 there exists a pseudo Abel sequence $(Z_e)_{e\geq e_0}$ for $X/C/\kappa$. To finish proving Theorem 4.9 we need to prove that under the additional Hypothesis 4.5, there exists an integer $\epsilon \geq e_0$ such that for every integer $e \geq \epsilon$ and for every general point $[\mathcal{O}_C(\Gamma)]$ of $\mathrm{Pic}^e_{C/\kappa}$, the integral scheme $Z_e \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$ is rationally connected, i.e., every strong desingularization is rationally connected. Recall that a *strong desingularization* of a quasi-projective variety X is a projective, birational morphism $\nu : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and ν is an isomorphisms over the smooth locus X_{smooth} of X. In characteristic 0 Hironaka proved that every quasi-projective variety has a strong desingularization.

Because of Corollary 6.7 in the last section, the "boundary is rationally connected modulo the interior": for every integer $e \geq e_0$ there exists an integer $\delta_0(e) \geq 2g-1$ such that for every integer $\delta \geq \delta_0(e)$, for every $[\mathcal{O}_C(\Gamma)]$ in $\mathrm{Pic}^{e+\delta}_{C/\kappa}$, every general pair of points in the same fiber $\mathrm{Porc}^{e+\delta,\delta}(X/C/\kappa)_Z \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$ are connected by a chain of rational curves whose nodes are all smooth points of $Z_{e+\delta}$. This shows that the fibers of the Abel map on $\mathrm{Porc}^{e+\delta,\delta}(X/C/\kappa)_Z$ are rationally chain connected modulo the interior, and the same holds after replacing the fiber by a strong desingularization.

We would be done if we could also show that the "interior is rationally connected modulo the boundary": for all integers $e \geq e_1$ and $\delta \geq \delta_1$, every sufficiently general point of $Z_{e+\delta} \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$ is contained in a chain of rational curves which also contains a general point of $\mathrm{Porc}^{e+\delta,\delta}(X/C/\kappa)_Z \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$ and whose nodes are all smooth points of $Z_{e+\delta} \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$. For then by concatenating chains, a general pair of points in $Z_{e+\delta} \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$ would be contained in a chain of rational curves whose nodes are all smooth points of $Z_{e+\delta} \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$. Thus the chain would lift to every strong desingularization of $Z_{e+\delta} \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$. And a smooth, projective variety is rationally connected if every sufficiently general pair of points is connected by a chain of rational curves, cf. [Kol96, Theorem IV.3.10.3]. Thus $Z_{e+\delta} \cap \alpha^{-1}([\mathcal{O}_C(\Gamma)])$ would be rationally connected.

This is how we will conclude the proof. In order to construct the rational chains connecting general points of the "interior" of $Z_{e+\delta}$ to $\mathrm{Porc}^{e+\delta,\delta}(X/C/\kappa) \cap Z_{e+\delta}$ we will first construct special "m-twisting" scrolls R in X. These are constructed by attaching to any scroll R_0 for X/C at finitely many fibers X_t of π a 2-twisting scroll for the constant family $\mathbb{P}^1_\kappa \times_\kappa X_t/\mathbb{P}^1_\kappa$, considered as a surface in X_t via projection. These 2-twisting scrolls exist by Hypothesis 4.5. After attaching such 2-twisting scrolls to R_0 at sufficiently many fibers and in sufficiently general normal directions, the reducible surface will deform to an m-twisting scroll. These m-twisting scrolls are relevant since pencils of divisors on m-twisting scrolls will give the rational curves needed to connect the interior of $Z_{e+\delta}$ to $\mathrm{Porc}^{e+\delta,\delta}(X/C/\kappa)_Z$. The following lemma proves that R_0 exists.

Lemma 7.1. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume Hypotheses 4.1 – 4.3 hold. For every integer e, every κ-point $[\sigma_0]$ of $\mathrm{Porc}^{e,0}(X/C/\kappa)$ is penned by a scroll R_0 for $X/C/\kappa$.*

Proof. By Hypothesis 4.3, the evaluation morphism

$$\mathrm{ev}_{0,1,1} : \overline{\mathrm{M}}_{0,1}(X/C,1) \to X$$

is smooth, projective and surjective with integral and rationally connected geometric fibers. Thus the same holds for the base-change morphism

$$\mathrm{pr}_2 : \overline{\mathrm{M}}_{0,1}(X/C,1) \times_{\mathrm{ev}_{0,1,1},X,\sigma_0} C \to C.$$

So this morphism is a rationally connected fibration over the curve C. By Theorem 2.1, this morphism has a section. As discussed in Definition 3.5, this section is equivalent to a scroll R_0 for $X/C/\kappa$. By construction, this scroll contains $\sigma_0(C)$. $\square$

Let (R, L) be an m-twisting scroll for $X/C/\kappa$, cf. Definition 3.6. By Bertini's theorem, a general member of the linear system $|L|$ on R is a smooth curve. By (1), this curve is of the form $\sigma_0(C)$ for a section $\sigma_0 : C \to R$ or $\pi|_R$.

Definition 7.2. Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Let $h : C' \to X$ be a porcupine with m quills and with body σ_0. A scroll R *perfectly pens* the porcupine h if $(R, [\sigma_0(C)])$ is an m-twisting scroll for $X/C/\kappa$.

A scroll R together with a section σ_0 penned by R determines a section

$$\zeta : C \to \overline{\mathrm{M}}_{0,1}(X/C,1)$$

of the projection $\pi_{0,1,1} : \overline{\mathrm{M}}_{0,1}(X/C,1) \to C$ as in Definition 3.5. In some other articles an m-twisting scroll is *defined* to be this associated section ζ of $\pi_{0,1,1}$, but in this article the definition above is more convenient. We can study the pullback under ζ of the vertical tangent bundles of the morphisms

$$\Phi : \overline{\mathrm{M}}_{0,1}(X/C,1) \to \overline{\mathrm{M}}_{0,0}(X/C,1)$$

and

$$\mathrm{ev}_{0,1,1} : \overline{\mathrm{M}}_{0,1}(X/C,1) \to X,$$

i.e., the duals of the locally free sheaves of relative differentials for these morphisms. Both in the construction of an m-twisting scroll starting from the scroll in Lemma 7.1 and in verifying Hypothesis 4.5 in Part 2, it is useful to characterize m-twisting scrolls in terms of the pullbacks $\zeta^* T_\Phi$ and $\zeta^* T_{\mathrm{ev}_{0,1,1}}$.

Lemma 7.3. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume Hypotheses 4.1 and 4.2 hold and assume that every line in every fiber X_t of π is a free line. Let $\sigma_0 : C \to X$ be a section of π and let R be a scroll penning σ_0. Let $\zeta : C \to \overline{M}_{0,1}(X/C, 1)$ be the associated section of $\pi_{0,1,1}$. For each integer $m > 0$, $(R, [\sigma_0(C)])$ is an m-twisting scroll if and only if the following hold.*

1. *The sheaves $\zeta^* T_\Phi$ and $\pi_* \mathcal{O}_R(\sigma_0(C))$ are both globally generated and non-special.*
2. *The sheaf $\zeta^* ev_{0,1,1}^* T_{X/C}$ is globally generated and non-special.*
3. *And the sheaf $\zeta^* T_{\mathrm{ev}_{0,1,1}} \otimes_{\mathcal{O}_C} \mathcal{O}_C(-\Gamma)$ is non-special for every invertible sheaf $\mathcal{O}_C(\Gamma)$ of degree $\leq m$.*

In the special case that g equals 0, if $\zeta^* T_\Phi$ is globally generated and non-special then so is $\pi_* \mathcal{O}_R(\sigma_0(C))$.

Proof. Denote by $\rho : R \to S$ the restriction of π. As is discussed in [HS05, Remark 4.4] for instance, $\zeta^* T_\Phi$ is isomorphic to $\sigma_0^* \mathcal{O}_R(L)$, and $\zeta^* T_{\mathrm{ev}_{0,1,1}}$ is isomorphic to $\rho_* N_{R/X}(-L)$. Of course $\zeta^* \mathrm{ev}_{0,1,1}^* T_{X/C}$ is the same as $N_{\sigma_0(C)/X}$. For every geometric point t of C, R_t is a free line in X_t, hence $h^q(R_t, N_{R_t/X_t})$ and $h^q(R_t, N_{R_t/X_t}(-\sigma_0(t)))$ both equal 0 for $q > 0$. Thus by the Leray spectral sequence, $h^2(R, N_{R/X}(-L) \otimes_{\mathcal{O}_R} \rho^* \mathcal{O}_C(-\Gamma))$ equals 0 and $H^1(R, N_{R/X}(-L) \otimes_{\mathcal{O}_R} \rho^* \mathcal{O}_C(-\Gamma))$ equals $H^1(C, \zeta^* T_{\mathrm{ev}_{0,1,1}} \otimes_{\mathcal{O}_C} \mathcal{O}_C(-\Gamma))$. Thus (3) above is equivalent to (3) of Definition 3.6. Similarly, (1) of Definition 3.6 holds if and only if $\rho_* \mathcal{O}_R(\sigma_0(C))$ is globally generated and non-special. So (1) above implies (1) of Definition 3.6. And (2) of Definition 3.6 holds if and only if $\rho_* N_{R/X}$ is globally generated and non-special.

There is a short exact sequence of $\mathcal{O}_R$-modules

$$0 \longrightarrow \mathcal{O}_R \longrightarrow \mathcal{O}_R(\sigma_0(C)) \longrightarrow \sigma_{0,*}\sigma_0^* \mathcal{O}_R(\sigma_0(C)) \longrightarrow 0$$

giving rise to a short exact sequence of $\mathcal{O}_C$-modules,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \rho_* \mathcal{O}_R(\sigma_0(C)) \longrightarrow \zeta^* T_\Phi \longrightarrow 0.$$

Thus if $\rho_* \mathcal{O}_R(\sigma_0(C))$ is globally generated and non-special, then so is $\zeta^* T_\Phi$. In particular, (1) of Definition 3.6 implies (1) of the lemma. In the special case that $g = 0$, $\mathcal{O}_C$ is non-special. Hence $\rho_* \mathcal{O}_R(\sigma_0(C))$ is globally generated and non-special if and only if $\zeta^* T_\Phi$ is globally generated and non-special.

Similarly there is a short exact sequence of $\mathcal{O}_C$-modules,

$$0 \longrightarrow \zeta^* T_{\mathrm{ev}_{0,1,1}} \longrightarrow \rho_* N_{R/X} \longrightarrow \sigma_0^* N_{R/X} \longrightarrow 0.$$

Assuming (3), the first term is globally generated and non-special. Thus the second term is globally generated and non-special if and only if the the third term is. There is also a short exact sequence

$$0 \longrightarrow \zeta^* T_\Phi \longrightarrow \zeta^* \mathrm{ev}_{0,1,1}^* T_{X/C} \longrightarrow \sigma_0^* N_{R/X}.$$

Assuming (1), $\zeta^* T_\Phi$ is globally generated and non-special. Thus the second term is globally generated and non-special if and only if the the third term is. But the third term of this sequence equals the third term of the previous sequence. Hence, assuming (1) and (3), (2) of Definition 3.6 is equivalent to (2) of the lemma. Therefore (1), (2) and (3) of Definition 3.6 are equivalent to (1), (2) and (3) above. □

We are interested in m-twisting scrolls because a porcupine with m quills which is perfectly penned in an m-twisting scroll is contained in a rational curve in $\Sigma(X/C/\kappa)$ intersecting the interior; this is the aim of the rest of this section. And then, by the next lemma, the same holds for all sufficiently general deformations of this m-quill porcupine.

Lemma 7.4. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume Hypotheses 4.1 and 4.3 hold. For every integer e and for every integer $m \geq 0$ there is an open subscheme of $Porc^{e,m}(X/C/\kappa)$ (possibly empty) whose κ-points are precisely those degree e porcupines $h : C' \to X$ with m quills and body $\sigma_0 : C \to X$ which are perfectly penned in a scroll R, i.e., such that $(R, [\sigma_0(C)])$ is m-twisting.*

Proof. Let $h : C' \to X$ be a porcupine with extended body (σ_0, D) where $D = \underline{t}_1 + \cdots + \underline{t}_m$ is an effective, reduced divisor of degree m. Let R be an scroll penning h such that $(R, [\sigma_0(C)])$ is m-twisting. There is a flag Hilbert scheme parameterizing pairs (R, h) of a scroll R in X and a porcupine h penned by R. This projects to the Hilbert scheme parameterizing scrolls R in X. Our first step is to prove that both the Hilbert schemes of scrolls in X is smooth at R, and the projection morphism from the flag Hilbert scheme is smooth at (R, h), so that the flag Hilbert scheme itself is smooth at (R, h).

First we show that $h^1(R, N_{R/X})$ equals 0, so that the Hilbert scheme of scrolls in X is smooth at R. There is a short exact sequence of $\mathcal{O}_R$-modules,

$$0 \longrightarrow N_{R/X}(-\sigma_0(C)) \longrightarrow N_{R/X} \longrightarrow \sigma_{0,*}\sigma_0^* N_{R/X} \longrightarrow 0.$$

By the associated long exact sequence of cohomology, it suffices to prove that both $h^1(R, N_{R/X}(-\sigma_0(C)))$ and $h^1(C, \sigma_0^* N_{R/X})$ equal 0. By Definition 3.6 with $\mathcal{O}_C(\Gamma) = \mathcal{O}_C$, $h^1(R, N_{R/X}(-\sigma_0(C)))$ equals 0. And $\sigma_0^* N_{R/X}$ is a quotient of $N_{\sigma_0(C)/X}$, thus $h^1(C, \sigma_0^* N_{R/X})$ equals 0 if $h^1(C, N_{\sigma_0(C)/X})$ equals 0. By hypothesis σ_0 is (g)-free, hence free. Therefore $h^1(C, N_{\sigma_0(C)/X})$ equals 0. Thus the Hilbert scheme of scrolls is smooth at R.

To prove smoothness of the projection morphism from the flag Hilbert scheme to the Hilbert scheme of scrolls at (R, h), it suffices to prove that $h^1(R, \mathcal{O}_R(h(C')))$ equals 0. Associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_R(\sigma_0(C)) \longrightarrow \mathcal{O}_R(h(C')) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{R_{t_i}}(1) \longrightarrow 0$$

there is a long exact sequence of cohomology giving

$$H^1(R, \mathcal{O}_R(\sigma_0(C))) \longrightarrow H^1(R, \mathcal{O}_R(h(C'))) \longrightarrow \bigoplus_{i=1}^n H^1(R_{t_i}, \mathcal{O}_{R_{t_i}}(1)) = 0.$$

Since $(R, [\sigma_0(C)])$ is an m-twisting surface, $h^1(R, \mathcal{O}_R(sigma_0(C)))$ equals 0. Therefore also $h^1(R, \mathcal{O}_R(h(C')))$ equals 0.

So the morphism from the flag Hilbert scheme to the Hilbert scheme of porcupines is a morphism between smooth schemes at (R, h). By the Jacobian criterion, to prove the morphism is smooth at (R, h) it suffices to prove the derivative map is surjective, cf. [Har77, Proposition III.10.4]. Chasing diagrams, the cokernel of the derivative map is a subspace of

$$H^1(R, N_{R/X}(-h(C'))) = H^1(R, N_{R/X}(-\sigma_0(C)) \otimes_{\mathcal{O}_C} \pi^* \mathcal{O}_C(-D)).$$

Since $(R, [\sigma_0(C)])$ is m-twisting and since D is a degree m divisor, this cohomology group is zero. Thus the morphism from the flag Hilbert scheme to the Hilbert scheme of porcupines is smooth at (R, h).

Each condition on (R,h) to be m-twisting is either a condition about the vanishing of cohomology of a locally free sheaf, or a condition about vanishing of cohomology together with global generation of the sheaf. But each of these is an open condition on the flag Hilbert scheme, cf. [Har77, Theorem III.12.11]. Thus there is a Zariski open neighborhood of (R,h) in the flag Hilbert scheme parameterizing those pairs such that $(R,[\sigma_0])$ is m-twisting. Because a smooth morphism of quasi-projective schemes is open, the image of this open neighborhood is an open neighborhood of h in the Hilbert scheme of porcupines. Therefore the set of porcupines which are penned by an m-twisting scroll contains a Zariski open neighborhood of h. Since this holds for every such porcupine h, the locus of porcupines which are penned by an m-twisting scroll is open in the Hilbert scheme of all porcupines. □

By Lemma 7.3 and by the proof of Lemma 7.4, we can finally prove the connection between Hypothesis 4.5 and existence of 2-twisting scrolls.

Corollary 7.5. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume Hypotheses 4.5 holds. Then there are finitely many closed points of C such that for every geometric point t of C whose image is not one of these, denoting the residue field by $\kappa(t)$, the following holds. There exists a scroll S for $\mathbb{P}^1_{\kappa(t)} \times X_t/\mathbb{P}^1_{\kappa(t)}/\kappa(t)$ and a section σ_S with $\sigma_S(\mathbb{P}^1_{\kappa(t)}) \subset S$ such that σ_S is free, in fact $\sigma_S^* T_{X_t/\kappa(t)}$ is ample, and $(S,[\sigma_S(\mathbb{P}^1_{\kappa(t)})])$ is 2-twisting, in fact $\zeta^* T_\Phi$ and $\zeta^* T_{ev_{0,1,1}}$ are both ample.*

Proof. Hypothesis 4.5 asserts that for the geometric generic point $t = \overline{\eta}$ of C, there exists a morphism

$$\zeta : \mathbb{P}^1_K \to \overline{\mathrm{M}}_{0,1}(Y/K, 1)$$

such that $\zeta^* T_\Phi$, $\zeta^* T_{\mathrm{ev}_{0,1,1}}$ and $\zeta^* \mathrm{ev}^*_{0,1,1} T_{Y/K}$ are all ample. This morphism is equivalent to a scroll S and a section σ_S with image in S. By Lemma 7.3, this is a 2-twisting scroll. By the same sort arguments as in the proof of Lemma 7.4, the existence of such a morphism ζ for the fiber X_t is an open condition on points of C. And since this open set contains the geometric generic point, it is a dense open set. Thus the complement of this open set is a set of finitely many closed points of C. □

Assume now that Hypotheses 4.1 – 4.5 hold. The next proposition and its corollary prove that the "interior is rationally connected modulo the bondary". First there is some more notation.

Let t be a general κ-point of C and let S and σ_S be as in Corollary 7.5. Let $h_S : C'_S \to \mathbb{P}^1_\kappa \times_\kappa X_t$ be a porcupine with body σ_S and with 2 quills being fibers of $S \to \mathbb{P}^1_\kappa$. Since S is free, also σ_S is free and hence (0)-free. Thus h_S is a porcupine with 2 quills. Since $(S,[\sigma_S(\mathbb{P}^1_\kappa)])$ is 2-twisting, h_S is perfectly penned by S. Denote by b_S the degree of σ_S with respect to $\mathcal{L}$.

By Corollary 6.7 there exists an integer a_S such that for every $a \geq a_S$, the porcupine obtained by attaching general quills to $\sigma_S(\mathbb{P}^1_\kappa)$ at a general points is contained in a chain of rational curves in $\Sigma^{b_S+a}(\mathbb{P}^1_\kappa \times_\kappa X_t/\mathbb{P}^1_\kappa/\kappa)$ which also contains a porcupine with constant body and such that all nodes of the chain are porcupines hence smooth points of $\Sigma^{b_S+a}(\mathbb{P}^1_\kappa \times_\kappa X_t/\mathbb{P}^1_\kappa/\kappa)$.

Proposition 7.6. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that Hypotheses 4.1 – 4.5 all hold. Let $(Z_e)_{e \geq e_0}$ be a pseudo Abel sequence as in Corollary 6.8,* *and denote Z_{e_0} by Z. Let a_S and b_S be as above. For every integer $m \geq 1$ and for every integer $a \geq a_S$ there exists an integer $c_\Delta \geq 0$ such that for every $c \geq c_\Delta$, denoting $e_0 + a + c \cdot b_S$ by e, there is a dense open subset V_e of $\mathrm{Porc}^{e,0}(X/C/\kappa)_Z$ such that for every κ-point $[\sigma_0]$ of V_e there exists a scroll R for $X/C/\kappa$ penning σ_0 and such that $(R, [\sigma_0(C)])$ is an m-twisting scroll. Hence every m-quill porcupine obtained from σ_0 by attaching m fibers in R is perfectly penned by R. And by Lemma 7.4, there is a nonempty open subset of $\mathrm{Porc}^{e+m,m}(X/C/\kappa)_Z$ parameterizing m-quill porcupines which are perfectly penned.*

Proof. Let σ be a general section parameterized by Z. By Lemma 7.1, σ is penned by a scroll R. Denote by ρ the projection

$$\rho : R \to C.$$

Let $a \geq a_S$ be a positive integer such that for a general divisor Δ of degree a on C, the divisor class $[\sigma(C)] + \rho^*\Delta$ on R is basepoint free and non-special, i.e., such that $(\rho_*\mathcal{O}_R(\sigma(C))) \otimes_{\mathcal{O}_C} \mathcal{O}_C(\Delta$ is globally generated and non-special. Let σ_Δ be a general section of ρ such that $[\sigma_\Delta(C)]$ is linearly equivalent to $[\sigma(C)] + \rho^*\Delta$ on R. Recall from Definition 3.4 that there are morphisms

$$\pi_{0,1,1} : \overline{\mathrm{M}}_{0,1}(X/C, 1) \to C$$

and

$$\mathrm{ev}_{0,1,1} : \overline{\mathrm{M}}_{0,1}(X/C, 1) \to X.$$

By Hypotheses 4.3, for every point t of C every line in the fiber $X_t = \pi^{-1}(t)$ is a free line. Thus both $\pi_{0,1,1}$ and $\mathrm{ev}_{0,1,1}$ are smooth morphisms. Denote by $T_{\pi_{0,1,1}}$ and $T_{\mathrm{ev}_{0,1,1}}$ the corresponding vertical tangent sheaves – the duals of the corresponding locally free sheaves of relative differentials. The pair (R, σ_Δ) determines a section

$$\tau_\Delta : C \to \overline{\mathrm{M}}_{0,1}(X/C, 1)$$

of $\pi_{0,1,1}$.

By the proof of Corollary 7.5 (really the argument from the proof of Lemma 7.4) for every general t' in C there exists a deformation into $\mathbb{P}^1_\kappa \times_\kappa X_{t'}$ of $h_S : C'_S \to \mathbb{P}^1_\kappa \times_\kappa X_t$ as a 2-quill porcupine, say $h_{S,t'} : C'_{S,t'} \to \mathbb{P}^1_\kappa \times_\kappa X_{t'}$ and such that one of the quills is $\rho^{-1}(t')$. And the deformation is penned in $\mathbb{P}^1_\kappa \times_\kappa X_{t'}$ by a 2-twisting deformation of S, say $S_{t'}$. The deformation $S_{t'}$ of S together with the body of $h_{S,t'}$ define a morphism

$$\tau_{t'} : \mathbb{P}^1_\kappa \to \overline{\mathrm{M}}_{0,1}(X_{t'}/\kappa, 1) = \pi^{-1}_{0,1,1}(t')$$

such that $\tau_{t'}(0)$ equals $\tau_\Delta(t')$. The condition that $(S, [\sigma_S(\mathbb{P}^1_\kappa)])$ is 2-twisting is equivalent to the condition that both $\tau_t^* T_{\pi_{0,1,1}}$ and $\tau_t^* T_{\mathrm{ev}_{0,1,1}}$ are ample sheaves on $\mathbb{P}^1_\kappa$.

Fix an invertible sheaf $\mathcal{O}_C(\Gamma_0)$ on C of degree $N \leq -m - g$. Of course $\tau_\Delta^*(T_{\pi_{0,1,1}} \otimes \pi^*_{0,1,1}\mathcal{O}_C(\Gamma_0))$, resp. $\tau_\Delta^*(T_{\mathrm{ev}_{0,1,1}} \otimes \pi^*_{0,1,1}\mathcal{O}_C(\Gamma_0))$, is isomorphic to $\tau_\Delta^* T_{\pi_{0,1,1}} \otimes \mathcal{O}_C(\Gamma_0)$, resp. $\tau_\Delta^* T_{\mathrm{ev}_{0,1,1}} \otimes \mathcal{O}_C(\Gamma_0)$. By [Kol96, Theorem II.7.9, Lemma II.7.10.1], there exists an integer c_Δ such that for all $c \geq c_\Delta$, after attaching teeth $\tau_t(\mathbb{P}^1)$ to the handle $\tau_\Delta(C)$ at c general points $\tau_\Delta(t)$, this comb in $\overline{\mathrm{M}}_{0,1}(X/C, 1)$ deforms to the image of a section of $\pi_{0,1,1}$, say

$$\tau_{\Delta,c} : C \to \overline{\mathrm{M}}_{0,1}(X/C, 1),$$

such that

$$h^1(C, \tau^*_{\Delta,c}(T_{\pi_{0,1,1}} \otimes \pi^*_{0,1,1}\mathcal{O}_C(\Gamma_0))) \text{ equals } 0$$

and

$$h^1(C, \tau_{\Delta,c}^*(T_{\mathrm{ev}_{0,1,1}} \otimes \pi_{0,1,1}^* \mathcal{O}_C(\Gamma_0))) \text{ equals } 0.$$

Denote by $R_{\Delta,c}$ the ruled surface and by $\sigma_{\Delta,c}$ the section of $R_{\Delta,c}$ corresponding to $\tau_{\Delta,c}$.

Let $\mathcal{O}_C(\Gamma)$ be any invertible sheaf on C of degree $\leq m$. Then $\mathcal{O}_C(-\Gamma - \Gamma_0)$ has degree $-N - m$, which is $\geq g(C)$. Thus it is effective, i.e., there exists an injective sheaf homomorphism

$$\mathcal{O}_C(\Gamma_0) \to \mathcal{O}_C(-\Gamma)$$

with torsion cokernel. Thus there exist injective sheaf homomorphisms with torsion cokernel

$$\tau_{\Delta,c}^*(T_{\pi_{0,1,1}} \otimes \pi_{0,1,1}^* \mathcal{O}_C(\Gamma_0)) \to \tau_{\Delta,c}^*(T_{\pi_{0,1,1}} \otimes \pi_{0,1,1}^* \mathcal{O}_C(-\Gamma))$$

and

$$\tau_{\Delta,c}^*(T_{\mathrm{ev}} \otimes \pi_{0,1,1}^* \mathcal{O}_C(\Gamma_0)) \to \tau_{\Delta,c}^*(T_{\mathrm{ev}} \otimes \pi_{0,1,1}^* \mathcal{O}_C(-\Gamma)).$$

Thus

$$h^1(C, \tau_{\Delta,c}^*(T_{\pi_{0,1,1}} \otimes \pi_{0,1,1}^* \mathcal{O}_C(-\Gamma)) \text{ equals } 0$$

and

$$h^1(C, \tau_{\Delta,c}^*(T_{\mathrm{ev}} \otimes \pi_{0,1,1}^* \mathcal{O}_C(-\Gamma)) \text{ equals } 0.$$

Chasing diagrams, this implies that the normal bundle N of $R_{\Delta,c}$ in X is globally generated, that $h^1(R_{\Delta,c}, N)$ equals 0, and that

$$h^1(R_{\Delta,c}, N(-[\sigma_{\Delta,c}(C)]) \otimes \rho_{\Delta,c}^* \mathcal{O}_C(-\Gamma)) \text{ equals } 0$$

for every invertible sheaf $\mathcal{O}_C(\Gamma)$ of degree $\leq m$. In other words, $R_{\Delta,c}$ together with $[\sigma_{\Delta,c}]$ is an m-twisting scroll. Therefore every porcupine penned by $R_{\Delta,c}$ with m quills and with body in the linear system $|\sigma_{\Delta,c}|$ is penned by an m-twisting scroll.

There is one issue: it is not immediately obvious that the section $\sigma_{\Delta,c}$ is parameterized by one of the components in our pseudo Abel sequence $(Z_e)_{e \geq e_0}$. In fact $\sigma_{\Delta,c}$ is a smoothing of the comb obtained from $\sigma_\Delta(C)$ by attaching c general deformations of $\sigma_S(\mathbb{P}^1)$. And $\sigma_\Delta(C)$ is itself a deformation of a porcupine with body $\sigma(C)$ and a quills. By the choice of a_S, and since $a \geq a_S$, $\sigma_\Delta(C)$ is parameterized by a point in our pseudo Abel sequence. And $\sigma_{\Delta,c}$ is a deformation of a stable section h_{initial} whose associated section equals $\sigma(C)$ and having a vertical components being free lines – the "line components" – and having c vertical components being deformations of $\sigma_S(\mathbb{P}^1)$ – the "$\sigma_S(\mathbb{P}^1)$-components". Since $\sigma(C)$ is (g)-free, any sub-stable section of h_{initial} is unobstructed and so is a smooth point of $\Sigma^e(X/C/k)$, where $e = e_0 + a + c \cdot b_S$.

Since a is $\geq a_S$, we may specialize a_S of the line components of h_{initial} to lie on one of the $\sigma_S(\mathbb{P}^1)$ components in a fiber X_t. This stable section is still unobstructed. And then we may deform the a_S line components on $\sigma_S(\mathbb{P}^1)$ to be general lines attached to $\sigma_S(\mathbb{P}^1)$, i.e., the vertical curve in X_t is now a porcupine with a_S quills and with body a deformation of $\sigma_S(\mathbb{P}^1)$. As explained at the beginning of the proof, a porcupine with body $\sigma_S(\mathbb{P}^1)$ and with a_S general quills is contained in a chain of rational curves which also contains a porcupine with $a_S + b_S$ quills and whose body is a constant section of $\mathbb{P}^1_\kappa \times_\kappa X_t$. And all of the nodes of this chain of rational curves are smooth points of the space of porcupines.

Thus $\sigma_{\Delta,c}$ is in the same irreducible component of $\Sigma^e(X/C/B)$ as the stable section obtained by removing a_S line components from h_{initial} and replacing one of the $\sigma_S(\mathbb{P}^1)$ components by a vertical curve which is itself a porcupine in X_t with constant body and with $a_S + b_S$ quills. Again everything is unobstructed, so we may deform the $a_S + b_S$ lines off of the constant body in X_t and back onto the original body $\sigma(C)$ of h_{initial}. Thus h_{initial} is in the same component as a new stable section, $h_{\text{successor}}$ which is the same as h_{initial} except that we have removed one of the $\sigma_S(\mathbb{P}^1)$ components and we have attached b_S additional line components (the constant body that remains in X_t is "contracted away" under the stabilization of this pre-stable curve).

Clearly we may repeat this with each $\sigma_S(\mathbb{P}^1)$ component. So, in the end, $\sigma_{\Delta,c}$ is in the same irreducible component as the porcupine with body $\sigma(C)$ and with $a+c\cdot b_S$ quills. In other words $\sigma_{\Delta,c}$ is in Z_e after all. $\square$

Corollary 7.7. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that Hypotheses 4.1 – 4.5 all hold. Let $(Z_e)_{e\geq e_0}$ be a pseudo Abel sequence as in Corollary 6.8, and denote Z_{e_0} by Z. For every integer $m \geq 0$ there exists an integer $e_1(m) \geq e_0$ such that for every integer $e \geq e_1(m)$ there is a dense open subset V_e of $Porc^{e,0}(X/C/\kappa)_Z$ such that for every κ-point $[\sigma_0]$ of V_e there exists a scroll R for $X/C/\kappa$ penning σ_0 and such that $(R, [\sigma_0(C)])$ is an m-twisting scroll.*

Proof. Let a vary among $a_S + i$ for $i = 0, \ldots, b_S - 1$. For each $i = 0, \ldots, b_S - 1$, by Proposition 7.6 applied to $a = a_S + i$, there exists an integer $c_{\Delta,i} \geq 0$ such that the corollary holds for every integer of the form $e = e_0 + a_S + i + c \ldots b_S$ with $c \geq c_{\Delta,i}$. Define c_0 to be the maximum of the integers $c_{\Delta,i}$ for $i = 0, \ldots, b_S - 1$. Then every integer $e \geq e_0 + a_S + c_0 \cdot b_S$ is of the form $\geq e_0 + a_S + i + c \cdot b_S$ for some $i = 0, \ldots, b_S - 1$ and some $c \geq c_{\Delta,i}$. Thus the corollary holds with $e_1(m) = e_0 + a_S + c_0 \cdot b_S$. $\square$

These m-twisting scrolls produce chains of rational curves which connect the boundary stratum $\mathrm{Porc}^{e+m,m}(X/C/\kappa)_Z$ to the interior of Z_{e+m}.

Proposition 7.8. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that Hypotheses 4.1 – 4.5 all hold. Let $(Z_e)_{e\geq e_0}$ be a pseudo Abel sequence as in Corollary 6.8, and denote Z_{e_0} by Z. Let $e \geq e_0$ be an integer, and let m be a positive integer such that a general porcupine in $Porc^{e+m,m}(X/C/\kappa)_Z$ is perfectly penned by an m-twisting scroll, i.e., $(R, [\sigma_0(C)])$ is an m-twisting scroll where σ_0 is the body of the porcupine. Then for every integer $0 \leq m' \leq m$ and for every integer $\delta \geq 0$, there is a nonempty open subset U of $Porc^{e+m+\delta,m'+\delta}(X/C/\kappa)_Z$ such that every κ-point of U is contained in a chain of rational curves in $Z_{e+m+\delta}$ which also contains a general point of $Porc^{e+m+\delta,m+\delta}(X/C/\kappa)_Z$ and all of whose nodes are smooth points of $Z_{e+m+\delta}$.*

Proof. Let $h : C' \to X$ be a general porcupine in $\mathrm{Porc}^{e+m,m}(X/C/\kappa)_Z$ with extended body (σ_0, D) where $D = \underline{t}_1 + \cdots + \underline{t}_m$ is a general, reduced divisor in C. And let R be a scroll penning h such that $(R, [\sigma_0(C)])$ is an m-twisting scroll. The hypotheses in the definition of m-twisting imply that $\sigma_0(C) + R_{t_1}$ on R moves in a linear system. And the general member of this linear system is the image of a new section $\sigma_1(C)$. The new porcupine $\sigma_1(C) + R_{t_2} + \cdots + R_{t_m}$ is a point of $\mathrm{Porc}^{e+m,m-1}(X/C/\kappa)_Z$. The pencil of divisors on R spanned by this porcupine and $h(C')$ gives a family of stable sections parameterized by the "pencil" $\mathbb{P}^1$, i.e., it gives a rational curve in the closure $\overline{\mathrm{Porc}^{e+m,m-1}(X/C/\kappa)_Z}$ which contains the new porcupine as well as the original porcupine in $\overline{\mathrm{Porc}^{e+m,\geq(m-1)}}(X/C/\kappa)_Z$.

Now $\mathrm{Porc}^{e+m,m}(X/C/\kappa)_Z$ has pure codimension 1 in $\mathrm{Porc}^{e+m,\geq(m-1)}(X/C/\kappa)_Z$ by Proposition 5.2. So since the rational curves in the closure $\overline{\mathrm{Porc}^{e+m,m-1}}(X/C/\kappa)_Z$ obtained from these pencils of divisors on m-twisting scrolls $(R,[\sigma_0])$ contain every sufficiently general point $h(C')$ of $\mathrm{Porc}^{e+m,m}(X/C/\kappa)_Z$ and yet are not contained in $\mathrm{Porc}^{e+m,m}(X/C/\kappa)_Z$ (since they also parameterize some porcupines with only $m-1$ quills), the union of these rational curves contains a nonempty open subset U of $\mathrm{Porc}^{e+m,m-1}(X/C/\kappa)_Z$.

Because $(R,[\sigma_0(C)])$ is an m-twisting scroll, also $(R,[\sigma_1(C)])$ is an $(m-1)$-twisting scroll. So we can repeat the argument with $\mathrm{Porc}^{e+m,m-1}(X/C/\kappa)_Z$ in place of $\mathrm{Porc}^{e+m,m}(X/C/\kappa)_Z$. Thus, by descending induction on m', for every $0\leq m'\leq m$ there exists a family of chains of rational curves in $\overline{\mathrm{Porc}^{e+m,m'}}(X/C/\kappa)_Z$ whose nodes are all smooth points of Z_{e+m} and whose general member contains both a general point of $\mathrm{Porc}^{e+m,m}(X/C/\kappa)_Z$ and a general point of $\mathrm{Porc}^{e+m,m'}(X/C/\kappa)_Z$.

Of course we can always attach more quills, say a further δ quills to $h(C')$ which need not be contained in R. And then as we vary the porcupine in a linear system on R, since the space of quills through a general point is rationally connected, Theorem 2.1 implies we can extend that pencil of porcupines to a pencil of porcupines with the δ quills attached. Thus we can also connect a general point of $\mathrm{Porc}^{e+m+\delta,m+\delta}(X/C/\kappa)_Z$ and a general point of $\mathrm{Porc}^{e+m+\delta,m'+\delta}(X/C/\kappa)_Z$ by a chain of rational curves in $Z_{e+m+\delta}$ whose nodes are all smooth points. $\square$

Corollary 7.9. *Let $X/C/\kappa$ and $\mathcal{L}$ be as in Notation 2.4. Assume that Hypotheses 4.1 – 4.5 all hold. Let $(Z_e)_{e\geq e_0}$ be a pseudo Abel sequence as in Corollary 6.8, and denote Z_{e_0} by Z. There exists an integer e_1 such that for every integer $e\geq e_1$ and for every integer $\delta>0$, there exists an irreducible, quasi-projective variety T parameterizing chains of rational curves in $Z_{e+\delta}$ all of whose nodes are smooth points of $Z_{e+\delta}$ and such that the geometric generic point of T parameterizes a chain which contains the geometric generic point of $Z_{e+\delta}$ as well as a geometric generic point of Porc$^{e+\delta,\delta}(X/C/\kappa)_Z$ (we don't say "the" geometric generic point because possibly this stratum is reducible, which in fact is irrelevant for what comes next).*

Proof. Applying Proposition 7.6 with m equal to 1, there exists an integer $e_1(1)\geq e_0$ such that for every $e\geq e_1(1)$, a general porcupine in $\mathrm{Porc}^{e+1,1}(X/C/\kappa)_Z$ is perfectly penned by a 1-twisting scroll. The corollary is proved for $e_1=e_1(1)$ by induction on δ.

By Proposition 7.8, for all $e\geq e_1(1)$, a general point of Z_{e+1} is connected by a chain of rational curves to a general point of $\mathrm{Porc}^{e+1,1}(X/C/\kappa)_Z$, and the nodes of the chains are all smooth points of Z_{e+1}. This establishes the base case $\delta=1$ of the induction.

By way of induction, assume $\delta>1$ and assume the claim is known for $\delta-1$. Since $\delta-1>0$, again Proposition 7.8 implies that for every $e\geq e_1(1)$, a general point of $\mathrm{Porc}^{e+1+(\delta-1),1+(\delta-1)}(X/C/\kappa)_Z$ is connected by a chain of rational curves to a general point of $\mathrm{Porc}^{e+1+(\delta-1),\delta-1}(X/C/\kappa)_Z$, and the nodes of the chains are all smooth points of $Z_{e+\delta}$. By the induction hypothesis applied to $e+1$ in place of e, a general point of $Z_{(e+1)+(\delta-1)}$ is connected to a general point of $\mathrm{Porc}^{(e+1)+(\delta-1),\delta-1}(X/C/\kappa)\cap Z_{e+\delta}$ by a chain of rational curves whose nodes are all smooth points of $Z_{e+\delta}$. Concatenating the chains gives a chain connecting a general point of $Z_{e+\delta}$ to $\mathrm{Porc}^{e+\delta,\delta}(X/C/\kappa)_Z$, and all the nodes are smooth points: the one new node is inside $\mathrm{Porc}^{e+\delta,\delta-1}(X/C/\kappa)_Z$ which is a smooth point of $Z_{e+\delta}$. Thus the claim is proved by induction on δ. $\square$

Finally we can complete the proof of the main theorem.

Proof of Theorem 4.9. By Lemma 4.11 it suffices to prove the theorem for uncountable, algebraically closed fields κ of characteristic 0. So we assume κ is such.

By Corollary 6.8, there exists a pseudo Abel sequence $(Z_e)_{e\geq e_0}$ for $X/C/\kappa$. To prove this is an Abel sequence it only remains to verify (ii) of Definition 4.8, i.e., to prove that the geometric generic fiber of

$$\alpha|_{Z_e} : Z_e \to \mathrm{Pic}^e_{C/\kappa},$$

which is integral by Corollary 6.8, is also rationally connected.

By [Kol96, Theorem IV.3.10.3], it suffices to prove that two general points in a strong desingularization of the geometric generic fiber are connected by a chain of rational curves in the strong desingularization. Every chain in the geometric generic fiber whose nodes are contained in the smooth locus lifts to a chain in the strong desingularization. Thus it suffices to prove that two general points in the geometric generic fiber are connected by a chain of rational curves in the geometric generic fiber whose nodes are all in the smooth locus.

By Corollary 7.9, there exists an integer $e_1 \geq e_0$ such that for all $e \geq e_1$ and all $\delta \geq 0$, every general point of $\mathrm{Porc}^{e+\delta,\delta}(X/C/\kappa) \cap Z_{e+\delta}$ is contained in a chain of rational curves in $Z_{e+\delta}$ which contains a general point of $Z_{e+\delta}$, and the nodes of the chain are all in the smooth locus. Fix one such integer, say $e = e_1$. Rationally chain connected points of $\Sigma^{e+\delta}(X/C/\kappa)$ are Abel equivalent. Thus every general pair of points in a general fiber $Z_{e+\delta} \cap \alpha^{-1}(\mathcal{O}_C(\Gamma))$ is connected by a chain of rational curves in $Z_{e+\delta} \cap \alpha^{-1}(\mathcal{O}_C(\Gamma))$ to a general pair of points in $\mathrm{Porc}^{e+\delta,\delta}(X/C/\kappa)_Z \cap \alpha^{-1}(\mathcal{O}_C(\Gamma))$, and the nodes are all in the smooth locus. Finally, by Corollary 6.7, for the fixed integer e_1 there exists an integer $\delta_0(e_1) \geq 2g-1$ such that for all integers $\delta \geq \delta_0(e_1)$, this general pair of points in $\mathrm{Porc}^{e_1+\delta,\delta}(X/C/\kappa)_Z \cap \alpha^{-1}(\mathcal{O}_C(\Gamma))$ is connected by a chain of rational curves in $Z_{e_1+\delta} \cap \alpha^{-1}(\mathcal{O}_C(\Gamma))$ whose nodes are all in the smooth locus. The concatenation of these chains is again a chain of rational curves whose nodes are all in the smooth locus, since $\mathrm{Porc}^{e_1+\delta,\delta}(X/C/\kappa)_Z$ is in the smooth locus of $Z_{e_1+\delta}$ (we are also assuming that $\mathcal{O}_C(\Gamma)$ is sufficiently general and we are using generic smoothness for the restriction of the morphism α to the smooth locus of $Z_{e_1+\delta}$). This concatenated chain connects two general points of $Z_{e_1+\delta} \cap \alpha^{-1}(\mathcal{O}_C(\Gamma))$. Therefore, for every integer $e \geq e_1 + \delta_0(e_1)$ and for every general $\mathcal{O}_C(\Gamma)$ in $\mathrm{Pic}^e_{C/\kappa}$, $Z_e \cap \alpha^{-1}(\mathcal{O}_C(\Gamma))$ is rationally connected. $\square$

8. Rational simply connected fibrations over a surface

Theorem 4.9 is important because of its application to the existence of rational sections of fibrations over surfaces.

Corollary 8.1. *Let k be an algebraically closed field of characteristic 0. Let S be a smooth, integral, projective surface over k. Let $f : X \to S$ be a proper, surjective morphism. Assume there exists a Zariski open subset U of S and an invertible sheaf $\mathcal{L}$ on $f^{-1}(U)$ such that*

(i) *$S - U$ is a finite collection of k-points of S,*

(ii) *the restriction*
$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$
is smooth,

(iii) *$\mathcal{L}$ is f-very ample on $f^{-1}(U)$, and*

(iv) *the restriction of f over a general member of a Lefschetz pencil of divisors on S satisfies Hypotheses 4.3, 4.4 and 4.5 of Section 4.*

Then there exists a rational section of f.

Proof. There exists a Lefschetz pencil of ample divisors on S whose basepoints are all contained in U. It is important to note that, since these are ample divisors they have positive self-intersection so that there is at least one basepoint. After replacing S by the blowing up of the base locus, replacing f and $\mathcal{L}$ by the pullbacks over the blowing up, and replacing U by its inverse image in the blowing up, we may assume in addition that there exists a surjective, projective morphism
$$r : S \to \mathbb{P}^1_k.$$
Denote the function field $k(\mathbb{P}^1_k)$ by κ (of course it is not algebraically closed). So Spec $\kappa \to \mathbb{P}^1_k$ is the generic point of $\mathbb{P}^1_k$. Denote by C the generic fiber of r, which is a smooth, projective, geometrically integral κ-curve. And denote by X_κ the fiber product,
$$X_\kappa = C \times_S X$$
together with its projection $\pi : X_\kappa \to C$. Then X_κ is a smooth, projective, geometrically integral κ-scheme and π is a projective, flat morphism of κ-schemes. Moreover the function field of S equals the function field of C and the generic fiber of f equals the generic fiber of π. Thus to prove that there exists a rational section of f, it suffices to prove that there exists a section of π.

The hypotheses on $X, \mathcal{L}, U$ and Y imply Hypotheses 4.1 – 4.5 for $X_\kappa/C/\kappa$, and $\mathcal{L}_\kappa$. By Theorem 4.9, there exists an Abel sequence $(Z_e)_{e\geq\epsilon}$ for $X_\kappa/C/\kappa$. In particular, the morphism
$$\alpha|_Z : Z_e \to \mathrm{Pic}^e_{C/\kappa}$$
has integral and rationally connected geometric generic fiber. As explained in [GHMS05] and [Sta06] (see [HX09, Theorem 1.2] for a quite different proof), the fiber of $\alpha|_Z$ over *every* κ-point of $\mathrm{Pic}^e_{C/\kappa}$ contains a projective, geometrically integral and geometrically rationally connected κ-scheme. In particular integer multiples of the base points of the pencil of Lefschetz divisors on S give κ-points of $\mathrm{Pic}^e_{C/\kappa}$ for every integer e. Thus there exists a closed subscheme of Z_e (contained in the fiber of α over one of these κ-points of $\mathrm{Pic}^e_{C/\kappa}$) which is geometrically integral and rationally connected. But by Theorem 2.1, for the function field $\kappa = k(\mathbb{P}^1_k)$, every projective κ-scheme which is geometrically integral and rationally connected has a κ-point. Thus for every $e \geq \epsilon$ there exists a κ-point of Z_e.

There is an issue: Z_e is the coarse moduli space of stable sections, it is not a fine moduli space. Thus this κ-point of Z_e may not correspond to a stable section which is defined over κ. Instead it corresponds to a Galois field extension κ'/κ, a stable section $h : C' \to X \otimes_\kappa \kappa'$ defined over κ', and a lifting of the action of $\mathrm{Gal}(\kappa'/\kappa)$ to an action on C' such that h is equivariant for the Galois actions. As explained following Definition 3.2, C' has a unique component C'_0 such that the projection $(\pi \otimes \mathrm{Id}) \otimes h : C_0' \to C_0 \otimes_\kappa \kappa'$ is an isomorphism i_0. Because h is equivariant for the Galois action, and because $\pi \otimes \mathrm{Id}$ is the base-change of a morphism π which is defined over κ, the isomorphism i_0 is equivariant. Thus the section $h \circ i_0^{-1} : C \otimes_\kappa \kappa' \to X \otimes_\kappa \kappa'$ is equivariant for the Galois action. By Galois descent this is the base-change of a morphism of κ-schemes, $\sigma_0 : C \to X$. And since the base-change of σ_0 to κ' is a section of the base-change of π, also σ_0 is a section of π. Therefore there exists a section of π, and hence there exists a rational section of f. □

Part 2. **Homogeneous spaces**

9. Rational simple connectedness of homogeneous spaces

An earlier draft of these notes contained a complete proof of the following beautiful result of He. This proof is now incorporated in the article [dJHS08]. The technique is a bit different that the other techniques in these notes. For this reason, and also for reasons of length, we refer the reader to [dJHS08] for the proof.

Theorem 9.1. *Let k be an algebraically closed field of characteristic 0. Let G be a connected, reductive algebraic group over k. Let P be a parabolic subgroup of G. There exists a k-morphism*

$$\zeta : \mathbb{P}^1 \to G/P$$

such that for every parabolic subgroup Q of G containing P, denoting the projection by

$$\pi : G/P \to G/Q,$$

$\zeta^* T_\pi$ is ample, where T_π is the dual of the sheaf Ω_π of relative differentials.

When Y is a projective homogeneous space G/R of Picard number 1, then "usually" $\overline{\mathrm{M}}_{0,1}(Y/K, 1)$ and $\overline{\mathrm{M}}_{0,0}(Y/K, 1)$ are also homogeneous spaces G/P and G/Q. Thus Theorem 9.1 implies Hypothesis 4.5 of Section 4. Unfortunately, there are some cases where $\overline{\mathrm{M}}_{0,1}(Y/K, 1)$ and $\overline{\mathrm{M}}_{0,0}(Y/K, 1)$ are not homogeneous under the natural G-action. Since $\overline{\mathrm{M}}_{0,0}(Y/K, 1)$ is proper, there will be a projective homogeneous space $M = G/P$ contained in $\overline{\mathrm{M}}_{0,0}(Y/K, 1)$. And there is a little trick to get what is needed. To make clear what is involved in the trick, it is presented in a bit more generality than strictly needed, and the necessary hypotheses are stated explicitly.

Let k be an algebraically closed field. Let Y and M be smooth, connected, quasi-projective k-schemes. Let $\mathcal{L}$ be an ample invertible sheaf on Y. Let

$$u : M \to \overline{\mathrm{M}}_{0,0}(Y, 1)_{\mathrm{free}}$$

be a 1-morphism corresponding to a diagram of k-schemes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & & \\ M & & \end{array}$$

where q is a smooth, projective morphism whose geometric fibers are isomorphic to $\mathbb{P}^1$, and where p is a morphism whose restriction to each geometric fiber of q is a free rational curve in Y. There is an associated 1-morphism

$$v : \mathcal{C} \to \overline{\mathrm{M}}_{0,1}(Y, 1)_{\mathrm{free}}.$$

Proposition 9.2. *Let*

$$\zeta : \mathbb{P}^1 \to \mathcal{C}$$

be a k-morphism. Let (R, s) be the pair of a scroll $R \subset \mathbb{P}^1 \times_k Y$ and a marked section $s : \mathbb{P}^1 \to R$ corresponding to the composite 1-morphism

$$v \circ \zeta : \mathbb{P}^1 \to \overline{M}_{0,1}(Y, 1)_{free}.$$

If p is smooth, if u is unramified, if $(p \circ \zeta)^* T_Y$ is globally generated, and if $\zeta^* T_p$ and $\zeta^* T_q$ are both ample, then $(R, [s(\mathbb{P}^1)])$ is a very twisting scroll in Y.

Proof. First of all, the scroll R is simply

$$R := \mathbb{P}^1 \times_{q \circ \zeta, M, q} \mathcal{C}$$

and $s : \mathbb{P}^1 \to R$ is simply

$$s = (\mathrm{Id}_{\mathbb{P}^1}, \zeta) : \mathbb{P}^1 \to \mathbb{P}^1 \times_{q \circ \zeta, M, q} \mathcal{C}.$$

In particular, $N_{R/\mathbb{P}^1 \times_k Y}$ equals the pullback of $N_{\mathcal{C}/M \times_k Y}$. Denote the projection to $\mathbb{P}^1$ by

$$f_R : R \to \mathbb{P}^1.$$

Then for every coherent sheaf $\mathcal{F}$ on $\mathcal{C}$, cohomology and base change implies the natural map

$$(q \circ \zeta)^* R^1 q_* \mathcal{F} \to R^1 f_{R,*}(\mathrm{pr}_{\mathcal{C}}^* \mathcal{F})$$

is an isomorphism, and also

$$(q \circ \zeta)^* q_* \mathcal{F} \to f_{R,*}(\mathrm{pr}_{\mathcal{C}}^* \mathcal{F})$$

is an isomorphism if the R^1 sheaves are zero.

Consider the commutative diagram of coherent sheaves on $\mathcal{C}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_q & \longrightarrow & T_{\mathcal{C}} & \longrightarrow & q^* T_M & \longrightarrow & 0 \\ & & {\scriptstyle dp}\downarrow & & \downarrow{\scriptstyle dp} & & & & \\ 0 & \longrightarrow & p^* T_Y & \longrightarrow & p^* T_Y & \longrightarrow & 0 & & \end{array}.$$

Applying the Snake Lemma, there is an associated short exact sequence

$$0 \longrightarrow T_p \longrightarrow q^* T_M \longrightarrow N_{\mathcal{C}/M \times_k Y} \longrightarrow 0.$$

There is an associated long exact sequence

$$0 \longrightarrow q_* T_p \longrightarrow q_* q^* T_M \longrightarrow q_* N_{\mathcal{C}/M \times_k Y} \longrightarrow R^1 q_* T_p \longrightarrow 0,$$

using the fact that $R^1 q_* q^* T_M$ is zero.

In fact, $q_* q^* T_M$ is canonically isomorphic to T_M, $q_* N_{\mathcal{C}/M \times_k Y}$ is canonically isomorphic to $u^* T_{\overline{\mathrm{M}}_{0,0}(Y,1)}$, and the sheaf homomorphism above is canonically isomorphic to du. Because of the hypothesis that u is unramified, $q_* T_p$ is zero, there is a canonical isomorphism

$$R^1 q_* T_p \cong N_{M/\overline{\mathrm{M}}_{0,1}(Y,1)},$$

and each of these (isomorphic) sheaves is locally free. By relative duality, the locally free sheaf $R^1 q_* T_p$ is dual to $q_*(\Omega_p \otimes_{\mathcal{O}_{\mathcal{C}}} \Omega_q)$.

Claim 9.3. *The sheaf $(q \circ \zeta)^* q_*(\Omega_p \otimes_{\mathcal{O}_{\mathcal{C}}} \Omega_q)$ is anti-ample, and thus $(q \circ \zeta)^* N_{M/\overline{\mathrm{M}}_{0,1}(Y,1)}$ is ample.*

The surface R is abstractly a Hirzebruch surface. Denote the projection to $\mathbb{P}^1$ by

$$f_R : R \to \mathbb{P}^1.$$

Denote by $\mathcal{E}$ the pullback of $\Omega_p \otimes_{\mathcal{O}_\mathcal{C}} \Omega_q$. The claim is equivalent to the assertion that $(f_R)_*\mathcal{E}$ is anti-ample.

The normal bundle $s^*N_{s(\mathbb{P}^1)/R}$ is canonically isomorphic to ζ^*T_q, which is ample by hypothesis. Therefore the divisor $s(\mathbb{P}^1)$ on R moves in a basepoint free linear system. In other words, $s(\mathbb{P}^1)$ deforms to a family $\{s_t\}_{t\in\Pi}$ of sections whose images cover a dense open subset of R. In particular, Items (0) and (1) in Definition 3.6 hold for $(R, [s(\mathbb{P}^1)])$.

Of course the pullback of $\Omega_p \otimes_{\mathcal{O}_\mathcal{C}} \Omega_q$ to R is a locally free sheaf $\mathcal{E}$ whose dual $\mathcal{E}^\vee$ is canonically isomorphic to the pullback of $T_p \otimes_{\mathcal{O}_\mathcal{C}} T_q$. And $s^*\mathcal{E}^\vee$ equals $\zeta^*T_p \otimes_{\mathcal{O}_{\mathbb{P}^1}} \zeta^*T_q$. By hypothesis, each of ζ^*T_p and ζ^*T_q is ample on $\mathbb{P}^1$. Thus $s^*\mathcal{E}^\vee$ is ample on $\mathbb{P}^1$. Since ampleness is an open condition, for general t in Π also $s_t^*\mathcal{E}^\vee$ is ample. Therefore, for general t in Π, $s_t^*\mathcal{E}$ is anti-ample.

For every t there is an evaluation morphism

$$e_t : (f_R)_*\mathcal{E} \to s_t^*\mathcal{E}.$$

Of course, since the curves $s_t(\mathbb{P}^1)$ cover a dense open subset of R, the only local section of $(f_R)_*\mathcal{E}$ in the kernel of *every* evaluation morphism e_t is the zero section. Since $(f_R)_*\mathcal{E}$ is a coherent sheaf, in fact for $N \gg 0$ and for $t_1, \dots, t_N$ a general collection of closed points of Π, the morphism

$$(e_{t_1}, \dots, e_{t_n}) : (f_R)_*\mathcal{E} \to \bigoplus_{i=1}^{N} s_{t_i}^*\mathcal{E}$$

is injective. Since $t_1, \dots, t_N$ are general points, the last paragraph implies every summand $s_{t_i}^*\mathcal{E}$ is anti-ample. Thus the direct sum is anti-ample. And a locally free sheaf admitting an injective sheaf homomorphism to an anti-ample sheaf is itself anti-ample. Therefore $(f_R)_*\mathcal{E}$ is anti-ample, proving Claim 9.3.

As usual, denote by

$$\mathrm{ev} : \overline{\mathrm{M}}_{0,1}(Y, 1) \to Y$$

the evaluation morphism. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{v} & \overline{\mathrm{M}}_{0,1}(Y,1) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ Y & \xrightarrow{=} & Y \end{array}$$

There is also a Cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{v} & \overline{\mathrm{M}}_{0,1}(Y,1) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \text{forgetful}} \\ M & \xrightarrow{u} & \overline{\mathrm{M}}_{0,0}(Y,1) \end{array}$$

It follows that $\overline{\mathrm{M}}_{0,1}(Y, 1)$ is smooth at every point of $v(\mathcal{C})$, v is unramified and $N_{\mathcal{C}/\overline{\mathrm{M}}_{0,1}(Y,1)}$ is canonically isomorphic to $q^*N_{M/\overline{\mathrm{M}}_{0,0}(Y,1)}$. By the first commutative diagram, there is a short exact sequence

$$0 \longrightarrow T_p \longrightarrow v^*T_{\mathrm{ev}} \longrightarrow N_{\mathcal{C}/\overline{\mathrm{M}}_{0,1}(Y,1)} \longrightarrow 0.$$

By hypothesis, $\zeta^* T_p$ is ample. And by Claim 9.3, $\zeta^* N_{\mathcal{C}/\overline{\mathrm{M}}_{0,1}(Y,1)}$ is ample, i.e., $(q \circ \zeta)^* R^1 q_* T_p$ is ample. Therefore also $(v \circ \zeta)^* T_{\mathrm{ev}}$ is ample. Since every fiber of q is a smooth, free curve in Y, $(v \circ \zeta)^* T_{\mathrm{ev}}$ is canonically isomorphic to

$$(f_R)_* N_{R/\mathbb{P}^1 \times_k Y}(-s(\mathbb{P}^1)).$$

Since this is very ample, there is a vanishing

$$h^1(\mathbb{P}^1, (f_R)_* N_{R/\mathbb{P}^1 \times_k Y}(-s(\mathbb{P}^1)) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(-2)) = 0.$$

Since the fibers of q are free, $N_{R/\mathbb{P}^1 \times_k Y}$ is f_R-relatively globally generated. Thus also $R^1(f_R)_* N_{R/\mathbb{P}^1 \times_k Y}(-s(\mathbb{P}^1))$ vanishes. And then, by the Leray spectral sequence,

$$h^1(R, N_{R/\mathbb{P}^1 \times_k Y}(-s(\mathbb{P}^1)) \otimes_{\mathcal{O}_R} (f_R)^* \mathcal{L}^\vee)$$

is zero for every invertible sheaf $\mathcal{L}$ on $\mathbb{P}^1$ of degree ≤ 2, i.e., it is zero for $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(2)$. In other words, $(R, [s(\mathbb{P}^1)])$ satisfies Item (3) of Definition 3.6.

Since $N_{R/\mathbb{P}^1 \times_k Y}$ is f_R-relatively globally generated, to prove $N_{R/\mathbb{P}^1 \times_k Y}$ is globally generated and to prove $h^1(R, N_{R/\mathbb{P}^1 \times_k Y})$ is zero, it suffices to prove

$$h^1(R, N_{R/\mathbb{P}^1 \times_k Y} \otimes_{\mathcal{O}_R} f_R^* \mathcal{O}_{\mathbb{P}^1}(-1)) \text{ equals } 0.$$

There is a short exact sequence

$$0 \to N_{R/\mathbb{P}^1 \times_k Y}(-s(\mathbb{P}^1)) \otimes_{\mathcal{O}_R} f_R^* \mathcal{O}_{\mathbb{P}^1}(-1) \to N_{R/\mathbb{P}^1 \times_k Y} \otimes_{\mathcal{O}_R} f_R^* \mathcal{O}_{\mathbb{P}^1}(-1) \to s^* N_{R/\mathbb{P}^1 \times_k Y} \otimes_{\mathcal{O}_{\mathbb{P}}^1} \mathcal{O}_{\mathbb{P}^1}(-1) \to 0.$$

By Item (3), the first term has no $h^1(R, -)$. The third term is $\zeta^* T_q \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(-1)$. Since $\zeta^* T_q$ is ample, $h^1(\mathbb{P}^1, \zeta^* T_q \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(-1))$ equals 0. Thus, by the long exact sequence of cohomology associated to the short exact sequence also

$$h^1(R, N_{R/\mathbb{P}^1 \times_k Y} \otimes_{\mathcal{O}_R} f_R^* \mathcal{O}_{\mathbb{P}^1}(-1)) \text{ equals } 0.$$

This proves Item (2) of Definition 3.6. Therefore $(R, [s(\mathbb{P}^1)])$ is a 2-twisting scroll. $\square$

There are a few easy lemmas.

Lemma 9.4. *Let k be an algebraically closed field of characteristic 0. Let Y be a smooth, connected, projective k-scheme of positive dimension which is a homogeneous space for a linear algebraic group scheme G over k. Assume there exists an invertible sheaf $\mathcal{L}$ on Y which is an ample generator for Pic(Y). Then $\mathcal{L}$ is very ample.*

Proof. It is a standard result that the homology classes of the closures of the Bruhat cells give an additive basis for the integral homology of Y. Let U be the open Bruhat cell in Y and denote by D the complement of U in Y. Since the Picard group of Y is $\mathbb{Z}$ generated by $\mathcal{L}$, $\mathcal{L}$ is isomorphic to $\mathcal{O}_Y(D)$. By homogeneity $|D|$ is basepoint free. Thus the complete linear system defines a morphism

$$f : X \to \mathbb{P}^N.$$

Since $\mathcal{L}$ is ample, f is finite. Since $\mathrm{char}(k)$ equals 0, f is generically étale to its image $f(Y)$. Again by homogeneity, $f(Y)$ is smooth and f is everywhere étale. But $f(Y)$ is rationally connected, and rationally connected varieties are simply connected, cf. [Deb01, Corollary 4.18]. Therefore f is an isomorphism from Y to $f(Y)$, i.e., $\mathcal{L}$ is very ample. $\square$

Corollary 9.5. *Let k be an algebraically closed field of characteristic* 0*. Let Y be a smooth, connected, projective k-scheme of positive dimension which is a homogeneous space for a linear algebraic group scheme G over k. Assume there exists an invertible sheaf $\mathcal{L}$ on Y which is an ample generator for Pic(Y). Then $\overline{M}_{0,1}(Y,1)_{free}$ equals all of $\overline{M}_{0,1}(Y,1)$, the evaluation morphism*

$$ev : \overline{M}_{0,1}(Y,1) \to Y$$

is smooth and projective, and there exists a very twisting scroll in Y (in particular, ev is surjective).

Proof. Since Y is projective, and since the radical R_G is solvable, the Lie-Kolchin theorem implies R_G fixes a point p of Y. Since R_G is normal in G, for every g in G, R_G equals gR_Gg^{-1}. Thus R_G also fixes $g \cdot p$. Since Y is a homogeneous space, R_G fixes every point of Y. In other words, the action of G on Y factors through the quotient $G \to G/R_G$. Thus, without loss of generality, assume G is connected and semisimple.

The action of G on Y determines a sheaf homomorphism

$$T_eG \otimes_k \mathcal{O}_Y \to T_Y$$

which is surjective because the action is separable and Y is homogeneous. Thus T_Y is globally generated. Therefore every smooth, rational curve in Y is free. This implies $\overline{\mathrm{M}}_{0,1}(Y,1)_{\mathrm{free}}$ equals $\overline{\mathrm{M}}_{0,1}(Y,1)$ and

$$\mathrm{ev} : \overline{\mathrm{M}}_{0,1}(Y,1) \to Y$$

is smooth (it is always projective).

Let P be the stabilizer subgroup of a point in Y, let B be a Borel subgroup of G contained in Y, and let T be a maximal torus in B. The data (G,B,T) determines a root system Φ. Denoting by I the set of simple roots in this root system, the parabolic subgroups of G containing B are in one-to-one, order-preserving correspondence with subsets of I. Because Y has Picard number 1, P is a maximal parabolic subgroup Thus P equals the parabolic subgroup P_{I_j} where $I_j = I - \{j\}$ for an element j of I. Thus Y is isomorphic as a k-scheme with G-action to G/P_{I_j}. As proved in [Coh95, §4.20] and [CC98, Lemma 3.1], the subvariety

$$L := P_{\{j\}} \cdot P_{I_j}/P_{I_j}$$

is a line in G/P_{I_j} with respect to $\mathcal{L}$, and containing the point

$$p := P_{I_j}/P_{I_j}.$$

(In fact this is a bit irrelevant. By standard theory, the classes of Schubert cycles give an additive basis for the homology of G/P. As discussed before, $\mathcal{L}$ equals $\mathcal{O}_Y(D)$ where D is the unique Schubert cycle which is a divisor. So Poincaré duality implies there exists a Schubert cycle L which is a curve and whose $\mathcal{L}$-degree equals 1, i.e., L is a line.)

The action of G on Y induces an action of G on $\overline{\mathrm{M}}_{0,0}(Y,1)$ and on $\overline{\mathrm{M}}_{0,1}(Y,1)$. The stabilizer of (L,p) in $\overline{\mathrm{M}}_{0,1}(Y,1)$ contains the Borel subgroup B, and thus is of the form P_{K_j} for a subset $K_j \subset I_j$. Since this is parabolic, the orbit $\mathcal{C}$ of (L,p) is a projective (hence closed) G-orbit. The image M of $\mathcal{C}$ in $\overline{\mathrm{M}}_{0,0}(Y,1)$ is also a projective G-orbit. Observe that P_j acts transitively on the subset $\{(L,q)|q \in L\}$ of $\overline{\mathrm{M}}_{0,1}(Y,1)$. Thus the fiber of the forgetful morphism

$$\overline{\mathrm{M}}_{0,1}(Y,1) \to \overline{\mathrm{M}}_{0,0}(Y,1)$$

over $[L]$ equals the fiber of $\mathcal{C} \to Y$ over $[L]$. By homogeneity it follows that

$$\mathcal{C} \text{ equals } \overline{\mathrm{M}}_{0,1}(Y,1) \times_{\overline{\mathrm{M}}_{0,0}(Y,1)} M.$$

So the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & & \\ M & & \end{array}$$

is a diagram of smooth, projective morphisms where every geometric fiber of q is a smooth, rational curve.

By Theorem 9.1, there exists a k-morphism,

$$\zeta : \mathbb{P}^1 \to \mathcal{C}$$

such that $\zeta^* T_p$ and $\zeta^* T_q$ are both ample sheaves. The morphism p is smooth by homogeneity. By homogeneity T_Y is globally generated so that also $(p \circ \zeta)^* T_Y$ is globally generated. And the inclusion $u : M \to \overline{\mathrm{M}}_{0,0}(Y,1)_{\mathrm{free}}$ is unramified. Thus, by Proposition 9.2, the composition

$$v \circ \zeta : \mathbb{P}^1 \to \overline{\mathrm{M}}_{0,1}(Y,1)_{\mathrm{free}}$$

is a very twisting scroll in Y. □

Corollary 9.6. *Let k be an algebraically closed field of characteristic 0. Let Y be a smooth, connected, projective k-scheme of positive dimension which is a homogeneous space for a linear algebraic group scheme G over k. Assume there exists an invertible sheaf $\mathcal{L}$ on Y which is an ample generator for Pic(Y). Then every geometric fiber of*

$$ev : \overline{M}_{0,1}(Y,1) \to Y$$

is nonempty and rationally connected.

Proof. By Corollary 9.5, ev is smooth and projective. Because Y is homogeneous and projective, every connected, finite, étale cover of Y is an isomorphism. In particular, using homogeneity, the finite part of the Stein factorization of ev is a connected, finite, étale cover of Y, and thus an isomorphism. Therefore every fiber of ev is connected.

Moreover, there exists a morphism

$$\zeta : \mathbb{P}^1 \to \overline{\mathrm{M}}_{0,1}(Y,1)$$

which is, among other things, very twisting relative to the morphism ev. Therefore [Sta04, Proposition 3.6] implies that a general fiber of ev is rationally connected (the proof of this proposition uses the characteristic 0 hypothesis). □

Lemma 9.7. *Let k be an algebraically closed field of characteristic 0. Let Y be a smooth, connected, projective k-scheme of positive dimension which is a homogeneous space for a linear algebraic group scheme G over k. Assume there exists an invertible sheaf $\mathcal{L}$ on Y which is an ample generator for Pic(Y). Then Y satisfies Hypothesis 4.4 of Section 4.*

Proof. The first part of the argument is essentially [Kol96, Corollary IV.4.14]. To try to be self-contained, we repeat the argument. By the techniques of Campana and Kollár-Miyaoka-Mori, there exists a quasi-projective k-scheme Q, a dense open subset U of Y, and a projective morphism

$$\phi : U \to Q$$

such that every fiber of ϕ is connected by chains of lines in Y, and a very general line in Y intersecting U is contained in a fiber of ϕ. But then by homogeneity, the maximal domain of definition U for such a quotient equals all of Y. Since every point of Y is contained in a line (by Corollary 9.5), every fiber of ϕ is positive-dimensional. But since Y has Picard number 1, the only morphism from Y with a positive-dimensional fiber is the constant morphism. Thus all of Y is a fiber of ϕ, i.e., every pair of points of Y is connected by a chain of lines in Y.

Let n_0 be an integer such that for all $n \geq n_0$,

$$\mathrm{ev} : \mathrm{Chn}_2(X/k, n) \to Y \times_k Y$$

is surjective. This is a G-equivariant morphism for the evident actions of G on the domain and target. By the Bruhat decomposition, there exists an open orbit $U = \Delta(G) \cdot (P/P, sP/P)$ in $Y \times_k Y$. Observe that the stabilizer $H = P \cap (sPs^{-1})$ of this point is a connected, linear algebraic group. In particular it is birationally rationally connected. Let F denote the orbit of ev over a point in U. There is a Cartesian diagram

$$\begin{tikzcd} G \times_k F \arrow[r] \arrow[d, "\mathrm{pr}_G"'] & \mathrm{ev}^{-1}(U) \arrow[d, "\mathrm{ev}"] \\ G \arrow[r] & U = G/H \end{tikzcd}$$

In particular, all fibers of

$$G \times_k F \to \mathrm{ev}^{-1}(U)$$

are isomorphic to the birationally rationally connected variety H.

Because

$$\mathrm{ev} : \overline{\mathrm{M}}_{0,1}(Y, 1) \to Y$$

is smooth and projective with rationally connected fibers, a simple induction argument proves that $\mathrm{Chn}_2(Y/k, n)$ is also rationally connected. Thus $\mathrm{ev}^{-1}(U)$ is birationally rationally connected. Because the target and fibers of the morphism

$$G \times_k F \to \mathrm{ev}^{-1}(U)$$

are both birationally rationally connected, also $G \times_k F$ is birationally rationally connected by [GHS03]. Since the image of a morphism from a birationally rationally connected variety is birationally rationally connected, also F is birationally rationally connected. Since F is projective, in fact F is rationally connected. By homogeneity, every fiber of

$$\mathrm{ev} : \mathrm{ev}^{-1}(U) \to U$$

is rationally connected. Thus Y is rationally simply connected by chains of free lines. □

Corollary 9.8. *Let κ be an algebraically closed field of characteristic* 0*. Let C be a smooth, geometrically connected, projective curve over κ. Let $\pi : X \to C$ be a proper, smooth morphism. And let $\mathcal{L}$ be an ample invertible sheaf on X. Assume that every geometric fiber of π is a connected, projective, positive-dimensional scheme which is homogeneous under a linear algebraic group G. And assume that the restriction of $\mathcal{L}$ to every fiber is an ample generator of the Picard group. The Hypotheses 4.1 – 4.5 of Section 4 hold.*

Proof. By Lemma 9.4, $\mathcal{L}$ is very ample. By Lemma 9.7, Y satisfies Hypothesis 4.4. By Corollary 9.5, Y satisfies Hypothesis 4.5. And by Corollary 9.6, π satisfies Hypothesis 4.3. □

Corollary 9.9. *Let κ be an algebraically closed field of characteristic* 0*. Let S be a smooth, integral, projective surface over κ. Let $f : X \to S$ be a proper, surjective morphism. Assume there exists a Zariski open subset U of S and an invertible sheaf $\mathcal{L}$ on $f^{-1}(U)$ such that*

(i) *$S - U$ is a finite collection of κ-points of S,*

(ii) *the restriction*
$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$
is smooth,

(iii) *$\mathcal{L}$ is f-ample, and*

(iv) *the fiber of f over every geometric point of U is a homogeneous space for a linear algebraic group, and the restriction of $\mathcal{L}$ to this fiber is a generator of the Picard group.*

Then there exists a rational section of f.

Proof. Because of Corollary 9.8, all the hypotheses of Corollary 8.1 are satisfied. Therefore Corollary 8.1 implies there exists a rational section of f. □

10. Discriminant avoidance

There is one more technique needed to deduce the main application. The statement is reviewed below. The proof is written up carefully in [dJS05], which has been submitted. In fact this technique is similar to techniques used in the works of Totaro and Edidin-Graham in their constructions of equivariant cohomology for reductive groups.

Let $\mathcal{G}$ be a reductive group scheme over some base scheme T. Let $\mathcal{X}$ be a smooth, projective T-scheme on which $\mathcal{G}$ acts. For every T-scheme S and every $\mathcal{G}$-torsor $\mathcal{T}$ over S, there is an associated S-scheme
$$\mathcal{X}_{\mathcal{T}} := \mathcal{X} \times_T \mathcal{T}/\mathcal{G},$$
the quotient by the free action of $\mathcal{G}$. Let U be a dense open subscheme of T. Let c be a nonnegative integer. Consider the following two properties of the datum $(T, \mathcal{G}, \mathcal{X})$ and the integer c.

Property 10.1. For every algebraically closed field over T, Spec $k \to T$, for every projective, integral k-scheme S of dimension c, and for every $\mathcal{G}$-torsor $\mathcal{T}$ over S, the projection
$$\mathcal{X}_{\mathcal{T}} \to S$$
admits a rational section.

Property 10.2. For every algebraically closed field over U, Spec $k \to U$, for every quasi-projective, integral k-scheme S of dimension c, and for every $\mathcal{G}$-torsor $\mathcal{T}$ over S, the projection

$$\mathcal{X}_{\mathcal{T}} \to S$$

admits a rational section.

The basic technique of "discriminant avoidance" proves the following.

Proposition 10.3. [dJS05] *If Property 1 holds, then Property 2 holds.*

Part 3. The Period-Index Theorem and Serre's "Conjecture II"

11. Statement of de Jong's theorem and Serre's conjectures

There is much to say about de Jong's Period-Index theorem and the general "period-index problem"; more in fact than I can cover here. One excellent source is [CT06]. I will just recall the statement of de Jong's theorem.

Theorem 11.1. [dJ04], [dJS05] *Let k be an algebraically closed field. Let K/k be the function field of a surface. For every division algebra D with center K and finite dimension $\dim_K(D) = n^2$, the order of $[D]$ in the Brauer group $Br(K)$ equals n.*

I will say a little more about Serre's conjectures. Let K be a field and let G be a linear algebraic group defined over K. A *left G-torsor* is a K-variety $\mathcal{T}$ together with a left action of G on $\mathcal{T}$ by K-morphisms,

$$m : G \times_K \mathcal{T} \to \mathcal{T}$$

such that $\mathcal{T} \otimes_K \overline{K}$ is isomorphic to $G \otimes_K \overline{K}$ as a $\overline{K}$-scheme with a left action of $G \otimes_K \overline{K}$. The *trivial G-torsor* is $\mathcal{T} = G$ with the usual left action. A G-torsor $\mathcal{T}$ is isomorphic to G as a G-torsor if and only if $\mathcal{T}$ has a K-point. J.-P. Serre formulated two conjectures about torsors for a semisimple algebraic K-group.

Conjecture 11.2 (Serre's "Conjecture I")**.** [Ser02] If G is connected and semisimple, and if K is a perfect field of *cohomological dimension* 1, then every G-torsor over K is trivial, i.e., every G-torsor has a K-point.

Conjecture 11.3 (Serre's "Conjecture II")**.** [Ser02, p. 137] If G is connected, simply connected and semisimple, and if K is a perfect field of *cohomological dimension* 2, then every G-torsor over K is trivial, i.e., every G-torsor has a K-point.

Remark 11.4. In [Ser95], Serre explains that the hypothesis that K is perfect is too strong in these conjectures. They should also hold if the perfect hypothesis is replaced by the hypothesis that $[K : K^p] \leq p^2$ and $H^3_p(K')$ is 0 for all finite, separable extensions K' of K. In particular, this new hypothesis holds when K is a function field of a surface over an algebraically closed field k, i.e., K is a finitely generated k-extension and tr.deg.(K/k) equals 2.

Of course I have not defined the "cohomological dimension". It is straightforward to define. But instead let me point out a special case. Let k be an algebraically closed field, let B be a finite type, integral k-scheme, and let K be $k(B)$. Then the cohomological dimension of K equals $\dim(B)$. Thus Serre's "Conjecture I" would apply if B were a curve, and Serre's "Conjecture II" would apply if B were a surface.

Following work in special cases by a number of authors, Steinberg completely proved Serre's Conjecture I in 1965, [Ste65]. Although there are many partial results in the direction of Serre's Conjecture II, the general case remains open. However, the works of many people have gradually reduced the general case for K the function field of a surface over an algebraically closed field k to the special case, for that same field K, where G is of type E_8; confer the works of Merkurjev and Suslin; E. Bayer and Parimala; Chernousov; and P. Gille. One nice summary of this work is [CTGP04, Theorem 1.2(v)]. The main application of Theorem 4.9 and Corollary 8.1 settles the "split case" of Serre's Conjecture II for function fields K.

Theorem 11.5. *Let k be an algebraically closed field and let K/k be the function field of a surface. Let G be a connected, simply connected, semisimple algebraic group over k. Every G-torsor over K is trivial.*

In particular if G_K is a simple algebraic group over K of type E_8 then G_K is itself split. Thus every G_K-torsor over K is trivial. So Serre's Conjecture II holds over function fields for groups of type E_8.

12. Reductions of structure group

There is a statement that implies both Theorem 11.5 and Theorem 11.1. Let k be an algebraically closed field. Let G be a (smooth) connected, simply connected, semisimple algebraic group over k, and let P be a (reduced) parabolic subgroup of G. The center Z_G of G is a finite group scheme which is contained in P. There is a maximal quotient $P \twoheadrightarrow T_P$ which is an algebraic torus. Denote by $Z_{G,P}$ the kernel of the induced homomorphism $Z_G \to T_P$. The natural action of G on G/P lifts canonically to a linear action on every invertible sheaf over G/P. The finite subgroup scheme $Z_{G,P}$ is the maximal subgroup acting trivially on G/P and on every invertible sheaf over G/P. Thus $G/Z_{G,P}$ is the maximal quotient of G acting on G/P whose action lifts to a linear action on every invertible sheaf over G/P.

Theorem 12.1. *Let K/k be the function field of a surface over k. For every torsor $\mathcal{T}$ for $G/Z_{G,P}$ over K, the associated K-variety $X = \mathcal{T}/P$ has a K-point. Equivalently, the torsor $\mathcal{T}$ has a reduction of structure group to $P/Z_{G,P}$.*

Proof. Let T denote Spec of the Witt ring of k, e.g., $T = \text{Spec } k$ if k has characteristic 0 and $T = \text{Spec } \mathbb{Z}_p$ if k equals $\mathbb{Z}/p\mathbb{Z}$. And denote by U the open subset of T which consists of the generic point only.

Associated to the root datum for G, we can construct a smooth, linear algebraic group scheme G_T over T. And associated to the parabolic P, we can construct a closed subgroup scheme P_T over T. The definition of $Z_{G,P}$ extends to give a finite, flat group scheme Z_{G_T,P_T} over T. The quotient group scheme $\mathcal{G} = G_T/Z_{G_T,P_T}$ is a reductive group scheme since it is T-flat and the closed fiber is reductive. And the T-scheme $\mathcal{X} = G_T/P_T$ is smooth and quasi-projective. Since the closed fiber is proper over k, $\mathcal{X}$ is projective over T. Thus $\mathcal{G}$ and $\mathcal{X}$ satisfy the hypotheses in Section 10. The goal is to prove Property 10.2 for $c = 2$. Because of Proposition 10.3, it suffices to prove Property 10.1 for $c = 2$. In particular, to prove the theorem it suffices to assume that k has characteristic 0.

Now we use an induction argument proposed by Philippe Gille. The induction is on the corank, $\text{rank}(G) - \text{rank}(P)$. The base case is when P is a maximal parabolic. Again applying Proposition 10.3, to prove the result for all $G/Z_{G,P}$-torsors over fraction fields K of surfaces over k, it suffices to prove the result for each torsor which is the generic fiber of a $G/Z_{G,P}$-torsor over a smooth, projective, connected surface S over k. In this case the S-scheme $\mathcal{X}_S = [\mathcal{T}_S \times (G/P)]/(G/Z_{G,P})$ satisfies the hypotheses of Corollary 9.9. Thus Corollary 9.9 implies the result in this case.

By way of induction, assume the corank is > 1 and the result is known for all smaller values of the corank. Since the corank is > 1, P is not a maximal parabolic. Let Q be a maximal parabolic containing P. Then $Z_{G,P}$ is contained in $Z_{G,Q}$. For every $G/Z_{G,P}$-torsor over K, by the base case, the associated $G/Z_{G,Q}$-torsor has a reduction of structure group to $Q/Z_{G,Q}$. Thus the original $G/Z_{G,P}$-torsor has a reduction of structure group to $Q/Z_{G,P}$ (observe $(G/Z_{G,P})/(Q/Z_{G,P})$ is the same as $(G/Z_{G,Q})/(Q/Z_{G,Q})$ since both are just G/Q).

Now Q has a filtration by normal subgroup schemes,

$$Q = Q_0 \supset Q_1 \supset Q_2,$$

where Q_2 is the unipotent radical of Q and where Q_0/Q_1 is the maximal quotient of Q which is of multiplicative type, i.e., isomorphic to $\mathbf{G}_{m,k}$. By Hilbert's Theorem 90, every Q_0/Q_1-torsor over K is trivial, thus there is a reduction of structure group to $Q_1/Z_{G,P}$ (by construction $Z_{G,P}$ is contained in $P \cap Q_1$). And over a characteristic 0 field, every torsor for a unipotent group is trivial. Thus this $Q_1/Z_{G,P}$-torsor has a reduction of structure group to $(P \cap Q_1)/Z_{G,P}$ if and only if the associated $Q_1/Z_{G,P}Q_2$ torsor has a reduction of structure group to $(P \cap Q_1)/Z_{G,P}(P \cap Q_2)$. But Q_1/Q_2 is again a semisimple, simply connected algebraic group, G' $(P \cap Q_1)/(P \cap Q_2)$ is a parabolic subgroup P', and $Z_{G',P'}$ equals the image of $Z_{G,P}$. Since the corank of P' in G' is 1 less than the corank of P in G, by the induction hypothesis every $G'/Z_{G',P'}$-torsor over the fraction field of a surface K has a reduction of structure group to $P'/Z_{G',P'}$. Thus every $G/Z_{G,P}$-torsor over K has a reduction of structure group to $P/Z_{G,P}$. Therefore the result is proved by induction on the corank. $\square$

Proof of Theorem 11.1. Let G be $\mathbf{SL}_{n,k}$. Let m be an integer $1 < m < n$ and which divides n. Let P be the maximal parabolic subgroup of $\mathbf{SL}_{n,k}$ consisting of upper block matrices with the upper right block of size m and another diagonal block of size $n - m$. The center Z_G of $\mathbf{SL}_{n,k}$ is the group scheme μ_n of n^{th} roots of unity. And $Z_{G,P}$ is the subgroup scheme μ_m.

One can prove that a torsor for $G/Z_G = \mathbf{PGL}_{n,k}$ has a reduction of structure group to $G/Z_{G,P}$ if and only if the order of the corresponding element in the Brauer group $H^2(\text{Gal}(K), \mu_n)$ divides m. And then, by Theorem 12.1, there is a reduction of structure group to $P/Z_{G,P}$.

Here is a reformulation in terms of central simple algebras. Let C be a central simple algebra with center K and with $\dim_K(C) = n^2$. Let $\mathcal{T}_K$ be the K-scheme whose set of A-points for each commutative K-algebra A equals the set of A-algebra isomorphisms of $C \otimes_K A$ with $\text{Mat}_{n\times n,A}$. Since the automorphism group of $\text{Mat}_{n\times n}$ is $\mathbf{PGL}_n$, $\mathcal{T}_K$ is a $\mathbf{PGL}_{n,k}$-torsor over K. By the previous paragraph, if the order of $[C]$ in the Brauer group of K divides m, then there is a reduction of structure group to $P/Z_{G,P}$. But this is the same thing as an isomorphism of C with $\text{Mat}_{m\times m,K} \otimes_K B$

for some central simple algebra B. In particular, if D is a division algebra over K with $\dim_K(D) = n^2$, then D is not isomorphic to $\mathrm{Mat}_{m\times m,K} \otimes_K B$. Thus the order of $[D]$ does not divide m. Since this holds for every proper divisor m of n, the conclusion is that the order of $[D]$ equals n. $\square$

Proof of Theorem 11.5. By the same argument as in the proof of Theorem 12.1, it suffices to prove the case when k has characteristic 0. Denote by B a Borel subgroup of G. Then $Z_{G,B}$ is the trivial group scheme. So by Theorem 12.1, every G-torsor over K has a reduction of structure group to a B-torsor over K. Denote by $R_u(B)$ the unipotent radical of B. Since B is connected and solvable, $B/R_u(B)$ is of multiplicative type, i.e., isomorphic to $\mathbf{G}_{m,k}^r$ where r is the rank of G. By Hilbert's Theorem 90, every torsor for $B/R_u(B)$ is trivial. Thus there is a reduction of structure group to $R_u(B)$. But every torsor for a unipotent group over a characteristic 0 field is trivial. Thus the B-torsor is trivial, and hence the original G-torsor was also trivial. $\square$

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